Two-buffer fluid models with multiple ON-OFF inputs and threshold assistance

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ABSTRACT

We consider a two-buffer fluid model with N ON-OFF inputs and threshold assistance, which is an extension of the same model with $N = 1$ in [18]. While the rates of change of both buffers are piecewise constant and dependent on the underlying Markovian phase of the model, the rates of change for Buffer 2 are also dependent on the specific level of Buffer 1. This is because both buffers share a fixed output capacity, the precise proportion of which depends on Buffer 1. The generalization of the number of ON-OFF inputs necessitates slight modifications in the original rules of output-capacity sharing from [18], and considerably complicates both the theoretical analysis and numerical computation of various performance measures.

Here, we give a short explanation on how to derive the marginal probability distribution of Buffer 1, and bounds for that of Buffer 2. In an upcoming paper, we describe the procedures in more details. Furthermore, restricting Buffer 1 to a finite size, we determine its marginal probability distribution in the specific case of $N = 1$, thus providing numerical comparisons to the corresponding results in [18] where Buffer 1 is assumed to be infinite. We also demonstrate how this imposed restriction effects the bounds of marginal probabilities for Buffer 2.

Categories and Subject Descriptors

I.6.5 [**Simulation and Modelling**]: Model development; G.3 [**Probability and Statistics**]: [queueing theory, stochastic processes]

General Terms

Performance, Measurement

1. INTRODUCTION

Stochastic fluid models have a wide range of real-life applications, such as industrial and computer engineering, actuarial science, environmental modeling and telecommunications. A Markov-modulated single-buffer fluid model is a two-dimensional Markov process $\{X(t), \varphi(t) : t \in \mathbb{R}^+\}$, where $X(t)$ is the continuous *level* of the buffer, and $\varphi(t)$ is the discrete *phase* of the underlying irreducible Markov chain that governs the rates of change. A practical and well-studied case is *piecewise constant* rates: the fluid is assumed to have a constant rate c_i when $\varphi(t) = i$, for i in a finite state space $\mathcal{S}$. The traditional approach for obtaining performance measures of Markov-modulated single-buffer fluids with piecewise constant rates is to use spectral analysis (see, among others, [17, 3, 19, 23, 12]). Over the last two decades, matrix analytic methods have gained a lot of attention as an alternative and algorithmically effective approach for analyzing these standard fluids (see, for instance, [22, 21, 1, 2, 6, 10, 5, 7, 11, 8]).

In this paper, we consider a two-buffer fluid model $\{X(t), Y(t), \varphi_1(t), \varphi_2(t) : t \in \mathbb{R}^+\}$, where $X(t) \geq 0$ and $Y(t) \geq 0$ represent the levels of Buffers 1 and 2, respectively. At a given time $t \geq 0$, the rates of change of Buffer 1 depend only on the underlying Markovian phase $\varphi_1(t)$; however, the rates of change of Buffer 2 depend on both $\varphi_2(t)$ and $X(t)$. This is because while each buffer receives its own input sources, both buffers share a fixed output capacity c, in proportion dependent on the level of Buffer 1. More specifically, Buffer j receives N ON-OFF input sources, each has exponentially distributed ON- and OFF- intervals at corresponding rates α_j and β_j, and continuously generates fluid at rate R_j during ON- intervals, for $j = 1, 2$. When the fluid level $X(t)$ of Buffer 1 is above a certain threshold $x^* > 0$, Buffer 1 is allocated the total shared output capacity c, leaving Buffer 2 without any; when $0 < X(t) < x^*$, Buffer j has output capacity c_j, $c_1 + c_2 = c$; and when $X(t) = 0$, Buffer 1 has output capacity $\min\{iR_1, c_1\}$, and Buffer 2 $c - \min\{iR_1, c_1\}$, where i is the number of inputs of Buffer 1 being on at the time t.

This theoretical model is an extension of the same model with $N = 1$ in [18], where the reader can find a comprehensive account of practical applications in communication networks. We note that when $X(t) = 0$, the rule for output-capacity allocation in our general N ON-OFF input model differs to that in the single ON-OFF input model in [18], which is to allocate the total capacity c to Buffer 2. The totality rule is logical for the single ON-OFF input: when there is only one ON-OFF input for each buffer, Buffer 1 is empty only when its input is off; in that case, Buffer 2 can receive the whole output capacity c, until the moment the input of

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SMCTOOLS 2011, May 16, Paris, France
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DOI 10.4108/icst.valuetools.2011.245848

Buffer 1 is on again. In our model where each buffer receives N ON-OFF inputs for $N \geq 1$, it is possible for Buffer 1 to be empty while i inputs are on, for $0 \leq i \leq \lfloor \frac{c_1}{R_1} \rfloor$. Under these circumstances, assigning the total output capacity c to Buffer 2 would immediately cause Buffer 1 to try to increase from level 0, consequently grabbing back c_1 amount of output capacity. However, as $i \leq \lfloor \frac{c_1}{R_1} \rfloor$, the output capacity c_1 would be sufficient to empty Buffer 1, forcing it to give away the whole output capacity c to Buffer 2, etc. Therefore, applying the original totality rule at $X(t) = 0$ for the generalized N ON-OFF input model would potentially lead to inconsistency.

The behavior described above at level 0 for Buffer 1 when $0 \leq i \leq \lfloor \frac{c_1}{R_1} \rfloor$ is referred to as being *sticky* [11], a property arisen when net rates of the buffer for the same Markovian phase but different levels are different in a particular way that makes it unable to go up or down, thus remaining stuck at a level until the background Markov chain switches to a non-sticky phase. In our model, by allocating iR_1 output capacity to Buffer 1 and $c - iR_1$ to Buffer 2 when $X(t) = 0$ and $0 \leq i \leq \lfloor \frac{c_1}{R_1} \rfloor$, we let Buffer 1 remain at level zero, while eliminating potential uncertainty and utilizing the total output capacity in the most effective way. For the same reason, when $X(t) = x^*$ and $\lceil \frac{c_1}{R_1} \rceil \leq i \leq \lfloor \frac{c}{R_1} \rfloor$, the output capacity is iR_1 for Buffer 1, and $c - iR_1$ for Buffer 2. While the stickiness, borne in the generalization of the number of IN-OUT inputs, necessitates only slight modifications in the output-capacity allocation policy, it considerably complicates the analysis and numerical computation of performance measures of the model. To deal with this complication, we employ a mixture of tools from both dominant approaches: spectral analysis and matrix analytic methods.

