



# Orbital Angular Momentum Microwave Generated by Free Electron Beam

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**Abstract.** Based on the theory of classical electrodynamics and quantum mechanics, we quantitatively deduce microwave carrying Orbital Angular Momentum (OAM) radiated from the moving free electron beams on different closed-curved trajectories. It shows that the non-relativistic free electrons can also transit quantized OAM to the microwave in addition to the relativistic cyclotron electrons in the magnetic field. This work indicates the effective way to construct the antennas to generate high OAM modes of the microwave by multi-electron radiation.

**Keywords:** Orbital Angular Momentum (OAM) · Free electron · Microwave · Magnetic field

## 1 Introduction

Electro-Magnetic (EM) wave has angular momentum, which can be considered as a fundamental physical quantity and a new degree of freedom in both classical and quantum electrodynamics [1, 2]. Moreover, the angular momentum can be divided into Spin Angular Momentum (SAM) and Orbital Angular Momentum (OAM). Unlike the linear momentum or SAM related to the polarization [3], OAM is the result of the spatial spiral distribution of electric field strength and phase, which is expected to be one of the candidate technologies for Beyond 5th Generation (B5G) and 6th Generation (6G) mobile communications because of its rotational degrees of freedom. Hence, the signals with different OAMs in the same carrier frequency are mutually orthogonal and propagate independently. This phenomenon can provide the benefits of the transmission with a very high spectrum efficiency [4]. A photon at optical frequency with OAM was originally discussed by Allen *et al.* with respect to a specific mode of EM wave called the Laguerre-Gaussian mode [5]. While in radio beams, Thidé *et al.* proposed to use antenna arrays for generating and detecting EM wave carrying both SAM

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This work has been supported by National Natural Science Foundation of China with project number 61731011.

and OAM called Bessel-Gaussian mode [6]. Since then, in addition to using the Uniform Circular Antenna (UCA) method to generate OAM wave at radio frequency [6,7]. Many studies are also considering the use of electron beams to generate OAM waves. In 2015, Asner *et al.* showed that a single electron that makes a relativistic cyclotron motion in a magnetic field can radiate EM waves, and the radiated EM wave energy has discrete properties [8]. Besides, Sawant *et al.* demonstrated through simulation and experiment that the use of relativistic electron beams in the gyrotron can generate microwaves carrying high-order OAM in 2017 [9].

According to the radiation mode, the OAM beam can be divided into the statistical state beam and the quantum state beam. The former has an equivalent spiral wave front and the single photon is a plane wave photon, while the latter is that each photon constituting the EM wave is a vortex photon and their wave fronts form a spiral wave front. The spatial phase modulation methods such as a Spiral Phase Plate (SPP) and a UCA belong to the statistical beam generation method. In contrast, the way in which vortex photons are directly radiated by electrons with quantized OAM belongs to the quantum beam generation method. The difference between the methods is whether the OAM of the EM wave is mapped to the gyrotron electron OAM. From the perspective of quantum mechanics, quantized EM wave is composed of photons in the form of harmonic oscillators [1], and the quantum state of a photon can be described by a multipole expansion of EM wave with a well-defined value of the energy ( $\hbar\omega$ , where  $\hbar$  and  $\omega$  denote reduced Planck constant and the angular frequency of the EM wave), parity (odd or even) and the total angular momentum (polarization or spin, and OAM), given by the corresponding quantum eigenvalue  $l(l+1)\hbar$  and OAM in a fixed direction (say  $z$ -axis normally and  $l$  is an integer denoting OAM mode number), given by the eigenvalue  $l\hbar$  [10]. When the frequency is constant, the even parity photon that does not carry OAM is called the electric dipole photon, while the odd parity photon that does not carry OAM is called the magnetic dipole photon. In addition, photons carrying  $l\hbar$  OAM are called  $2^{(|l|+1)}$ -pole photons. For instance, photons with OAM of  $\hbar$  are called quadrupole photons. Because photons are the medium of electron transfer interaction, the OAM of photons can also be naturally obtained by the OAM of electrons. However, for a single dipole antenna (electrons can only vibrate in one direction in the antenna and carry no angular momentum), the EM wave radiation carrying a high order OAM are unlikely to occur because the angular momentum  $L$  selection rule for dipole approximation radiation is  $\Delta L = 0, \pm 1$  [11]. In order to directly radiate microwave carrying OAM, it is necessary to construct electrical multipole radiation and change the selection rule. Therefore, there are two ways to modify the selection rule: (1) single relativistic electron radiation [8]; (2) multi-electron radiation [1].

For the first method, the single relativistic electron can produce pure OAM photons, but the cost is high, generally requiring at least the speed of the electron to reach half the speed of light. In 2017, Katoh *et al.* theoretically showed that a single free electron in circular motion will radiate the EM wave with OAM [12]. When the speed of the electron is relativistic, the radiation field contains

harmonic components and the photons of  $l$ -th harmonic carry  $l\hbar$  total angular momentum including  $\pm\hbar$  SAM and  $(l \mp 1)\hbar$  OAM. So it is theoretically and experimentally proved that a single electron can emit pure twisted photons rather than a beam carrying OAM [12, 13]. However, there is only transmitter in the aforementioned method. In other words, there lacks corresponding receiver. The relativistic electron means that the speed of free electron approaches the speed of light, which is difficult for popularization and application in practice.

