On the choice of the stochastic comparison method for multidimensional Markov chains analysis*

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ABSTRACT

The $\preceq_\Phi$ stochastic comparison ($\preceq_\Phi \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$) of multidimensional Continuous Time Markov Chains (CTMC)s is an efficient but a complex method for the performability evaluation of computer systems. Different techniques can be applied for the stochastic comparison of Markov chains. The coupling is an intuitive method, and may be applied by comparing the evolution of sample paths due to events to establish the $\preceq_{st}$ ordering. The increasing set method is based on the comparison of transition rates for a family of increasing sets. It is a more general formalism as it can be applied for all of these orderings : $\preceq_{st}$, $\preceq_{wk}$, and $\preceq_{wk^*}$. The goal of this paper is to identify the relationships between these orderings, in order to determine the method to apply for establishing comparisons between models. Altough the $\preceq_{st}$ ordering between random variables implies $\preceq_{wk}$ and $\preceq_{wk^*}$ orderings, this result could not be generalized to the comparison of stochastic processes. However even the $\preceq_{st}$ ordering does not exist between processes, the $\preceq_{wk}$ and the $\preceq_{wk^*}$ constraints could be satisfied. In this paper, we aim to give the intuition to choose the most suitable method with respect to the underlying performability study.

1. INTRODUCTION

Markovian models are largely used for the quantitative analysis of computer networks and systems. Unfortunately, the mathematical analysis of multidimensional Markov chains could be very difficult due to the state space explosion problem. In performability and verification studies, one is interested in the stationary and the transient behaviors of the underlying system. If there is not a specific method to compute the transient and the stationary distributions like matrix-geometric, product form solutions, then it could be very hard or impossible to compute as the state space increases exponentially.

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Stochastic comparisons have been proposed in order to overcome this problem [13, 11, 9]. Various applications of stochastic comparison of queueing networks with many related references are presented in [14]. The basic idea consists in bounding a complex process by another process easier to analyze. Moreover, this method can be applied for the performance evaluation of different kinds of network architectures [10].

Several stochastic orderings can be defined on a multidimensional state space corresponding to different comparison relations of the underlying distributions. We denote by $\preceq_\Phi$ the stochastic ordering, corresponding to one of the stochastic ordering: $\{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$ generated by families of increasing sets (see Massey [8]). The best-known is the strong stochastic (sample path) ordering ($\preceq_{st}$) which yields comparisons of increasing functionals (the expectations of all increasing functions of probability distributions) [11]. The weak ordering ($\preceq_{wk}$) is equivalent to tail distribution comparisons, while the weak* ordering ($\preceq_{wk^*}$) leads to cumulative distribution function comparisons [8, 12]. When the strong ordering could not be defined, weaker orderings may be an alternative as they require less constraints, so they may provide more precise bounds.

Different methods can be used in order to compare processes: increasing sets method, and the coupling method. Increasing sets method is a general formalism, allowing us to define the strong stochastic ordering: $\preceq_{st}$, and also weak orderings: $\preceq_{wk}$, and $\preceq_{wk^*}$, by means of different families of increasing sets. For the $\preceq_{st}$ ordering, the coupling method governed by the events is proposed to compare realizations of the processes [5, 7, 2]. The coupling [15] means the joint construction of two or more random variables (or processes) to deduce properties or relations between them. These relations could be the stochastic domination as it has been used in the present paper, or properties such as the asymptotic stationarity which could be derived using the coupling time [15]. The coupling of processes can be also done by means of the compensation of jumps such that realizations stay in a set K [4], in order to obtain inequalities on transition rates.

Both the coupling and the increasing set methods lead indeed to the comparison of transition rates. The stochastic monotonicity is related to the stochastic comparison which provides one of the sufficient condition to establish stochastic comparisons (see Massey [8] and Muller-Stoyan [11]).

The goal of this paper is to explain how to apply stochastic comparison methods with respect to the applied stochastic ordering. We also aim to establish relationships between the stochastic orderings. Although the $\preceq_{st}$ ordering between random variables implies the comparisons in the sense of $\preceq_{wk}$ and $\preceq_{wk^*}$ orderings, we show in this paper that this is not true for the comparison of Markov processes. Moreover, the $\preceq_{st}$ monotonicity of a Markov chain, does not imply ($\preceq_{wk}$ and $\preceq_{wk^*}$) monotonicity. These results are important in order to know which method we could apply according to the kind of the performance measure to compute.