The rest of the paper is organized as follows: in Section 2, we formulate the model mathematically. Assuming that both buffer sizes are infinite, we derive the marginal probability distribution of Buffer 1 in Section 3.1, and bounds for those of Buffer 2 in Section 3.2.

2. REFERENCE MODEL

Consider a four-dimensional Markov process $\{X(t), Y(t), \varphi_1(t), \varphi_2(t) : t \in \mathbb{R}^+\}$, where $X(t) \geq 0$ and $Y(t) \geq 0$ are the levels in buffers 1 and 2, respectively, and for $j = 1, 2$ and $N \geq 1$, $\varphi_j(t)$ represents the phase of the background irreducible Markov chain for Buffer j with finite state space $\mathcal{S} = \{0, \ldots, N\}$; state $i \in \mathcal{S}$ indicates that i ON-OFF inputs are on. The generator T_j for $\{\varphi_j(t)\}$ is

$$T_j = \begin{bmatrix} * & N\beta_j & & & \\ \alpha_j & * & (N-1)\beta_j & & \\ & \ddots & \ddots & \ddots & \\ & & (N-1)\alpha_j & * & \beta_j \\ & & & N\alpha_j & * \end{bmatrix},$$

with each diagonal element $*$ defined appropriately such that each row sum of T_j is 0. For $i_1, i_2 \in \mathcal{S}$, we denote by $\dot{x}_{i_1}$ and $\dot{y}_{i_2}$ the respective net rates for Buffer 1 in phase i_1 and Buffer 2 in phase i_2. For $X(t) > x^*$ and $Y(t) > 0$,

$$\begin{aligned} \dot{x}_{i_1} &= i_1 R_1 - c, \\ \dot{y}_{i_2} &= i_2 R_2, \end{aligned}$$

for $X(t) = x^*$ and $Y(t) > 0$:

$$\begin{aligned} \dot{x}_{i_1} &= 0 \quad \text{for } \lceil \frac{c_1}{R_1} \rceil \leq i_1 \leq \lfloor \frac{c}{R_1} \rfloor, \\ &= i_1 R_1 - c_1 \quad \text{otherwise}, \\ \dot{y}_{i_2} &= i_2 R_2 - (c - i_1 R_1) \quad \text{for } \lceil \frac{c_1}{R_1} \rceil \leq i_1 \leq \lfloor \frac{c}{R_1} \rfloor, \\ &= i_2 R_2 - c_2 \quad \text{otherwise}, \end{aligned}$$

for $0 < X(t) < x^*$ and $Y(t) > 0$:

$$\begin{aligned} \dot{x}_{i_1} &= i_1 R_1 - c_1 \\ \dot{y}_{i_2} &= i_2 R_2 - c_2, \end{aligned}$$

and for $X(t) = 0$ and $Y(t) > 0$:

$$\begin{aligned} \dot{x}_{i_1} &= 0 \quad \text{for } 0 \leq i_1 \leq \lfloor \frac{c_1}{R_1} \rfloor, \\ &= i_1 R_1 - c_1 \quad \text{otherwise}, \\ \dot{y}_{i_2} &= i_2 R_2 - (c - i_1 R_1) \quad \text{for } 0 \leq i_1 \leq \lfloor \frac{c_1}{R_1} \rfloor, \\ &= i_2 R_2 - c_2 \quad \text{otherwise}. \end{aligned}$$

For $Y(t) = 0$, $\dot{y}_{i_2}$ is the maximum between 0 and the net rate of Buffer 2 in $i_2 \in \mathcal{S}$ when $Y(t) > 0$.

We assume that $NR_j > c$, that $\frac{c_j}{R_j}, \frac{c}{R_j} \notin \mathbb{N}$, and that the system is positive recurrent.

3. INFINITE BUFFERS WITH MULTIPLE ON-OFF INPUTS

3.1 Analysis for Buffer 1

To analyze Buffer 1 when $N = 1$, Mahabhashyam *et al.* [18] consider an equivalent system of two standard single sub-buffers, each with a single ON-OFF input, one sub-buffer with constant output capacity c_1 and the other with constant output capacity c. Decomposing Buffer 1 in this fashion, the authors show that the marginal probability distribution of Buffer 1 can be obtained by appropriately combining the average time of going up from x^* and then going down to x^* in Sub-buffer 1, and the average time of going down from x^* and then going up to x^* in Sub-buffer 2. The authors determine analytic expressions for the former average time by using, from [20], the busy period distribution of a standard single buffer with one exponential ON-OFF input and constant output capacity, and for the latter by establishing a pair of partial differential equations, transferred into ordinary differential equations and then solved by a spectral decomposition technique.

In this paper, for general $N \geq 1$, we analyze Buffer 1 by modeling it as a level-dependent fluid and applying matrix analytic methods. With this approach, while it is not simple to obtain closed-form expressions for $N \geq 2$, we can obtain various performance measures numerically using fast convergent algorithms (see, most relevantly, [5, 11] and the references therein). The focus of this section is the marginal probability distribution for Buffer 1.

We refer to $X(t) = 0$ and $X(t) = x^*$ as *boundaries* $\circ$ and $*$, and $0 < X(t) < x^*$ and $X(t) > x^*$ as *bands* 1 and 2. While T_1 governs the transitions of $\{\varphi_1(t)\}$ for all $X(t) \geq 0$, the

rate of Buffer 1 in the same phase varies between boundaries and bands. Therefore, we partition $\mathcal{S}$ differently for each boundary and each band. We denote, respectively, by $\mathcal{S}_d^{(\bullet)}$, $\mathcal{S}_s^{(\bullet)}$ and $\mathcal{S}_u^{(\bullet)}$ the sets of states with negative, zero and positive net rates when Buffer 1 is at boundary $\bullet$, for $\bullet \in \{\circ, *\}$, and by $\mathcal{S}_-^{(k)}$ and $\mathcal{S}_+^{(k)}$ the sets of states with negative and positive net rates when Buffer 1 is in band k, for $k = 1, 2$. Then, $\mathcal{S} = \mathcal{S}_s^{(\circ)} \cup \mathcal{S}_u^{(\circ)} = \mathcal{S}_-^{(1)} \cup \mathcal{S}_+^{(1)} = \mathcal{S}_d^{(*)} \cup \mathcal{S}_s^{(*)} \cup \mathcal{S}_u^{(*)} = \mathcal{S}_-^{(2)} \cup \mathcal{S}_+^{(2)}$, with