For the second method, the model of non-relativistic free electrons continuously circling in different closed-curved trajectories are proposed in this paper. In addition, we theoretically demonstrate the EM wave with OAM can be emitted from classical electrodynamics and quantum mechanics theory. Furthermore, the corresponding receiver is designed and the motion of the free electron is theoretically demonstrated to move in different trajectories, which can be utilized for OAM demultiplexing.

## 2 Preliminary Knowledge

The EM wave with OAM has a helical wave front, an azimuthal term  $e^{il\phi}$ , and an OAM of  $l\hbar$  per photon, where  $\phi$  is the azimuthal angle. Moreover, the microwave OAM signals propagating along the  $z$ -axis can be expressed in the cylindrical coordinate  $(r, \theta, \phi)$  as

$$\mathbf{U}_{kl}(r, \phi, \theta) = \boldsymbol{\varepsilon} \frac{A_l e^{-i(\omega_0 t + \phi_l)}}{r} J_l(k_{\perp} r \sin \theta) e^{-i(k_{\parallel} z + l\phi)}, \quad (1)$$

where  $J_l(\cdot)$  denotes Bessel function of the order  $l$ ,  $\boldsymbol{\varepsilon}$  denotes the polarization vector,  $\theta$  denotes the pitch angle,  $\perp$  and  $\parallel$  stand for the transverse and propagation vector component, and satisfy the wave number relationship  $k^2 = k_{\perp}^2 + k_{\parallel}^2$ .  $A_l e^{-i(\omega_0 t + \phi_l)}$  is the modulated signal in the traditional main channel, where  $A_l$  and  $\phi_l$  are the amplitude and phase of the signal respectively. According to Eq. (1), it can be easily seen that different OAM signals are orthogonal to each other, i.e.,  $\mathbf{U}_{kl} \mathbf{U}_{k'l'} = \delta_{kk'} \delta_{ll'}$ , where  $\delta$  is Dirac function, which means that signals with different OAM mode numbers at the same frequency can transmit in the channel without interference. In addition, the OAM mode is a new freedom of EM wave, and it can be combined with the digital modulation of conventional EM wave to improve transmission performance.

According to Refs. [14–16], OAM-based microwave transmission systems can enjoy the benefit of the mode division multiplexing for short-range communications. The emerging physical layer solution for such short-range line-of-sight wireless communication provides a relatively low detection complexity and high spectral efficiency. Moreover, it can be used for the OAM transmission system through the partial phase plane index modulation receiving method. Usually, OAM can add new degrees of freedom to the traditional wireless microwave transmission systems. An OAM-based Multiple Input Multiple-Output (MIMO) transmission system was proposed in Ref. [17]. When the transmission distance is greater than a certain distance, the capacity of OAM-MIMO is larger than that

of the traditional MIMO. The expression of OAM-MIMO transmission capacity can be written as

$$C = \log_2 \det \left[ \mathbf{I}_N + \frac{P}{\sigma^2 N_{\text{OAM}}} \mathbf{H}_{\text{OAM}} \mathbf{H}_{\text{OAM}}^H \right], \quad (2)$$

where  $\mathbf{H}_{\text{OAM}} \in \mathbb{C}^{M \times N}$  is the OAM channel matrix,  $\mathbf{I}_N$  is the  $N \times N$  unit matrix,  $(\cdot)^H$  is the conjugate transition operation,  $\det[\cdot]$  represents the determinant of a square matrix,  $P$  is the total transmission power,  $N_{\text{OAM}}$  denotes the number of OAM modes, and  $\sigma^2$  is the variance of the Gaussian white noise [14]. OAM as a new dimension can also be combined with the modulation and coding methods, expand European space and improve communication performance, such as with the Low-Density Parity-Check (LDPC) coding [18].

The angular momentum  $\mathbf{J}$  including SAM  $\mathbf{S}$  and OAM  $\mathbf{L}$  of the EM wave  $(\mathbf{E}, \mathbf{B})$  in vacuum can be expressed as

$$\mathbf{J} = \mathbf{S} + \mathbf{L} = \frac{1}{\mu_0 c^2} \int_V \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) dV = \frac{1}{4\pi c} \int_V [(\mathbf{E} \times \mathbf{A}) + \mathbf{E} \cdot (\mathbf{r} \times -i\nabla \cdot \mathbf{A})] dV, \quad (3)$$

where  $\mu_0$ ,  $c$ ,  $\mathbf{r}$ ,  $\mathbf{A}$ ,  $\nabla$  and  $V$  denote the permeability in vacuum, the speed of light, the field location vector, the vector potential, the Laplace operator and the integrated full volume. The first term of Eq. (3) denotes the SAM of the EM wave, which is well known as the photon spin and polarization. The second term of Eq. (3) contains the total OAM operator  $\mathbf{r} \times -i\nabla$  item, which is consistent with the OAM operator in quantum mechanics  $\mathcal{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times -i\hbar\nabla$ , where  $\mathbf{p}$  is the momentum operator.