This paper is organized as follows: we first introduce the stochastic comparisons of Markov chains, and then we present the different methods: the coupling method and the increasing sets method. In section 5, we identify the relationships among different stochastic orderings defined between Markov chains in order to choose which method could be applied. In section 6, we summarize the application of stochastic comparison for performance evaluation studies. Finally, we conclude and give comments for further research.

2. STOCHASTIC COMPARISONS

We present the main concepts about stochastic comparisons for random variables and Markov chains.

2.1 Stochastic ordering theory

Let E be a discrete, and countable state space, and $\preceq$ be at least a preorder (reflexive, transitive but not necessarily an anti-symmetric binary relation) on E. We suppose that E is a multidimensional state space, where each component is discrete, which is suitable for the representation of queueing networks. We consider two random variables X and Y defined respectively on E, and their probability measures given respectively by the probability vectors p and q where $p[i] = Prob(X = i)$, $\forall i \in E$ (resp. $q[i] = Prob(Y = i)$, $\forall i \in E$).

The well-known sample path ordering $\preceq_{st}$ is defined as follows [11, 4]:

Definition 1. *$X \preceq_{st} Y \Leftrightarrow E[(f(X))] \leq E[(f(Y))]$ $\forall f : E \to \mathbb{R}^+$ $\preceq -$increasing whenever the expectations exist.*

There exist other orderings implying weaker constraints. In [8], the weak ordering $\preceq_{wk}$ has been defined, it is equivalent to tail distribution comparisons, and $\preceq_{wk^*}$ serves the same role for cumulative distribution functions. In the case of multidimensional state spaces, the increasing set formalism is a general formalism, used in the stochastic ordering theory. Let $\Gamma \subseteq E$, we denote by:

$$\Gamma \uparrow = \{y \in E \mid y \succeq x, \ for \ some \ x \in \Gamma\} \tag{1}$$

An increasing set Γ is a set such that for any element x, all elements which are greater than x are also in Γ. It is defined formally as follows [8]:

Definition 2. *$\Gamma \subseteq E$ is called an increasing set if and only if $\Gamma = \Gamma \uparrow$.*

Three stochastic orderings have been defined from different families of increasing sets (see Massey [9, 8]). The first family is $\Phi_{st}(E)$ which is defined from all the increasing sets of E:

$$\Phi_{st}(E) = \{\text{all increasing sets on } E\}. \tag{2}$$

Other families $\Phi_{wk}(E)$ and $\Phi_{wk^*}(E)$ are defined from particular kinds of increasing sets. For $x \in E$, let:

$$\{x\} \uparrow = \{y \in E, \ y \succeq x\}$$

and

$$\{x\} \downarrow = \{y \in E, \ y \preceq x\}$$

So $\Phi_{wk}(E)$ is defined by:

$$\Phi_{wk}(E) = \{\{x\} \uparrow, \ x \in E\} \tag{3}$$

and $\Phi_{wk^*}(E)$ by:

$$\Phi_{wk^*}(E) = \{E - \{x\} \downarrow, \ x \in E\} \tag{4}$$

Let us remark here that when the state space is totally ordered, we have: $\Phi_{wk^*}(E) = \Phi_{wk}(E) = \Phi_{st}(E)$. If $\Phi(E)$ represents one of these families $\{\Phi_{st}(E), \Phi_{wk}(E), \Phi_{wk^*}(E)\}$, then a stochastic ordering $\preceq_\Phi$ representing $\{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}$ $\}$ can be defined as follows (see [8]):

Definition 3.

$$X \preceq_\Phi Y \Leftrightarrow \sum_{x\in\Gamma} p[x] \leq \sum_{x\in\Gamma} q[x], \forall \Gamma \in \Phi(E) \tag{5}$$

Obviously, in Definition 3, the ordering $\preceq_\Phi$ between random variables ($X \preceq_\Phi Y$) is equivalent to the comparison of corresponding probability vectors ($p \preceq_\Phi q$). We have the following inclusion relations between the families [8]:

Proposition 1.