$$\mathcal{S}_s^{(\circ)} = \mathcal{S}_-^{(1)} = \mathcal{S}_d^{(*)} = \mathcal{S}_-^{(2)} = \{0, \ldots, \lfloor \frac{c_1}{R_1} \rfloor\},$$
$$\mathcal{S}_u^{(\circ)} = \mathcal{S}_+^{(1)} = \mathcal{S}_+^{(2)} = \{\lceil \frac{c_1}{R_1} \rceil, \ldots, N\},$$
$$\mathcal{S}_s^{(*)} = \{\lceil \frac{c_1}{R_1} \rceil, \ldots, \lfloor \frac{c}{R_1} \rfloor\}, \mathcal{S}_u^{(*)} = \{\lceil \frac{c}{R_1} \rceil, \ldots, N\},$$

For each band k, we partition T_1 into sub-matrices $T_{\ell m}^{(k)}$, of which each element $[T_{\ell m}^{(k)}]_{ij}$ recording transitions going from $i \in \mathcal{S}_\ell^{(k)}$ to $j \in \mathcal{S}_m^{(k)}$, and denote by $C_\ell^{(k)}$ a diagonal absolute net rate matrix for $i \in \mathcal{S}_\ell^{(k)}$:

$$C_-^{(1)} = \begin{bmatrix} |-c_1| & & & \\ & |R_1 - c_1| & & \\ & & \ddots & \\ & & & |\lfloor \frac{c_1}{R_1} \rfloor R_1 - c_1| \end{bmatrix},$$

$$C_+^{(1)} = \begin{bmatrix} \lceil \frac{c_1}{R_1} \rceil R_1 - c_1 & & & \\ & (\lceil \frac{c_1}{R_1} \rceil + 1) R_1 - c_1 & & \\ & & \ddots & \\ & & & N R_1 - c_1 \end{bmatrix},$$

$$C_-^{(2)} = \begin{bmatrix} |-c| & & & \\ & |R_1 - c| & & \\ & & \ddots & \\ & & & |\lfloor \frac{c}{R_1} \rfloor R_1 - c| \end{bmatrix},$$

$$C_+^{(2)} = \begin{bmatrix} \lceil \frac{c}{R_1} \rceil R_1 - c & & & \\ & (\lceil \frac{c}{R_1} \rceil + 1) R_1 - c & & \\ & & \ddots & \\ & & & N R_1 - c \end{bmatrix}.$$

We illustrates the relationships between the large cast of characters in Figure 1.

Exploiting Markov-renewal arguments, da Silva Soares and Latouche [11, Theorem 4.2] prove that the stationary density vector of a Markov-modulated level-dependent single-buffer fluid queue can be obtained by properly combining limiting densities from above and below each boundary (when possible) and steady state probability masses at these boundaries. To obtain the necessary limiting densities and probability masses, we consider the jump chain $\{J_n : n \geq 0\}$ of the process $\{X(t), \varphi_1(t)\}$ restricted to the set of boundary states $\mathcal{B} = \{(\bullet, i) : \bullet \in \{\circ, *\}, i \in \mathcal{S}\}$. We note that this jump chain will also be useful for obtaining bounds on marginal probabilities of Buffer 2, as described in Section 3.2. By [11, Theorem 4.4] and block-partitioned according to $\mathcal{B} = (\circ, \mathcal{S}_u^{(\circ)}) \cup (\circ, \mathcal{S}_s^{(\circ)}) \cup (*, \mathcal{S}_u^{(*)}) \cup (*, \mathcal{S}_s^{(*)}) \cup (*, \mathcal{S}_d^{(*)})$, the $(2N+1) \times (2N+1)$ transition matrix Ω of $\{J_n\}$ is

$$\Omega = \left[\begin{array}{cc|ccc} \cdot & \Psi_{us}^{(\circ)} & \Lambda_{uu}^{(\circ,*)} & \Lambda_{us}^{(\circ,*)} & \cdot \\ P_{su}^{(\circ)} & P_{ss}^{(\circ)} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \Psi_{us}^{(*)} & \Psi_{ud}^{(*)} \\ \cdot & \cdot & P_{su}^{(*)} & P_{ss}^{(*)} & P_{sd}^{(*)} \\ \cdot & \hat{\Lambda}_{ds}^{(*,\circ)} & \hat{\Psi}_{du}^{(*)} & \hat{\Psi}_{ds}^{(*)} & \cdot \end{array}\right], \quad (1)$$

where $\Psi_{um}^{(\bullet)}, \hat{\Psi}_{\ell m}^{(*)}, \Lambda_{\ell m}^{(\circ,*)}, \hat{\Lambda}_{\ell m}^{(*,\circ)}$ and $P_{sm}^{(\bullet)}$ denote various *first passage probability* matrices, with

$[\Psi_{um}^{(\bullet)}]_{ij}$ = probability of returning to $\bullet$ and in $j \in \mathcal{S}_m^{(\bullet)}$, after initially increasing from $\bullet$ and in $i \in \mathcal{S}_u^{(\bullet)}$, while avoiding a higher boundary (if exists),

$[\hat{\Psi}_{dm}^{(*)}]_{ij}$ = probability of returning to x^* and in $j \in \mathcal{S}_m^{(\bullet)}$, after initially decreasing from x^* and in $i \in \mathcal{S}_d^{(\bullet)}$, while avoiding level 0,

$[\Lambda_{um}^{(\circ,*)}]_{ij}$ = probability of reaching x^* and in $j \in \mathcal{S}_m^{(*)}$, while avoiding level 0 after initially increasing from there in $i \in \mathcal{S}_u^{\circ}$,

$[\hat{\Lambda}_{ds}^{(*,\circ)}]_{ij}$ = probability of reaching level 0 and in $j \in \mathcal{S}_s^{(\circ)}$, while avoiding x^* after initially decreasing from there in $i \in \mathcal{S}_d^{*}$, and

$[P_{sm}^{(\bullet)}]_{ij}$ = probability of going from $i \in \mathcal{S}_s^{(\bullet)}$ to $j \in \mathcal{S}_m^{(\bullet)}$, in one transition.