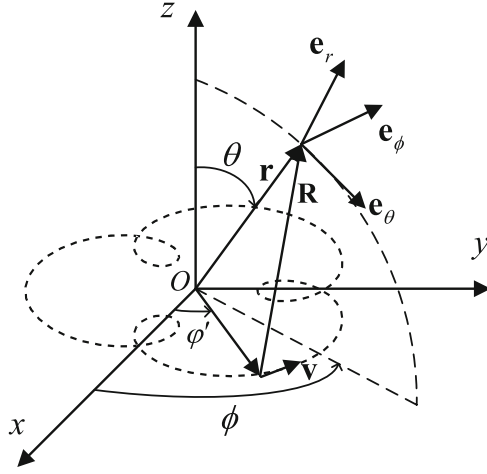
While the angular momentum is the basic property of the field and matter, except that the photons constituting the microwave can have angular momentum, the EM wave radiated by the electrons can also have the angular momentum. The whole radiation process satisfies the angular momentum conservation and the selection rule. Hence, electrons carrying angular momentum can radiate microwaves that carry both SAM and OAM.

### 3 System Model

As shown in Fig. 1, a non-relativistic electron rotates around the  $z$ -axis at an angular velocity  $\omega$  in the Cartesian coordinate system, and its velocity is  $\mathbf{v}$ . The dotted line indicates the conceptual trajectory of the moving electron with different OAM. The acceleration of the electron can be expressed as  $a_0 = |\dot{\mathbf{v}}|$ . According to Ref. [19], the expression of the radiation produced by the motion of the electronic acceleration can be described by Liénard-Wiechert field and the second order term  $o(1/r^2)$  called generalized Coulomb field is ignored:

$$\mathbf{E}_e(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{e}_r \times (\mathbf{e}_r \times \dot{\mathbf{v}})}{r'} \approx \frac{q}{4\pi\epsilon_0 c^2 r} (\mathbf{e}_r \cdot \dot{\mathbf{v}}) \cdot \mathbf{e}_r - \dot{\mathbf{v}}, \quad (4)$$

$$\mathbf{B}_e(\mathbf{r}, t) = \mathbf{e}_r \times \mathbf{E}_e/c, \quad (5)$$



**Fig. 1.** A non-relativistic electron rotates around the  $z$ -axis.

where  $\mathbf{r}'(t') = \mathbf{r} - \mathbf{r}_0(t')$ ,  $r' = |\mathbf{r}'(t')| \approx r$ ,  $\mathbf{e}_r = \mathbf{r}/r$ , which is the unit vector between the observation point and the position of the electron at the retarded time,  $\mathbf{r}_0$  denotes the position vector of the electron,  $t' = t - |\mathbf{r} - \mathbf{r}'|/c$  is the retarded time,  $q$  denotes the charge of the electron, and  $\epsilon_0$  denotes the permittivity of vacuum. In addition,  $\mathbf{E}_e$  and  $\mathbf{B}_e$  are the EM radiation from the single electron to the observer at  $\mathbf{r}$ .

In order to calculate  $\mathbf{E}_e$  and  $\mathbf{B}_e$ , the expression of  $\mathbf{e}_r$ ,  $\mathbf{r}_0$  and  $\dot{\mathbf{v}}$  can be obtained from the geometric relationship in the spherical coordinate system of Fig. 1 as follows:

$$\mathbf{e}_r = \{\sin \theta \cos \phi \mathbf{n}_x + \sin \theta \sin \phi \mathbf{n}_y + \cos \theta \mathbf{n}_z\}, \quad (6)$$

$$\mathbf{r}_0(t) = r_0 \left\{ \cos \omega t e^{i\varphi(t)} \mathbf{n}_x + \sin \omega t e^{i\varphi(t)} \mathbf{n}_y \right\}, \quad (7)$$

$$\dot{\mathbf{v}}(t) \approx a_0 \left\{ -\cos \omega t e^{i\varphi(t)} \mathbf{n}_x - \sin \omega t e^{i\varphi(t)} \mathbf{n}_y \right\}, \quad (8)$$

where  $\mathbf{n}_x$ ,  $\mathbf{n}_y$  and  $\mathbf{n}_z$  are unit vectors of  $x$ ,  $y$ , and  $z$  axes,  $r_0$  denotes the constant relevant to the positional parameters and  $\varphi(t)$  denotes the phase of the electron. By substituting Eqs. (6) and (8) into Eq. (4), we can calculate the specific expression of the electric field in the Cartesian coordinate system:

$$\begin{cases} E_{ex} = \frac{E_e}{r} e^{i\varphi(t)} [-\sin^2 \theta \sin \phi \sin(\omega t - \phi) + \cos^2 \theta \cos \omega t] \\ E_{ey} = \frac{E_e}{r} e^{i\varphi(t)} [+ \sin^2 \theta \cos \phi \sin(\omega t - \phi) + \cos^2 \theta \sin \omega t] \\ E_{ez} = \frac{E_e}{r} e^{i\varphi(t)} [-\sin \theta \cos \theta \cos(\omega t - \phi)] \\ E_e = \frac{qa_0}{4\pi\epsilon_0 c^2} \end{cases}. \quad (9)$$