- $\Phi_{wk}(E) \subset \Phi_{st}(E)$
- $\Phi_{wk^*}(E) \subset \Phi_{st}(E)$

From these inclusion relations, we remark that the $\preceq_{st}$ ordering is generated by means of the largest family of increasing sets $\Phi_{st}(E)$. So $\preceq_{wk}$ and $\preceq_{wk^*}$ stochastic orderings are weaker than the $\preceq_{st}$ ordering. Thus the following implications among stochastic orderings are easily obtained for the probability vectors p and q (or random variables X and Y) [8]:

Proposition 2.

$$p \preceq_{st} q \Rightarrow p \preceq_{wk} q \ \ and \ \ p \preceq_{wk^*} q \tag{6}$$

This relationship between stochastic orderings could not be generalized to the comparison of Markov chains, that we will illustrate in section 5.

2.2 Markov chain comparisons

We focus on the stochastic comparisons of multidimensional Continuous Time Markov Chains (CTMC)s.

Let $\{X_1(t), t \geq 0\}$ (resp. $\{X_2(t), t \geq 0\}$) be a CTMC taking values on E, with infinitesimal generator Q_1 (resp. Q_2). The $\preceq_\Phi$ stochastic comparison in the sense of ($\preceq_\Phi \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$) is defined as follows [8]:

Definition 4. *$\{X_1(t), t \geq 0\}$ is said to be less in the sense of the stochastic order $\preceq_\Phi$ than $\{X_2(t), t \geq 0\}$ written as ($\{X_1(t), t \geq 0\} \preceq_\Phi \{X_2(t), t \geq 0\}$), if and only if:*

$$X_1(0) \preceq_\Phi X_2(0) \Longrightarrow X_1(t) \preceq_\Phi X_2(t), \;\; \forall t > 0$$

The stochastic comparison of Discrete Time Markov Chains (DTMC)s can be similarly defined, by comparing the chains at each step n ($n \in \mathbf{N}$). We can define the $\preceq_\phi$ stochastic comparison of time-homogeneous CTMCs using their infinitesimal generators [8]:

Definition 5. *$\{X_1(t), t \geq 0\} \preceq_\Phi \{X_2(t), t \geq 0\}$ if and only if for all probability vectors p and q in E, we have:*

$$p \preceq_\Phi q \Longrightarrow p \; exp(tQ_1) \preceq_\Phi q \; exp(tQ_2)$$

Using the probability transition matrices we can define the comparison of DTMCs. Let $\{X_1(n), n \geq 0\}$ (resp. $\{X_2(n), n \geq 0\}$) be a time-homogeneous DTMCs with probability transition matrix P_1 (resp. P_2):

Definition 6. *$\{X_1(n), n \geq 0\} \preceq_\Phi \{X_2(n), n \geq 0\}$ if and only if for all probability vectors p and q in E, we have:*

$$p \preceq_\Phi q \Longrightarrow pP_1 \preceq_\Phi q \; P_2$$

For the stochastic comparison of processes, the stochastic monotonicity is a property that is usually used in order to simplify the verification for stochastic comparisons. It is defined as an increasing (or decreasing) in time [11] of the process:

Definition 7. *$\{X(t), t \geq 0\}$ is said to be $\preceq_\Phi$ monotone, if and only if:*

$$X(t) \preceq_\Phi X(t+\tau), \forall t \geq 0, \forall \tau \geq 0. \tag{7}$$

Note that Definition 7 can be also given for DTMCs, as an increasing (or decreasing) with $n \in \mathbb{N}$. The monotonicity of time-homogeneous chains can be also defined using the probability transition matrix.

Definition 8. *$\{X_1(n), n \geq 0\}$ is said to be $\preceq_\Phi$ monotone, if and only if for all probability vectors p and q in E, we have:*

$$p \preceq_\Phi q \Longrightarrow pP_1 \preceq_\Phi q \; P_1$$

Remark that the comparison and the monotonicity of CTMCs can be defined using the uniformized Markov chains [11]. We say that $\{X(t), t \geq 0\}$ is uniformizable if and only if:

$$|Q| < \infty$$

where

$$|Q| = sup_{x \in E} Q(x,x)$$

Thus, we can define the uniformized Markov chain $\{X^\lambda(n), n \geq 0\}$ with the stochastic matrix P^λ such that:

$$P^\lambda = I + \frac{1}{\lambda} Q$$

where $\lambda \geq 2 * |Q|$.