For the remainder of the paper, we denote by I the identity matrix and by $\mathbf{1}$ the vector of ones of the appropriate size. Let Δ be a diagonal matrix with $[\Delta]_i = [T_1]_{ii}$ and define

$$P = I - \Delta^{-1} T_1. \quad (2)$$

Then, clearly each of $P_{su}^{(\circ)}, P_{ss}^{(\circ)}, P_{su}^{(*)}, P_{ss}^{(*)}$ and $P_{sd}^{(*)}$ is a submatrix of P:

$$P = \begin{bmatrix} P_{ss}^{(\circ)} & P_{su}^{(\circ)} \\ P_{uu}^{(\circ)} & P_{us}^{(\circ)} \end{bmatrix} = \begin{bmatrix} P_{dd}^{(*)} & P_{ds}^{(*)} & P_{du}^{(*)} \\ P_{sd}^{(*)} & P_{ss}^{(*)} & P_{su}^{(*)} \\ P_{ud}^{(*)} & P_{us}^{(*)} & P_{uu}^{(*)} \end{bmatrix}.$$

Respectively, $\Psi_{us}^{(\circ)}, [\Lambda_{us}^{(\circ,*)}, \Lambda_{uu}^{(\circ,*)}], \hat{\Lambda}_{ds}^{(*,\circ)}$ and $[\hat{\Psi}_{ds}^{(*)}, \hat{\Psi}_{du}^{*}]$ are equal to $\Psi_{+-}^{(1)}, \Lambda_{++}^{(1)}, \hat{\Lambda}_{--}^{(1)}$ and $\hat{\Psi}_{-+}^{(1)}$, the corresponding first passage probability matrices for the level-independent fluid queue $\{M_1(t), \rho_1(t) : t \in \mathbb{R}^+\}$ with finite size x^*, state space $\mathcal{S}_-^{(1)} \cup \mathcal{S}_+^{(1)}$, generator T_1 and rate matrices $C_-^{(1)}$ and $C_+^{(1)}$. By [10, Theorem 5.2],

$$\begin{bmatrix} \Lambda_{++}^{(1)} & \Psi_{+-}^{(1)} \\ \hat{\Psi}_{-+}^{(1)} & \hat{\Lambda}_{--}^{(1)} \end{bmatrix} = \begin{bmatrix} e^{\hat{U}_1 x^*} & \Psi_1 \\ \hat{\Psi}_1 & e^{U_1 x^*} \end{bmatrix} \begin{bmatrix} I & \Psi_1 e^{U_1 x^*} \\ \hat{\Psi}_1 e^{\hat{U}_1 x^*} & I \end{bmatrix}^{-1}, \quad (3)$$

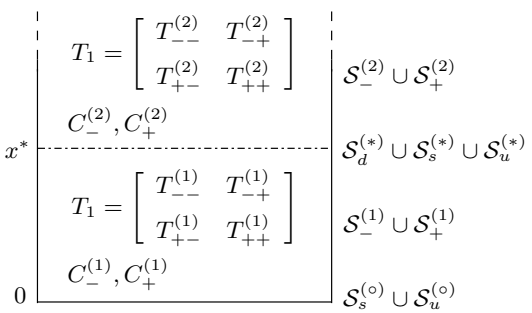

Figure 1: Buffer 1

where Ψ_1 is the minimum nonnegative solution to the Riccati equation

$$(C_+^{(1)})^{-1}T_{+-}^{(1)} + (C_+^{(1)})^{-1}T_{++}^{(1)}\Psi_1 + \Psi_1(C_-^{(1)})^{-1}T_{--}^{(1)} + \Psi_1(C_-^{(1)})^{-1}T_{-+}^{(1)}\Psi_1 = 0, \quad (4)$$

$\hat{\Psi}_1$ is the minimum nonnegative solution to the Riccati equation

$$(C_-^{(1)})^{-1}T_{-+}^{(1)} + (C_-^{(1)})^{-1}T_{--}^{(1)}\hat{\Psi}_1 + \hat{\Psi}_1(C_+^{(1)})^{-1}T_{++}^{(1)} + \hat{\Psi}_1(C_+^{(1)})^{-1}T_{+-}^{(1)}\hat{\Psi}_1 = 0, \quad (5)$$
$$U_1 = (C_-^{(1)})^{-1}T_{--}^{(1)} + (C_-^{(1)})^{-1}T_{-+}^{(1)}\Psi_1,$$

and

$$\hat{U}_1 = (C_+^{(1)})^{-1}T_{++}^{(1)} + (C_+^{(1)})^{-1}T_{+-}^{(1)}\hat{\Psi}_1.$$

Similarly, $[\Psi_{us}^{(*)}, \Psi_{ud}^{(*)}] = \Psi_2$, which is the first passage probability matrix for the infinite level-independent fluid queue $\{M_2(t), \rho_2(t) : t \in \mathbb{R}^+\}$ with state space $\mathcal{S}_-^{(2)} \cup \mathcal{S}_+^{(2)}$, the generator T_1 and rate matrices $C_-^{(2)}$ and $C_+^{(2)}$. By [22], the matrix Ψ_2 is the minimum nonnegative solution to the Riccati equation

$$(C_+^{(2)})^{-1}T_{+-}^{(2)} + (C_+^{(2)})^{-1}T_{++}^{(2)}\Psi_2 + \Psi_2(C_-^{(2)})^{-1}T_{--}^{(2)} + \Psi_2(C_-^{(2)})^{-1}T_{-+}^{(2)}\Psi_2 = 0. \quad (6)$$

Applying fast convergent algorithms described in [9, 4], we can solve Riccati equations (4), (5) and (6) to obtain $\Psi_1, \hat{\Psi}_1$ and Ψ_2, and consequently Ω.