The instantaneous energy flow resulting from the Liénard-Wiechert field is given by the Poynting vector

$$\mathbf{S} = \frac{q^2 a_0^2}{16\pi^2 \varepsilon_0 c^3 r^2} [\cos^2 \theta + \sin^2 \theta \sin^2 (\omega t - \phi)] \mathbf{e}_r. \quad (10)$$

When  $\omega t = \phi$ , setting the angle between the acceleration and the radiation direction as  $\Theta$ , the energy radiated into per solid angle  $\Omega$  during each cycle of the rotating electron is

$$\frac{dP}{d\Omega} = r^2 \mathbf{S} \cdot \mathbf{e}_r = \frac{q^2 a_0^2}{16\pi^2 \varepsilon_0 c^3} \cos^2 \theta = \frac{q^2 a_0^2}{16\pi^2 \varepsilon_0 c^3} \sin^2 \Theta. \quad (11)$$

Integrating Eq. (11) over the all solid angles, we get the non-relativistic electron generalization of Larmor's formula:

$$P = \int_0^{2\pi} \int_0^\pi \frac{q^2 a_0^2}{16\pi^2 \varepsilon_0 c^3} \sin^2 \Theta \sin \Theta d\Theta d\phi = \frac{q^2 a_0^2}{6\pi \varepsilon_0 c^3}. \quad (12)$$

Now considering that the total radiation field produced by an electron in one cycle, the expression of the radiation fields at any time generated by the electron is Eq. (9), but their phases are different from each other. Therefore, the total electric field expression after coherent superposition is

$$\begin{aligned} \mathbf{E}_{\text{total}} &= \sum \mathbf{E}_e e^{ikR} e^{-i\omega(t-\mathbf{e}_r \cdot \mathbf{R}/c)} \\ &\approx e^{i(kr-\omega t+l\phi)} \int_0^{2\pi} |\mathbf{E}_e| e^{il(\varphi-\phi)} e^{i\beta \sin \theta \cos(\varphi-\phi)} d\varphi', \end{aligned} \quad (13)$$

where  $k$  denotes the wave number,  $\mathbf{R}$  denotes the vector from the electron to the observer,  $R = |\mathbf{R}|$ ,  $\omega \mathbf{e}_r \cdot \mathbf{R}/c \approx \beta \sin \theta \cos(\varphi - \phi)$ ,  $\varphi' = \omega t$  and  $\beta = v/c$  denotes the ratio of the speed of the electron to  $c$ . When  $l = 0$  and  $\varphi = \varphi'$ , the integral of Eq. (13) can be obtained as follows:

$$\begin{cases} E_x = \frac{E_e J_l(\beta \sin \theta) \cos^2 \theta \pi e^{i(kr-\omega t+l\phi)}}{r} \left( i e^{i\phi} + \frac{i}{e^{i\phi}} \right) \\ E_y = \frac{E_e J_l(\beta \sin \theta) \cos^2 \theta \pi e^{i(kr-\omega t+l\phi)}}{r} \left( e^{i\phi} + \frac{1}{e^{i\phi}} \right) \\ E_z = \frac{E_e J_l(\beta \sin \theta) \sin \theta \cos \theta \pi e^{i(kr-\omega t+l\phi)}}{r} \left( i - \frac{1}{i} \right) \end{cases}. \quad (14)$$

According to Eqs. (9) and (14), the coherent radiation of the electric field does not affect the total angular momentum including SAM and OAM. It can be seen that SAM is +1 and OAM is  $l$ , because  $\mathbf{E}_{\text{total}}$  contains the  $e^{il\phi}$  item and the phase difference between  $E_x$  and  $E_y$  is  $90^\circ$ . In addition, the total angular

momentum along the  $z$ -axis remains unchanged while propagating. When  $l \geq 1$ , the trajectory of the electron is not a perfect circle, so it is difficult to get the integral expression of Eq. (13). However, the phase of  $\mathbf{E}_{\text{total}}$  can be proven to be  $\exp[i(kr - \omega t + l\phi + c_l)]$  by computer numerical integration, where  $c_l$  denotes a constant associated with  $l$ . Through the OAM operator in the cylindrical coordinate system  $\mathcal{L}_z = -i\hbar\partial/\partial\phi$ , the OAM of each photon in the EM wave can be calculated as  $l\hbar$ .