The stochastic comparison of CTMCs $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ can be defined using Definition 5, and also Definition 6 through the corresponding uniformized Markov chains $\left\{X_1^\lambda(n), n \geq 0\right\}$ and $\left\{X_2^\lambda(n), n \geq 0\right\}$. Next, we focus on the methods to establish stochastic orderings between the Markov chains. First, we present the coupling method.

3. THE COUPLING METHOD

The coupling method is a well-known method for the comparison of probability measures and Markov chains (see Lindvall [5]). We present first this method for the $\preceq_{st}$ stochastic comparisons, and then to establish the $\preceq_{st}$ monotonicity.

The $\preceq_{st}$ comparison of CTMCs is equivalent to the definition of a coupled version of the processes in order to compare their sample paths. For the coupling of $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$, we define two CTMCs on E, $\left\{\widehat{X_1}(t), t \geq 0\right\}$ and $\left\{\widehat{X_2}(t), t \geq 0\right\}$ such that:

- $\left\{\widehat{X_1}(t), t \geq 0\right\}$ has the same infinitesimal generator as $\{X_1(t), t \geq 0\}$.
- $\left\{\widehat{X_2}(t), t \geq 0\right\}$) has the same infinitesimal generator as $\{X_2(t), t \geq 0\}$.

The $\preceq_{st}$ comparison can be obtained using the coupling of the processes to check if the sample paths are ordered [5]:

Theorem 1. *$\{X_1(t), t \geq 0\} \preceq_{st} \{X_2(t), t \geq 0\}$ if and only if there exists a coupling $\{(\widehat{X_1}(t), \widehat{X_2}(t)), t \geq 0\}$, such that $\forall \omega \in \Omega$:*

$$\widehat{X_1}(0)(w) \preceq \widehat{X_2}(0)(w) \Rightarrow \widehat{X_1}(t)(w) \preceq \widehat{X_2}(t)(w), \; \forall t > 0. \tag{8}$$

The coupling can be also applied in order to check the monotonicity of a chain. In fact, the strong monotonicity is equivalent to the coupling of the chain with itself (see Lindvall [5]). In order to establish the monotonicity of $\{X(t), t \geq 0\}$, we define two chains:

$$\{\widehat{X}(t),\ t \geq 0\} \text{ and } \{\widehat{X'}(t),\ t \geq 0\}$$

governed by the same infinitesimal generator matrix as $\{X(t), t \geq 0\}$, representing different realizations of $\{X(t), t \geq 0\}$ with different initial conditions. The theorem of the monotonicity using the coupling is as follows (see Lindvall [5, 6]):

THEOREM 2. $\{X(t), t \geq 0\}$ *is said to be* $\preceq_{st}$ *-monotone if and only if there exists the coupling* $\{(\widehat{X}(t), \widehat{X'}(t)),\ t \geq 0\}$ *such that* $\forall \omega \in \Omega$ *:*

$$\widehat{X}(0)(\omega) \preceq \widehat{X'}(0)(\omega) \Rightarrow \widehat{X}(t)(\omega) \preceq \widehat{X'}(t)(\omega),\ \forall t > 0.$$

In the case of Markovian discrete event models, one can compare the evolutions of the realizations due to events. In the case of multidimensional Markov chains, the coupling is more difficult to apply (see [5]) since the system is naturally endowed by a partial order, and governed by more events.

4. INCREASING SET METHOD

In this section, we give conditions for the comparability and the monotonicity of Markov chains by means of the increasing sets method. We highlight those that can be applied only for the $\preceq_{st}$ ordering, and those that can be applied for the three orderings : $\preceq_{st}$, $\preceq_{wk}$ and $\preceq_{wk^*}$.

The stochastic comparisons can be checked using the probability transition matrices for DTMCs. In the case of the $\preceq_{st}$ ordering, it can be checked by comparing the rows of the matrices for comparable states x and y such that $x \preceq y$ [8, 11].