We denote by $\boldsymbol{m} = [\boldsymbol{p}_s^{(\circ)}, \boldsymbol{p}_s^{(*)}]$ the probability mass vector of Buffer 1 at the set of boundary sticky states $\mathcal{K} = \{(\bullet, \zeta) : \bullet \in \{\circ, *\}, \zeta \in \mathcal{S}_s^{(\bullet)}\}$, and define $\mathcal{E}^{(*)} = \{(*, \zeta) : \zeta \in \mathcal{S}_u^{(*)} \cup \mathcal{S}_d^{(*)}\}$ and $\mathcal{E}^{(\circ)} = \{(\circ, \zeta) : \zeta \in \mathcal{S}_u^{(\circ)}\}$. Note that $\mathcal{K} = \mathcal{B} - \{\mathcal{E}^{(*)} \cup \mathcal{E}^{(\circ)}\}$. Then, the transition matrix $\Omega^{(*)}$ of the censored fluid queue on $\{\mathcal{B} - \mathcal{E}^{(*)}\}$ is

$$\Omega^{(*)} = \begin{bmatrix} \cdot & \Psi_{us}^{(\circ)} & \Lambda_{us}^{(\circ,*)} \\ P_{su}^{(\circ)} & P_{ss}^{(\circ)} & \cdot \\ \cdot & \cdot & P_{ss}^{(*)} \end{bmatrix} + \begin{bmatrix} \Lambda_{uu}^{(\circ,*)} & \cdot \\ \cdot & \cdot \\ P_{su}^{(*)} & P_{sd}^{(*)} \end{bmatrix} \begin{bmatrix} I & -\Psi_{ud}^{(*)} \\ -\hat{\Psi}_{ud}^{(*)} & I \end{bmatrix}^{-1} \begin{bmatrix} \cdot & \cdot & \Psi_{us}^{(*)} \\ \cdot & \hat{\Lambda}_{ds}^{(*,\circ)} & \hat{\Psi}_{ds}^{(*)} \end{bmatrix}. \quad (7)$$

Consequently, the transition matrix of the censored fluid queue on $\mathcal{K}$ is

$$\Omega^{(\circ)} = \Omega_{\mathcal{K}\mathcal{K}}^{(*)} + \Omega_{\mathcal{K}\mathcal{E}^{(\circ)}}^{(*)} \left\{ I - \Omega_{\mathcal{E}^{(\circ)}\mathcal{E}^{(\circ)}}^{(*)} \right\}^{-1} \Omega_{\mathcal{E}^{(\circ)}\mathcal{K}}^{(*)}, \quad (8)$$

and its generator matrix is

$$\Theta = \Delta^{(s)}(I - \Omega^{(\circ)}),$$

where $\Delta^{(s)}$ is the diagonal matrix with $[\Delta^{(s)}]_{ii} = [T_1]_{ii}$ for $0 \le i \le \lfloor \frac{c}{R_1} \rfloor$. By [11, Theorem 4.5], Θ is the generator of the censored fluid queue on $\mathcal{S}^{(\circ)} \cup \mathcal{S}^{(*)}$, and

$$\boldsymbol{m} = \kappa[\boldsymbol{x}_s^{(\circ)}, \boldsymbol{x}_s^{(*)}], \quad (9)$$

where $[\boldsymbol{x}_s^{(\circ)}, \boldsymbol{x}_s^{(*)}]$ is a solution of $[\boldsymbol{x}_s^{(\circ)}, \boldsymbol{x}_s^{(*)}]\Theta = \boldsymbol{0}$ and

$$\kappa = \left\{ [\boldsymbol{x}_s^{(\circ)}, \boldsymbol{x}_s^{(*)}]\boldsymbol{1} + \int_0^{x^*} \boldsymbol{y}_1(x)\boldsymbol{1}dx + \int_{x^*}^{\infty} \boldsymbol{y}_2(x)\boldsymbol{1}dx \right\}^{-1},$$

with

$$\boldsymbol{y}_2(x) = \{\boldsymbol{u}C_+^{(2)}N_+^{(2)}(x - x^*)\}(C^{(2)})^{-1}, \quad (10)$$
$$\boldsymbol{y}_1(x) = \{\boldsymbol{x}_s^{(\circ)}T_{-+}^{(1)}N_+^{(1)}(0, x) + \boldsymbol{d}C_-^{(1)}N_-^{(1)}(x^*, x)\}(C^{(1)})^{-1}, \quad (11)$$

the vectors $\boldsymbol{u}$ and $\boldsymbol{d}$ are the solution of

$$\boldsymbol{d} = \{\boldsymbol{x}_s^{(*)}T_{s-}^{(2)} + \boldsymbol{u}C_+^{(2)}\Psi_2\}(C_-^{(1)})^{-1}, \quad (12)$$
$$\boldsymbol{u} = \{\boldsymbol{x}_s^{(*)}T_{s+}^{(2)} + \boldsymbol{x}_s^{(\circ)}T_{-+}^{(1)}\Lambda_{uu}^{(\circ,*)} + \boldsymbol{d}C_-^{(1)}\hat{\Psi}_{-+}^{(1)}\}(C_+^{(2)})^{-1}, \quad (13)$$

$N_+^{(2)}(w)$ is a sub-matrix of $N^{(2)}(w) = [N_+^{(2)}(w), N_-^{(2)}(w)]$, the matrix of expected number of visits to level $w > 0$, while avoiding 0 after initially increasing from there, for the infinite fluid queue $\{M_2(t), \rho_2(t)\}$, and $N_-^{(1)}(x^*, w)$ is the matrix of expected number of visits to level $w < x^*$, after initially decreasing from x^* and while avoiding both x^* and 0, for the finite fluid queue $\{M_1(t), \rho_1(t)\}$.