Overall, when the electrons are moving on different trajectories, whose acceleration expression contains the space related item  $e^{il\varphi(t)}$  with different OAM mode numbers  $l$ , the EM wave radiated by the electron carries both SAM and OAM. In other words, electron beams rotating around the  $z$ -axis can radiate vortex microwave carrying OAM, which leads to that OAM transits from the free electrons to the microwave and the total angular momentum remains conserved.

When each electron of the electron beam moves on a corresponding different trajectory, each electron generates a radiation field around it, and when the electric fields generated by the electron beams are superposed on each other, EM wave of different OAMs can be generated. Moreover, as the number of OAM modes increases, the center hole of the electric field also increases. In summary, by utilizing the two-dimensional motion characteristics of electrons and the superposition characteristics of electron-radiating electric fields, it is possible to theoretically generate OAM microwaves of arbitrary modes, which can greatly expand the radiation freedom of conventional dipole antennas.

**Table 1.** Microwave radiation parameters

Parameter	Value	Dimension
Electron density $\rho$	$1 \times 10^{10}$	$\text{cm}^{-3}$
Speed of electron $v$	$1 \times 10^7$	m/s
Electron Beam current $I$	1.0	mA
Acceleration of electron $a_0$	$2.199 \times 10^{18}$	$\text{m/s}^2$
Microwave frequency $f$	35.0	GHz
Speed of light $c$	$3 \times 10^8$	m/s
Charge of electron $q$	$1.6 \times 10^{-19}$	C
Permittivity of vacuum $\epsilon_0$	$8.854 \times 10^{-12}$	F/m
Planck constant $h$	$6.626 \times 10^{-34}$	J·s

To calculate the radiated power, an optional parameter list is shown in Table 1 when the frequency of photons carrying OAM radiated by the non-relativistic free electron is in the microwave band. Obviously, the radiated power of a single free electron is  $P_e = 2.754 \times 10^{-17}$  W according to Eq. (12), and the average number of photons radiated by a single electron per unit time can be calculated as  $N_p = P_e/(hf)$ . Of course, the average number is about  $1.188 \times 10^6 \text{ s}^{-1}$  for microwave photons, which is much smaller than one photon. When the electron

density in the free space or the electron beam current increases, the radiation power increases. To calculate the radiation field generated by the free electron beam, an electron beam or a single electron can be considered as the equivalent current, using the full space-time Fourier transform of the current density of the single point charge:

$$\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{r}}_0(t) \delta[\mathbf{r} - \mathbf{r}_0(t)], \quad (15)$$

where  $\dot{\mathbf{r}}_0(t)$  denotes the velocity of the charge and  $\delta[\mathbf{r} - \mathbf{r}_0(t)]$  denotes the spatial unit impulse function [20]. The expression of the velocity is:

$$\dot{\mathbf{r}}_0(t) = d \{ \sin \omega_0 t \mathbf{n}_x + \cos \omega_0 t \mathbf{n}_y + 0 \mathbf{n}_z \}, \quad (16)$$

where  $d$  denotes a constant. According to [20], the full space-time Fourier transform of the current density can be expressed as follows:

$$\mathbf{J}(\mathbf{k}, \omega) = \int q\dot{\mathbf{r}}_0(t) e^{-i\mathbf{k}\cdot\mathbf{r}_0} e^{i\omega t} dt, \quad (17)$$

By substituting Eq. (16) into Eq. (17), the frequency domain expression of the current density can be described as:

$$|\mathbf{J}(\mathbf{k}, \omega)| = \frac{\sqrt{2}q\omega_0 d}{2} J_0(kd \cos \alpha) [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], \quad (18)$$

where  $\alpha$  is the angle between  $\mathbf{v}$  and  $\mathbf{k}$ . According to Eq. (18), a conclusion can be drawn that there is a radiation field only if  $k = \omega_0/c$ . Moreover, the current density can be substituted into Maxwell's equations to solve the radiation field:

$$\mathbf{E}(\mathbf{k}, \omega) = \frac{-i\omega\mu_0\mathbf{n} \times [\mathbf{n} \times \mathbf{J}(\mathbf{k}, \omega)]}{k^2 - (\omega^2/c^2)}, \quad (19)$$

where  $\mathbf{n} = \mathbf{k}/k$ ,  $\mu_0$  denotes the permeability of vacuum. That is to say, the calculation result and expression form are equivalent to Eq. (4). When the quantity of electrons contained in the electron beam is  $N$ , The total radiation field of the electron beam can be expressed as  $\sum_{n=1}^N \mathbf{E}(\mathbf{k}, \omega)\varphi_n$ , where  $\varphi_n$  denotes the phase of each electron radiation field.

On the other hand, the reason why electrons rotating around the  $z$ -axis can radiate high-order OAM microwaves is that electrons also carry the quantized angular momentum in addition to the classical angular momentum  $\mathbf{L} = \mathbf{r} \times (m_e\mathbf{v})$ . It is well known that there are two solutions to the Schrödinger equation for quantum mechanics. One is the time-independent solution, the other is the time-dependent solution, which can be regarded as the linear superposition of the time-independent solutions. The reason why rotating electrons carry the quantized OAM is that the wave function of the electron is a standing wave function along the tangential direction of the closed trajectory, which is a solution of the time-independent Schrödinger equation.