THEOREM 3. $\{X_1(n), n \geq 0\} \preceq_{st} \{X_2(n), n \geq 0\}$ *if and only if:*

$$\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y,$$

$$\sum_{z \in \Gamma} P_1(x, z) \leq \sum_{z \in \Gamma} P_2(y, z).$$

In the case of CTMCs, we have also the theorem using infinitesimal generators [8, 11]:

THEOREM 4. $\{X_1(t), t \geq 0\} \preceq_{st} \{X_2(t), t \geq 0\}$ *if and only if*

$$\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y \mid x, y \in \Gamma \text{ or } x, y \notin \Gamma,$$

$$\sum_{z \in \Gamma} Q_1(x, z) \leq \sum_{z \in \Gamma} Q_2(y, z).$$

Note that in Theorem 4, we have to consider the states $x \preceq y$ such that either they are both in the increasing sets or not, because of the negative term in the diagonal. These theorems could not be generalized to the other weaker orderings $\preceq_{wk}$ and $\preceq_{wk^*}$. In the general case of the $\preceq_\Phi$ comparisons, the comparison of CTMCs is also established using inequality conditions on the generators [8]. Furthermore, the monotonicity may be also checked as a sufficient condition for the comparison.

Let $\{X_1(t), t \geq 0\}$ (resp. $\{X_2(t), t \geq 0\}$) be a CTMC defined on E with infinitesimal generator Q_1 (resp. Q_2). We have the following theorem (see Massey [8]).

THEOREM 5. $\{X_1(t), t \geq 0\} \preceq_\Phi \{X_2(t), t \geq 0\}$*, if the following conditions are satisfied:*

1. $X_1(0) \preceq_\Phi X_2(0)$
2. $\{X_1(t), t \geq 0\}$*. or* $\{X_2(t), t \geq 0\}$ *is* $\preceq_\Phi$ *monotone.*
3. *Comparison of infinitesimal generators* Q_1 *and* Q_2*:*

$$\forall \Gamma \in \Phi(E),\ \forall x \in E,\ \sum_{z \in \Gamma} Q_1(x, z) \leq \sum_{z \in \Gamma} Q_2(x, z).$$

As we can see in Theorem 5, the $\preceq_\Phi$ monotonicity of one of the processes is a sufficient condition (condition (2)) to establish stochastic comparisons. In the case of the $\preceq_{st}$ ordering, the stochastic monotonicity of DTMCs can be obtained from the following theorem :

THEOREM 6. $\{X_1(n), n \geq 0\}$ *is* $\preceq_{st}$ *monotone if and only if:*

$$\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y,$$

$$\sum_{z \in \Gamma} P_1(x, z) \leq \sum_{z \in \Gamma} P_1(y, z).$$

Unfortunately, this property could not be generalized to the $\preceq_\Phi$ monotonicity. In [8], it is shown that $\preceq_\Phi$ monotonicity of a process is equivalent to the $\preceq_\Phi$ monotonicity of the generator which is expressed in terms of specific operators. Moreover, the $\preceq_{wk}$ monotonicity can be obtained from the Mobius monotonicity.

5. CHOICE OF STOCHASTIC COMPARISON METHOD

The coupling is an intuitive method but applicable only for the $\preceq_{st}$ ordering. The increasing set method can be applied for all orderings $\preceq_\phi \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$, however it is rather difficult for the $\preceq_{st}$ ordering since one must take into account all the increasing sets. We aim in this section to establish relationships between these stochastic orderings for Markov chains. This will lead to choose the most suitable stochastic comparison method to apply. First, we study the comparison of CTMCs, and secondly the monotonicity property.

5.1 Strong and weak comparisons

First, we can remark that the $\preceq_{st}$ ordering between the chains does not imply $\preceq_{wk}$ and $\preceq_{wk^*}$ orderings. We suppose that the $\preceq_{st}$ ordering exists between the two chains: $\{X_1(n), n \geq 0\}$ and $\{X_2(n), n \geq 0\}$. From Definition 6, we have

$$p \preceq_{st} q \Longrightarrow pP_1 \preceq_{st} q\ P_2. \quad (9)$$

We suppose that $p \preceq_{wk} q$, then we have two cases:

1. if $p \preceq_{st} q$, then from Equation (9) we have: $pP_1 \preceq_{st} qP_2$, which implies that $pP_1 \preceq_{wk} qP_2$ due to the increasing sets family inclusions (see Proposition 1).
2. if $p \not\preceq_{st} q$ then we could have $pP_1 \not\preceq_{st} qP_2$, so we could not deduce that $pP_1 \preceq_{wk} qP_2$.