By [21, Theorems 2.1 and 2.2],

$$N^{(2)}(w) = e^{K_2 w}[\ I \quad \Psi_2\ ], \quad (14)$$

with

$$K_2 = (C_+^{(2)})^{-1}T_{++}^{(2)} + \Psi_2(C_-^{(2)})^{-1}T_{-+}^{(2)}.$$

Denote by $N_+^{(1)}(0, w)$ the matrix of expected number of visits to level $w < x^*$, after initially increasing from 0 and while avoiding both 0 and x^*, for the finite fluid queue $\{M_1(t), \rho_1(t)\}$; then, by [10, Lemma 4.1],

$$\begin{bmatrix} N_+^{(1)}(0, w) \\ N_-^{(1)}(x^*, w) \end{bmatrix} = \begin{bmatrix} I & e^{K_1 x^*}\Psi_1 \\ e^{\hat{K}_1 x^*}\hat{\Psi}_1 & I \end{bmatrix}^{-1} \begin{bmatrix} e^{K_1 w} & e^{K_1 w}\Psi_1 \\ e^{\hat{K}_1(x^* - w)}\hat{\Psi}_1 & e^{\hat{K}_1(x^* - w)} \end{bmatrix}, \quad (15)$$

with

$$K_1 = (C_+^{(1)})^{-1}T_{++}^{(1)} + \Psi_1(C_-^{(1)})^{-1}T_{-+}^{(1)},$$
$$\hat{K}_1 = (C_-^{(1)})^{-1}T_{--}^{(1)} + \hat{\Psi}_1(C_+^{(1)})^{-1}T_{+-}^{(1)}.$$

After obtaining $\boldsymbol{m}$, we can determine the limiting densities $\boldsymbol{\pi}_+(x_+^*)$ from above x^* and $\boldsymbol{\pi}_-(x_-^*)$ from below x^*, as they are the respective counterparts of $\boldsymbol{u}$ and $\boldsymbol{d}$; that is, they are the solution of the system (12) and (13), with $\boldsymbol{x}_s^{(\circ)}$ and $\boldsymbol{x}_s^{(*)}$ replaced respectively by $\boldsymbol{p}_s^{(\circ)}$ and $\boldsymbol{p}_s^{(*)}$ [11, Theorem 4.3]. In turn, the density vector $\boldsymbol{\pi}(x)$ of Buffer 1 is

$$\boldsymbol{\pi}(x) = \boldsymbol{y}_1(x) \quad \text{for } 0 < x < x^*, \quad (16)$$
$$= \boldsymbol{y}_2(x) \quad \text{for } x > x^*, \quad (17)$$

with $\boldsymbol{u}, \boldsymbol{d}$, and $\boldsymbol{x}_s^{\circ}$, being replaced by $\boldsymbol{\pi}_+(x_+^*), \boldsymbol{\pi}_-(x_-^*)$ and $\boldsymbol{p}_s^{(\circ)}$ [11, Theorem 4.2]. Therefore, the marginal distribution

function of Buffer 1 is

$$P(X_1 \le x) = \boldsymbol{p}_s^{(\circ)}\mathbf{1} + \int_0^x \boldsymbol{\pi}(x)dx \quad \text{for } 0 < x < x^*,$$
$$= [\boldsymbol{p}_s^{(\circ)}, \boldsymbol{p}_s^{(*)}]\mathbf{1} + \int_0^x \boldsymbol{\pi}(x)dx \quad \text{for } x \ge x^*.$$

3.2 Analysis for Buffer 2

To derive the marginal probability distribution for Buffer 2 is not easy. Due to its output capacity being dependent on $X(t)$, when analyzed as a standalone process $\{Y(t), \varphi_2(t) : t \in \mathbb{R}^+\}$, Buffer 2 does not enjoy the Markovian property that $\{X(t), \varphi_1(t) : t \in \mathbb{R}^+\}$ has. Gautam *et al.* [15] give bounds for the stationary distribution of fluid models with semi-Markov inputs and constant outputs. To apply these results, we first need to transform Buffer 2 into an equivalent fluid queue with semi-Markov inputs and a constant output. We achieve the transformation by employing a *compensating source*, a concept developed by Elwalid and Mitra [13] and extended in Mahabhashyam *et al.* [18]. The role of a compensating source is to add the exact amount of input for maintaining a constant output, while keeping all the time the fluid level the same as that of the original, output-varying, buffer. In our case, the compensating source is to maintain a constant output of c for the equivalent fluid.

Consider a virtual fluid queue $\{Z(t), A(t), \varphi_2(t) : t \in \mathbb{R}^+\}$ which is equivalent to Buffer 2 and has two independent sources of input: N exponential ON-OFF inputs and the compensating source. Here, $Z(t) \ge 0$ is the level, $A(t)$ the semi-Markov process that drives the compensating source, and $\varphi_2(t)$ the irreducible Markov chain controlling ON-OFF inputs, with state space $\mathcal{S}$ and generator T_2. The semi-Markov process $A(t)$ has state space $\mathcal{B}$, the set of boundary states for the jump chain $\{J_n\}$ defined in Section 3.1 for the analysis of Buffer 1, as the output capacity of Buffer 2, and consequently the compensating source, changes each time $X(t)$ is in a boundary state. Specifically, the input rates $\dot{a}_{\bullet,i}$ of the compensating source are

$$\begin{aligned}\dot{a}_{\bullet,i} &= iR_1 \quad \text{for } (\bullet, i) \in (\circ, \mathcal{S}_s^{(\circ)}) \cup (*, \mathcal{S}_s^{(*)}),\\ &= c_1 \quad \text{for } (\bullet, i) \in (\circ, \mathcal{S}_u^{(\circ)}) \cup (*, \mathcal{S}_d^{(*)}),\\ &= c_2 \quad \text{for } (\bullet, i) \in (*, \mathcal{S}_u^{(*)}).\end{aligned}$$

For $n \ge 0$, let S_n be the time of the nth jump epoch in $A(t)$ with $S_n = 0$, Z_n the state of $A(t)$ immediately after the nth jump, and $\Omega(t)$ the kernel of $A(t)$, where

$$[\Omega(t)]_{ij} = P(S_1 \le t, Z_1 = j | Z_0 = i).$$

It is clear that

$$\Omega(\infty) = \Omega, \tag{18}$$

the transition matrix of the jump chain $\{J_n\}$, given by (1). We denote by $\widetilde{\Omega}(s)$ the matrix of Laplace-Stieltjes transforms of S_1, and in general, by $\widetilde{D}(s)$ and $\bar{D}(s)$ the respective LST counterparts of the sub-matrices D and $\hat{D}$ of Ω. The matrices $\widetilde{P}_{su}^{(\circ)}(s), \widetilde{P}_{ss}^{(\circ)}(s), \widetilde{P}_{su}^{(*)}(s)$, $\widetilde{P}_{ss}^{(*)}(s)$, and $\widetilde{P}_{sd}^{(*)}(s)$ are sub-matrices of $\widetilde{P}(s)$, where