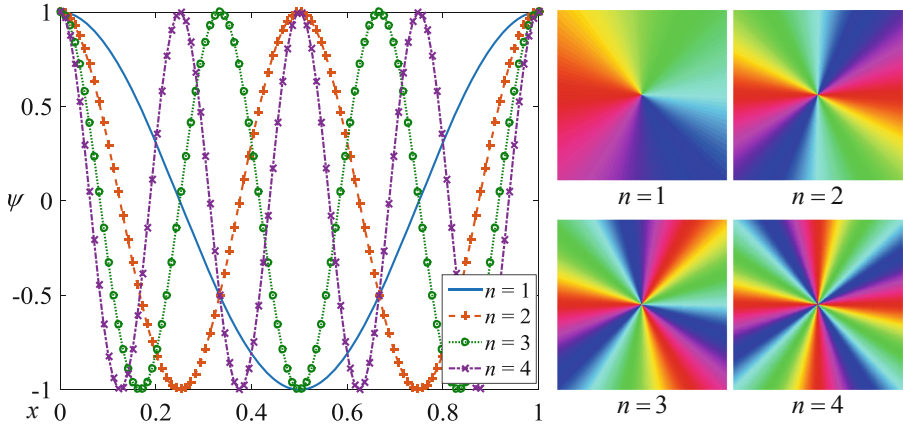
Assuming that the circumference of the electron's motion is  $L$ , its wave function  $\psi(x)$  satisfies the following conditions:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \quad (20)$$

$$\psi[x + L/(|l| + 1)] = \psi(x), \quad (21)$$

where  $m$ ,  $V(x)$  and  $E$  denote the mass, the potential function and the energy of the electron, respectively. Besides, it is self-evident that  $V(x)$  does not change with time and is always zero. The influence of EM wave generated by electrons on themselves is ignored. For example, an electron is making circular motion at the uniform velocity in a constant magnetic field and there is not any force in the direction of movement of the electron. Therefore, the general solution of the electron wave function is  $\psi(x) = Ae^{ikx} + Be^{-ikx}$ , where  $k = \sqrt{2mE}/\hbar$ ,  $A$  and  $B$  are both undetermined coefficients of the solution. Furthermore, we can get a series of solutions of the time-independent Schrödinger equation:

$$\psi(x) = A \exp\left[\frac{\pm i2n(|l| + 1)\pi x}{L}\right] \quad (n = 1, 2, 3, \dots). \quad (22)$$



**Fig. 2.** Normalized ring standing wave function and quantum angular momentum formed by rotating lectrons.

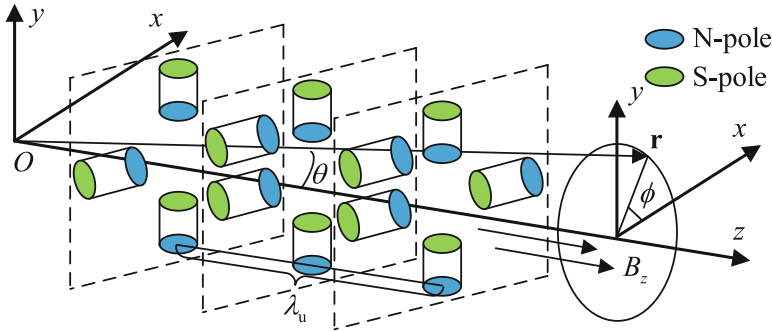
Moreover, it is easy to know that  $A = 1/\sqrt{L}$  from the normalization condition of the wave function  $\int |\Psi(x, t)|^2 dx = 1$ . It can be seen from Eq. (22) clearly that all solutions of the time-independent Schrödinger equation form standing waves of length  $L$  and are orthogonal to each other with the ring phase from  $0$  to  $2n\pi$ . In the polar coordinate system, the expression of Eq. (22) is  $A \exp(in\varphi + in|l|\varphi)$  according to  $x \approx \varphi/(2\pi)$ . As shown in Fig. 2, when the wave functions of the

electron are in states with different  $n$  and  $l = 1$ , the electron forms a standing wave with an integer multiple of  $n$  in the direction of electron motion. At the same time, the electron forms a spiral phase in the two-dimensional plane and has a quantization angular momentum  $(n + |l|) \hbar$  along the  $z$ -axis.

Finally, the expression of the energy eigenvalue is:

$$E_n = \frac{\hbar^2}{2m} \left( \frac{2\pi n + 2\pi n |l|}{L} \right)^2 = \frac{2\pi^2(n + n |l|)^2 \hbar^2}{mL^2}. \tag{23}$$

Besides,  $\psi_1$  has the lowest energy, which is called the ground state, and the energy of the other excited states is proportional to the increasing of  $(n + n |l|)^2$ . When electrons radiate microwaves, there will be quantized angular momentum transitions in addition to the energy transitions from the electron to the microwave beam.