Therefore if $\{X_1(n), n \geq 0\} \preceq_{st} \{X_2(n), n \geq 0\}$, then we could not deduce that $\{X_1(n), n \geq 0\} \preceq_{wk} \{X_2(n), n \geq 0\}$. We now give an example in order to clearly show this result.

5.1.1 Example

We consider $E = \{(0,0),(0,1),(1,0),(1,1)\}$ and the componentwise ordering on the state space. Let $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ be two Markov processes on E with infinitesimal generators Q_1 and Q_2, where the states are ordered in the lexicographic order.

$$\mathbf{Q}_1 = \begin{pmatrix} -0.5 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0.25 & 0 & -0.5 & 0.25 \\ 0 & 0.25 & 0.25 & -0.5 \end{pmatrix}$$

$$\mathbf{Q}_2 = \begin{pmatrix} -0.5 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0.25 & -0.5 & 0.25 \\ 0 & 0.25 & 0.25 & -0.5 \end{pmatrix}$$

We define the uniformized Markov chains $\{X_1^\lambda(n), n \geq 0\}$ and $\{X_2^\lambda(n), n \geq 0\}$ for the stochastic comparison.

Let take $\lambda = |Q_1| + |Q_2| = 1$ which is greater than $|Q_1|$ and $|Q_2|$. Thus:

$$P_1^\lambda = I + Q_1 = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

and

$$P_2^\lambda = I + Q_2 = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

We give for the stochastic comparison ($\preceq_{st}$ and $\preceq_{wk}$) the following families of increasing sets:

- $\Phi_{wk}(E) = \{\{(0,0)\}\uparrow, \{(0,1)\}\uparrow, \{(1,0)\}\uparrow, \{(1,1)\}\uparrow\}$
- $\Phi_{st}(E) = \Phi_{wk}(E) \cup (E - \{(0,0)\}\downarrow)$.

where

- $\{(0,0)\}\uparrow = E$
- $\{(0,1)\}\uparrow = \{(0,1),(1,1)\}$
- $\{(1,0)\}\uparrow = \{(1,0),(1,1)\}$
- $\{(1,1)\}\uparrow = \{(1,1)\}$
- $E - \{(0,0)\}\downarrow = \{(0,1),(1,0),(1,1)\}$.

We can observe that $\forall \Gamma \in \Phi_{st}(E)$:

$$\sum_{z\in\Gamma} P_1^\lambda(x,z) \leq \sum_{z\in\Gamma} P_2^\lambda(y,z), \forall x \preceq y$$

so we deduce that:

$$\{X_1^\lambda(n), n \geq 0\} \preceq_{st} \{X_2^\lambda(n), n \geq 0\}$$

and also:

$$\{X_1(t), t \geq 0\} \preceq_{st} \{X_2(t), t \geq 0\}$$

We now show that:

$$\{X_1(t), t \geq 0\} \not\preceq_{wk} \{X_2(t), t \geq 0\}$$

To apply Definition 6, we take two probability vectors p and q such that $p = (0, 0.5, 0.5, 0)$, and $q = (0.5, 0, 0, 0.5)$. We can remark that $p \preceq_{wk} q$ if we apply in Definition 3 the family of increasing sets $\Phi_{wk}(E)$ defined previously. On the other hand, for the increasing set $\Gamma_1 = \{(0,1),(1,0),(1,1)\} \in \Phi_{st}(E)$, the sense of inequality is reversed as $\sum_{x\in\Gamma_1} p[x] = 1 > \sum_{x\in\Gamma_1} q[x] = 0.5$. So the probability vectors p and q are such that $p \preceq_{wk} q$ and $p \not\preceq_{st} q$.

If we make the product by the matrices, we obtain:

$$pP_1^\lambda = (0.125, 0.25, 0.5, 0.125)$$

and:

$$qP_2^\lambda = (0.25, 0.25, 0.25, 0.25)$$

If we take the increasing set $\Gamma = \{(1,0),(1,1)\} \in \Phi_{wk}(E)$, we have:

$$\sum_{x\in\Gamma} pP_1^\lambda[x] = 0.625$$

and:

$$\sum_{x\in\Gamma} qP_2^\lambda[x] = 0.5$$

We deduce that:

$$pP_1 \not\preceq_{wk} qP_2$$

This result is important since if we have the $\preceq_{st}$ ordering between Markov processes, then we could not deduce that weaker orderings such as $\preceq_{wk}$ exists. In the case when the $\preceq_{st}$ ordering can not be established between the chains, then one must check if weaker orderings exist or not by applying increasing sets method (see Theorem 5).