$$\widetilde{P}(s) = (sI - \Delta)^{-1}(T_1 - \Delta). \tag{19}$$

To obtain the remaining sub-matrices of $\widetilde{\Omega}(s)$, we follow an analogous analysis to that described in Section 3.1. The matrices $\widetilde{\Psi}_{us}^{(\circ)}(s), [\widetilde{\Lambda}_{us}^{(\circ,*)}(s), \widetilde{\Lambda}_{uu}^{(\circ,*)}(s)], \bar{\Lambda}_{ds}^{(*,\circ)}(s)$ and $[\bar{\Psi}_{ds}^{(*)}(s), \bar{\Psi}_{du}^{(*)}]$ are equal to $\widetilde{\Psi}_{+-}^{(1)}(s), \widetilde{\Lambda}_{++}^{(1)}(s), \bar{\Lambda}_{--}^{(1)}(s)$ and $\bar{\Psi}_{-+}^{(1)}(s)$, the corresponding matrices of the LSTs of *first passage times* for $\{M_1(t), \rho_1(t)\}$. By [8, Theorem 3], for s such that $\mathrm{Re}(s) > 0$,

$$\begin{bmatrix} \widetilde{\Lambda}_{++}^{(1)}(s) & \widetilde{\Psi}_{+-}^{(1)}(s) \\ \bar{\Psi}_{-+}^{(1)}(s) & \bar{\Lambda}_{--}^{(1)}(s) \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} e^{\bar{U}_1(s)x^*} & \widetilde{\Psi}_1(s) \\ \bar{\Psi}_1(s) & e^{\widetilde{U}_1(s)x^*} \end{bmatrix} \begin{bmatrix} I & \widetilde{\Psi}_1(s)e^{\widetilde{U}_1(s)x^*} \\ \bar{\Psi}_1(s)e^{\bar{U}_1(s)x^*} & I \end{bmatrix}^{-1}, \tag{21}$$

where $\widetilde{\Psi}_1(s)$ is the minimum nonnegative solution to the Riccati equation

$$\begin{aligned}&(C_+^{(1)})^{-1}(T_{+-}^{(1)} - sI) + (C_+^{(1)})^{-1}(T_{++}^{(1)} - sI)\widetilde{\Psi}_1(s)\\ &+ \widetilde{\Psi}_1(s)(C_-^{(1)})^{-1}(T_{--}^{(1)} - sI)\\ &+ \widetilde{\Psi}_1(s)(C_-^{(1)})^{-1}(T_{-+}^{(1)} - sI)\widetilde{\Psi}_1(s) = 0,\end{aligned} \tag{22}$$

$\bar{\Psi}_1(s)$ is the minimum nonnegative solution to the Riccati equation

$$\begin{aligned}&(C_-^{(1)})^{-1}(T_{-+}^{(1)} - sI) + (C_-^{(1)})^{-1}(T_{--}^{(1)} - sI)\bar{\Psi}_1(s)\\ &+ \bar{\Psi}_1(s)(C_+^{(1)})^{-1}(T_{++}^{(1)} - sI)\\ &+ \bar{\Psi}_1(s)(C_+^{(1)})^{-1}(T_{+-}^{(1)} - sI)\bar{\Psi}_1(s) = 0,\end{aligned} \tag{23}$$

$$\widetilde{U}_1(s) = (C_-^{(1)})^{-1}(T_{--}^{(1)} - sI) + (C_-^{(1)})^{-1}(T_{-+}^{(1)} - sI)\widetilde{\Psi}_1(s),$$

and

$$\bar{U}_1(s) = (C_+^{(1)})^{-1}(T_{++}^{(1)} - sI) + (C_+^{(1)})^{-1}(T_{+-}^{(1)} - sI)\bar{\Psi}_1(s).$$

Similarly, $[\widetilde{\Psi}_{us}^{(*)}(s), \widetilde{\Psi}_{ud}^{(*)}(s)] = \widetilde{\Psi}_2(s)$, which is the matrix of the LST of first passage times for $\{M_2(t), \rho_2(t)\}$. By [6, Theorem 1], the matrix $\widetilde{\Psi}_2(s)$ is the minimum nonnegative solution to the Riccati equation

$$\begin{aligned}&(C_+^{(2)})^{-1}(T_{+-}^{(2)} - sI) + (C_+^{(2)})^{-1}(T_{++}^{(2)} - sI)\widetilde{\Psi}_2(s)\\ &+ \widetilde{\Psi}_2(s)(C_-^{(2)})^{-1}(T_{--}^{(2)} - sI)\\ &+ \widetilde{\Psi}_2(s)(C_-^{(2)})^{-1}(T_{-+}^{(2)} - sI)\widetilde{\Psi}_2(s) = 0.\end{aligned} \tag{24}$$

Bean *et al.* [7] give efficient algorithms for solving (22), (23) and (24) to obtain $\widetilde{\Psi}_1(s), \bar{\Psi}_1(s)$ and $\widetilde{\Psi}_2(s)$, and consequently $\widetilde{\Omega}(s)$.

Before we state the bounds for Buffer 2, we need to define *effective bandwidths* and *failure rate functions*. For $v > 0$, the *effective bandwidth* $\mathsf{eb}(v)$ of an input that generates $F(t)$ amount of fluid in time t (see, for example, [12, 16]) is defined to be

$$\mathsf{eb}(v) = \lim_{t\to\infty} \frac{1}{vt} \log E[e^{vF(t)}].$$

By [3, 12], the effective bandwidth $\mathsf{eb}_N(v)$ of a single exponential ON-OFF source for fixed v is

$$\mathsf{eb}_e(v) = \frac{R_2 v - \alpha_2 - \beta_2 + \sqrt{(R_2 v - \alpha_2 - \beta_2)^2 + 4\beta_2 R_2 v}}{2v}. \tag{25}$$

To obtain the effective bandwidth $\mathsf{eb}_c(v)$ for the compensating source, we begin by defining $\Phi(v, u)$ [15, equation (86)] to be the matrix with sub-matrices

$$[\Phi(v, u)]_{(\bullet,i),(\bar{\bullet},i')} = [\widetilde{\Omega}(v(u - \dot{a}_{\bullet,i}))]_{(\bullet,i),(\bar{\bullet},i')}. \tag{26}$$