**Fig. 3.** The conceptual figure of the magnetic field structure that produces different electron beam flower ring trajectories.

### 4 Method of Generating Different Trajectories

As shown in Fig. 3, the uniform magnetic field of the  $z$ -axis and the spiral wave magnetic field in the  $xOy$  plane can produce different electron beam flower ring trajectories, which causes the electrons to radiate EM wave carrying OAM. It is assumed that the electron moves along the  $z$ -axis at the initial moment from the origin of the coordinate. The direction of the spatial undulating magnetic field is along the  $x$ -axis and the  $y$ -axis. The vertical components of the magnetic induction  $B_x$  and  $B_y$  are both the sine function related to the space, and the free electron initially moves along the  $z$ -axis. Moreover, the space period of the N-pole and S-pole magnet arrays is  $\lambda_u$ , and the number of the pole pairs is  $M$ . Hence, the expression of the magnetic field generated by the magnet arrays is

$$\mathbf{B} = \left\{ B_0 \cos \left( \frac{2\pi z}{\lambda_u} \right), B_0 \sin \left( \frac{2\pi z}{\lambda_u} \right), B_z \right\}, \tag{24}$$

where  $B_0$  is the maximum value of the sinusoidal magnetic field along the  $x$ -axis and  $y$ -axis. The magnetic field Lorentz force causes the free electrons to acquire acceleration and generate radiation, and it can be observed that the EM field generated by free electrons is along the positive direction of the  $z$ -axis. Since free electrons make a uniform circular motion in the uniform magnetic field of the  $z$ -axis, we consider the uniaxial undulating magnetic field firstly. Assuming  $B_z = 0$ , the equation of motion of moving electrons in the Cartesian coordinate system from Newton's second law can be written as follows:

$$\left\{ \begin{array}{l} m\dot{v}_x = -qv_z B_0 \sin \frac{2\pi z}{\lambda_u} \\ m\dot{v}_y = -qv_z B_0 \cos \frac{2\pi z}{\lambda_u} \\ m\dot{v}_z = qv_x B_0 \sin \frac{2\pi z}{\lambda_u} + qv_y B_0 \cos \frac{2\pi z}{\lambda_u} \\ v_0^2 = v_x^2 + v_y^2 + v_z^2 \\ \dot{v}_x = \frac{dv_x}{dz} \cdot \frac{dz}{dt} = v_z \frac{dv_x}{dz}, \dot{v}_y = \frac{dv_y}{dz} \cdot \frac{dz}{dt} = v_z \frac{dv_y}{dz} \end{array} \right. , \quad (25)$$

where  $v_*$ ,  $\dot{v}_*$  and  $v_0$  denote the speed, acceleration of the electron at  $*$ -axis and the initial speed of the electron respectively. According to Eq. (25), the expression of the electron velocity can be solved as

$$\left\{ \begin{array}{l} v_x = K_y v_0 \cos \omega_0 t \\ v_y = K_x v_0 \sin \omega_0 t \\ v_z = v_0 [1 - (K_x^2 + K_y^2) / 4 + (K_x^2 - K_y^2) / 4] \cos 2\omega_0 t, \\ K_x = K_y = K = \frac{q\lambda_u B_0}{2\pi m v_0} \end{array} \right. , \quad (26)$$

where  $\omega_0 \approx 2\pi v_0 / \lambda_u$  denotes the angular frequency of radiated EM field,  $K_x$  and  $K_y$  denote the deflection parameters of the  $x$ -axis and  $y$ -axis. In addition, the average electron speed of the  $z$ -axis is  $\bar{v}_z = v_0 (1 - K^2/2)$ , and the acceleration of the electron cyclotron motion is  $a_0 \approx qB_0 v_0 / m = \omega_0^2 R$ , which means constructing a virtual magnetic field  $B_0$  along the  $z$ -axis and  $R$  is the electron cyclotron radius in the virtual magnetic field. When there is only the spiral wave magnetic field in the  $xOy$  plane, the electrons are rotated by a spiral magnetic field that periodically changes in the horizontal and vertical directions, and a circular motion that rotates around the central axis of the magnetic field can generate circularly polarized EM wave.