Since the monotonicity is a sufficient condition for the $\preceq_{\Phi}$ comparison, it is also important to study this property for the different kinds of stochastic orderings.

5.2 Strong and weak monotonicity

In this subsection, we show that the $\preceq_{st}$ monotonicity does not imply the $\preceq_{wk}$ monotonicity. From Definition 8, the $\preceq_{st}$ monotonicity is defined as:

$$p \preceq_{st} q \Longrightarrow pP_1 \preceq_{st} q\ P_1 \tag{10}$$

If $p \preceq_{wk} q$, and $p \not\preceq_{st} q$ then by applying Equation (10), we could not deduce a relationship between pP_1 and qP_1. To illustrate this remark, we give the following example.

5.2.1 Example

Let $\{X(t), t \geq 0\}$ be a CTMC with the following infinitesimal generator Q:

$$Q = \begin{pmatrix} -0.5 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0.25 & 0 & -0.5 & 0.25 \\ 0 & 0.25 & 0.25 & -0.5 \end{pmatrix}$$

we define $\{X^{\lambda}(n), n \geq 0\}$ be the uniformized Markov chain where : $\lambda = 1 = 2|Q|$, with probability transition matrix:

$$P^{\lambda} = I + Q = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

We can easily remark that : $\{X^{\lambda}(n), n \geq 0\}$ is $\preceq_{st}$ monotone:

$$\forall \Gamma \in \Phi_{st}(E), \forall x \preceq y \in E, \sum_{z \in \Gamma} P^{\lambda}(x, z) \leq \sum_{z \in \Gamma} P^{\lambda}(y, z),$$

However, $\{X^{\lambda}(n), n \geq 0\}$ is not $\preceq_{wk}$ monotone. If we take $p = (0, 0.5, 0.5, 0)$ and $q = (0.5, 0, 0, 0.5)$, we remark that $p \preceq_{wk} q$. We obtain $pP^{\lambda} = (0.125, 0.25, 0.5, 0.125)$, and $qP^{\lambda} = (0.25, 0.25, 0.25, 0.25)$. We remark that for $\Gamma = \{(1, 0), (1, 1)\}$, we have:

$$\sum_{x \in \Gamma} pP^{\lambda}[x] = 0.625 > \sum_{x \in \Gamma} qP^{\lambda}[x] = 0.5$$

Thus, $\{X^{\lambda}(n), n \geq 0\}$ is not $\preceq_{wk}$ monotone which implies that $\{X(t), t \geq 0\}$ is not $\preceq_{wk}$ monotone.

These results are important in order to decide which method to choose for establishing stochastic comparison of Markov chains. Although the coupling of sample paths generates the $\preceq_{st}$ ordering, it can not be applied for the $\preceq_{wk}$ and the $\preceq_{wk^*}$ orderings. The same remark is valid for the monotonicity. Next, we explain how to apply the stochastic comparison methods for performance evaluation studies.

6. PERFORMANCE EVALUATION

We suppose that the considered system is modelled by a multidimensional CTMC denoted by $\{X_1(t), t \geq 0\}$ taking values in E. The considered performance or reliability measure such as the loss probability, the availability is defined as a functional at time t over a subset of states $A \subset E$:

$$R_1(t) = \sum_{x \in A} \Pi_1(x, t) f(x) \tag{11}$$

where $f : E \rightarrow \mathbb{R}^+$ is an increasing reward function and $\Pi_1(x, t)$ is the probability to be in the state x at time t. For $t \rightarrow \infty$, if the process has a stationary behavior, then we denote by $\Pi_1(x)$ the stationary probability to be in state x, and R_1 represents the measure of interest computed from the stationary probability distribution Π_1.