Denote by $e(D)$ the maximal real eigenvalue of a matrix D, then by [15, Sections 4 and 5], the effective bandwidth $\mathsf{eb}_c(v)$ for fixed v is the unique positive solution to the equation

$$e(\Phi(v, \mathsf{eb}_c(v))) = 1. \tag{27}$$

Let η be the minimum positive solution to

$$\mathsf{eb}_c(\eta) + N\mathsf{eb}_e(\eta) = c. \tag{28}$$

By [15, Section 6.2], for $i, j \in \mathcal{B}$, the *failure rate function* $\lambda_{ij}(x)$ of the compensating source is

$$\lambda_{ij}(x) = \frac{[\Omega(t)]'_{ij}}{[\Omega]_{ij} - [\Omega(t)]_{ij}}. \tag{29}$$

The failure rate $\lambda_{ij}(x)$ is said to be an increasing failure rate (IFR) if $\lambda_{ij}(x) \uparrow x$, and a decreasing failure rate (DFR) if $\lambda_{ij}(x) \downarrow x$. For $i \in \mathcal{B}$, we denote by τ_i the expected sojourn time in i of $A(t)$

$$\tau_i = -\sum_{j \in \mathcal{B}} \widetilde{\Omega}'_{ij}(0), \tag{30}$$

by $\boldsymbol{\omega}$ the stationary vector associated with Ω, $\boldsymbol{\omega}\Omega = \mathbf{1}$ and $\boldsymbol{\omega}\mathbf{1} = 1$, by $\boldsymbol{p}$ the vector with elements

$$p_i = \frac{\omega_i \tau_i}{\sum\limits_{j \in \mathcal{B}} \omega_i \tau_j}, \tag{31}$$

and by $\boldsymbol{h}$ the left eigenvector of $\Phi(\eta, \mathsf{eb}_c(\eta))$ corresponding to eigenvalue one, $\boldsymbol{h}\Phi(\eta, \mathsf{eb}_c(\eta)) = \boldsymbol{h}$. We define H_c as

$$H_c = \sum_{i \in \mathcal{B}} \left[\frac{h_i}{\eta(\dot{a}_i - \mathsf{eb}_c(\eta))} \right] \left[\sum_{j \in \mathcal{B}} [\Phi(\eta, \mathsf{eb}_c(\eta))]_{ij} - 1 \right], \tag{32}$$

and $\Psi_{\max}(i,j)$ and $\Psi_{\min}(i,j)$ for $i, j \in \mathcal{B}$, depending on whether $\lambda_{ij}(x)$ is IFR or DFR and whether $\dot{a}_i > \mathsf{eb}_c(\eta)$, as in Table 1, using [15, Theorem 7].

Applying [15, Theorem 6] and then simplifying using [14, Section 4.2.4], we obtain the following result.

THEOREM 1. *For $x > 0$,*

$$K_* e^{-\eta x} \leq P(X_2 > x) \leq K^* e^{-\eta x}, \tag{33}$$

where

$$K_* = \tag{34}$$

$$\frac{\left[\dfrac{R_2}{\mathsf{eb}_e(\eta)\alpha_2}\right]^N H_c}{\max\limits_{s,(i,j)} \left[\dfrac{\alpha_2 + \beta_2}{\alpha_2 \beta_2}\right]^s \left[\dfrac{(\alpha_2 + \beta_2)(R_2 - \mathsf{eb}_e(\eta))}{\mathsf{eb}_e(\eta)\alpha_2^2}\right]^{(N-s)} \Psi_{\max}(i,j)},$$

and

$$K^* = \tag{35}$$

$$\frac{\left[\dfrac{R_2}{\mathsf{eb}_e(\eta)\alpha_2}\right]^N H_c}{\min\limits_{s,(i,j)} \left[\dfrac{\alpha_2 + \beta_2}{\alpha_2 \beta_2}\right]^s \left[\dfrac{(\alpha_2 + \beta_2)(R_2 - \mathsf{eb}_e(\eta))}{\mathsf{eb}_e(\eta)\alpha_2^2}\right]^{(N-s)} \Psi_{\min}(i,j)},$$

with $i, j \in \mathcal{B}$ and $1 \leq s \leq N$ such that $\dot{a}_i + sR_2 > c$ and $[\Omega]_{i,j} > 0$.

4. ACKNOWLEDGEMENT

This work has been subsidized by the ARC grant AUWB-08/13-ULB 5 financed by the Ministère de la Communauté française de Belgique. The second author also gratefully acknowledges the hospitality of the Mathematical Institute at University of Wrocław, where part of this work has been done.

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Table 1: Values of $\Psi_{\max}(i,j)$ and $\Psi_{\min}(i,j)$.

	IFR, $\dot{a}_i > \mathsf{eb}_c(\eta)$	IFR, $\dot{a}_i \leq \mathsf{eb}_c(\eta)$	DFR, $\dot{a}_i > \mathsf{eb}_c(\eta)$	DFR, $\dot{a}_i \leq \mathsf{eb}_c(\eta)$
$\Psi_{\max}(i,j)$	$\frac{[\Phi(\eta,\mathsf{eb}_c(\eta))]_{ij}\tau_i h_i}{[\Omega]_{ij}p_i}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(\dot{a}_i-\mathsf{eb}_c(\eta)))}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(\dot{a}_i-\mathsf{eb}_c(\eta)))}$	$\frac{[\Phi(\eta,\mathsf{eb}_c(\eta))]_{ij}\tau_i h_i}{[\Omega]_{ij}p_i}$
$\Psi_{\min}(i,j)$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(\dot{a}_i-\mathsf{eb}_c(\eta)))}$	$\frac{[\Phi(\eta,\mathsf{eb}_c(\eta))]_{ij}\tau_i h_i}{[\Omega]_{ij}p_i}$	$\frac{[\Phi(\eta,\mathsf{eb}_c(\eta))]_{ij}\tau_i h_i}{[\Omega]_{ij}p_i}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(\dot{a}_i-\mathsf{eb}_c(\eta)))}$

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