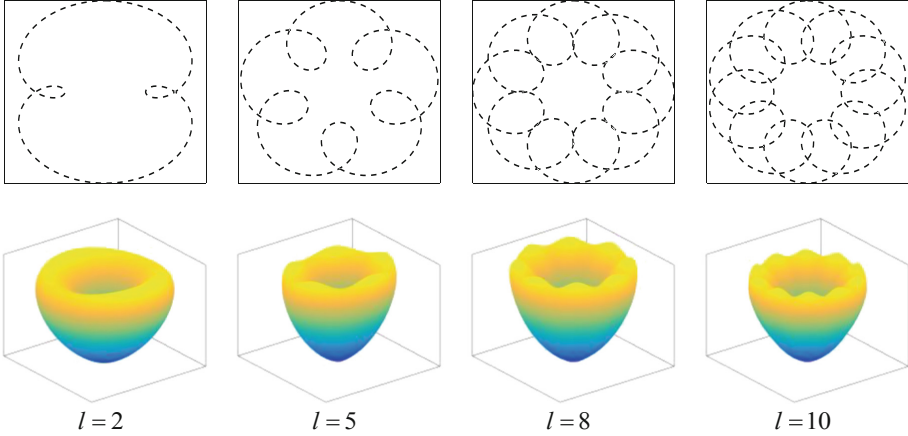
Since the wavelength of the radiated EM wave is much larger than the radius of the electronic spiral motion along the  $z$ -axis, the number of magnets  $M$  is large, and the observation point is along the  $z$ -axis direction, and the angular momentum along the  $z$ -axis is invariant by the Lorentz transformation. The electron can be regarded as a circular motion and the distance of the  $z$ -axis movement can be ignored. When  $B_z = lB_0$  and  $l$  take different integer values, the electrons can circulate around the  $z$ -axis in a double-center rotation mode

and construct different flower ring trajectories projected on the  $xOy$  plane, which can radiate EM wave with different OAM modes.

Specifically, we can refer to the method mentioned in Ref. [9] to make the electrons with the whirling motion in the waveguide. The waveguide adopts the rotating  $TE_{mn}$  mode, and the OAM wave is radiated into the free space. The circularly polarized EM wave expression in the waveguide is as follows:

$$\begin{cases} E_r = -E_0 m J_m(kr \sin \theta) \exp[i(kz + m\phi - \omega t)] / (kr \sin \theta) \\ E_\theta = E_0 J'_m(kr \sin \theta) \exp[i(kz + m\phi - \omega t)] \\ B_r = E_0 k \cos \theta J'_m(kr \sin \theta) \exp[i(kz + m\phi - \omega t)] / \omega \\ B_\theta = -E_0 k \cos \theta m J_m(kr \sin \theta) \exp[i(kz + m\phi - \omega t)] / (kr \omega \sin \theta) \\ B_z = E_0 k \sin \theta J_m(kr \sin \theta) \exp[i(kz + m\phi - \omega t)] / \omega \end{cases} \quad (27)$$

where  $J'_m(*)$  denotes the derivative of  $m$ -order first type Bessel function,  $m$  and  $n$  denote two quantum numbers indicating that the circular waveguides are independent of each other. It can be seen from the formula that the operating frequency  $\omega$  of the waveguide is determined by the waveguide size and the quantum number. The total angular momentum of the waveguide operating mode is  $m\hbar$ , where the SAM is  $\hbar$ , and the OAM is  $(m - 1)\hbar$ . Therefore, the circular waveguide is a good medium for transmitting the high order OAM wave. Moreover, the OAM radiation and reception can be accomplished by means of a circular waveguide and two-dimensionally moving free electrons.



**Fig. 4.** Electron beams with different flower ring trajectories radiating the microwave with different OAM modes.

## 5 Example

As shown in Fig. 4, when each electron of the electron beam moves on a same trajectory, each electron generates a radiation field. When the electric fields gen-

erated by the electron beams are superposed on each other, EM wave of different OAM modes can be generated. In summary, by utilizing the two-dimensional motion characteristics of electrons and the superposition characteristics of multi-electron radiation fields, it is possible to theoretically generate OAM microwaves of arbitrary modes, which can greatly expand the radiation freedom of antennas. On the contrary, it is also possible to use rotating electrons to detect OAM of a radiated EM field. Since OAM microwaves can directly transfer angular momentum to free electrons rather than the bound electrons of the antenna in the free space, which provides a method for directly detecting electric field strength and OAM for OAM microwave applications.

## 6 Discussion

By constructing an elaborately designed magnetic field structure, the electrons at the transmitting end have a bi-center rotation acceleration, and the electrons can rotate around the  $z$ -axis with different trajectories to radiate EM waves with different high-order OAMs. In order to receive the multiplexed OAM wave generated by the cyclotron electron, UCA can be used to demultiplex the OAM wave. However, the EM wave angular momentum that is mapped to two-dimensional moving electron beam angular momentum proposed in the paper provides a possible solution for directly detecting OAM wave.

Compared with the traditional OAM-MIMO system with the UCA or the SPP method [6, 15, 17], the proposed method in this paper can change the problem that the single antenna can not directly radiate the OAM wave. Specifically, the use of a helical magnetic field and electrons in the free space, as well as a waveguide system that conforms to the OAM mode for the OAM wave transmission can take advantage of OAM as a new dimension, not just as an additional degree of freedom of the MIMO system.

## 7 Conclusion

In this paper, we propose a new method for generating microwave beams carrying OAM with non-relativistic multi-electron flower ring trajectories and we can use the spiral wave magnetic field and the uniform magnetic field to generate different trajectories, which provides a new theoretical solution for existing microwave OAM applications. In the future, we may use superconducting quantum circuits or electron tubes to generate microwave beams carrying OAM experimentally. Furthermore, our work may be important for future mobile communications such as B5G and 6G.

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