If $\Pi_1(t)$ (or Π_1) has not a closed form solution, it would be very difficult or impossible to compute $R_1(t)$ (or R_1). We propose to apply the stochastic comparison method, which means to bound (upper or lower bound) $\{X_1(t), t \geq 0\}$ by $\{X_2(t), t \geq 0\}$, in the sense of the $\preceq_{\Phi}$-ordering:

$$\{X_1(t), t \geq 0\} \preceq_{\Phi} \{X_2(t), t \geq 0\} \tag{12}$$

or

$$\{X_2(t), t \geq 0\} \preceq_{\Phi} \{X_1(t), t \geq 0\} \tag{13}$$

such that $\{X_2(t), t \geq 0\}$ is easier to analyze (see Fourneau et al. [3]), The bounding model may have a closed form solution or defined in a reduced state space which makes easier the computation of the bounding distribution $\Pi_2(t)$.

Therefore, the bound on the measure of interest $R_2(t)$ is computed from the bounding distribution $\Pi_2(t)$, such that:

$$R_1(t) \leq R_2(t) \textit{ or } R_2(t) \leq R_1(t) \tag{14}$$

It is not trivial to choose $\preceq_{\Phi} \in \{\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}\}$ since each stochastic ordering generates different inequalities between underlying probability distributions. First, one has to determinate to which family of increasing set the subset A belongs. In fact, $R_1(t)$ is computed as a functional over this subset (Equation 11). Therefore, A is the subset on which we must check the probability constraints.

In the former sections, we have explained how to apply stochastic comparison methods in order to compute performance measure bounds. First, A must be an increasing set ($A \in \Phi_{st}(E)$) that yield to establish inequalities on functionals. Moreover, let us remark that it follows from Proposition 1 that, if $A \in \Phi_{\{wk, wk*\}}(E)$ then $A \in \Phi_{st}(E)$ also.

1. If A is an any increasing set ($A \in \Phi_{st}(E)$), then we apply the coupling method in order to check if the $\preceq_{st}$ ordering exists. So we have two cases:

    (a) If the $\preceq_{st}$ ordering exists, then for the computation of transient probability distributions bounds, we have to take the initial probability vectors p and q such that $p \preceq_{st} q$ in Definition (5). And this for any kind of increasing set A. As an example, if $A \in \Phi_{wk}(E)$, then we could not take $p \preceq_{wk} q$ as $\preceq_{st}$ comparisons do not imply $\preceq_{wk}$ comparisons.

(b) If the $\preceq_{st}$ ordering does not exist, then we consider weaker orderings :

i. If $A \in \Phi_{wk}(E)$ then we test whether $\preceq_{wk}$ ordering exists by applying Theorem 5. We check if the $\preceq_{wk}$ monotonicity exists (not the $\preceq_{st}$ monotonicity because it does not imply the $\preceq_{wk}$ monotonicity)

ii. If $A \in \Phi_{wk^*}(E)$ then we test whether the $\preceq_{wk^*}$ ordering exists using the same steps as the former case.

2. If $A \notin \Phi_{st}(E)$, then for an upper bound (resp. lower bound) we define an increasing set B such that $A \subset B$ (resp. $B \subset A$), and we return to (1) by considering B instead of A.

For instance, if we consider a queueing network formalism and component-wise order on the state space, loss probabilities can be derived from the $\preceq_{wk}$ ordering (tail distributions) while the resource utilizations and the availability can be derived from $\preceq_{wk^*}$ comparisons. The $\preceq_{st}$ ordering imposes more constraints, which may degrade the quality of bounds. However, it allows us to build bounds for several performance measures, such as the loss probabilities and the resource utilization. In [1], we have defined different bounding systems from $\preceq_{st}$ and $\preceq_{wk}$ orderings. As the loss probability bounds can be derived from these two kinds of stochastic orderings, we can compare the quality of the bounding systems and take the most precise one.

7. CONCLUSION

In this paper, we have explained how to compare multidimensional Markov chains by means of the increasing sets formalism and the coupling method. Stochastic comparisons have been applied since many years in different applied probability domains. The efficiency of this method has been also shown to overcome state space explosion problem in performance and dependability analysis of Markovian models. However, it is not always easy to apply this approach. We aim in this paper to show how it can be applied for performability studies with regard to different stochastic orderings and the measure of interest. Thus, we discuss the relationships between the different stochastic orderings by giving some illustrative examples. An algorithm approach is proposed in order to build bounding models with respect to the performability measure.

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