System Content Analysis for a Two-Class Queue Where Service Times in a Busy Period Depend on the Presence of Class-2

Sara Sasaninejad(✉), Joris Walraevens, Hossein Moradi,
and Sabine Wittevrongel

SMACS Research Group, Department of Telecommunications and Information
Processing, Ghent University, Ghent, Belgium
{`Sara.Sasaninejad`,`Joris.Walraevens`,`Hossein.Moradi`,
`Sabine.Wittevrongel`}`@ugent.be`

Abstract. Since many real-world queueing systems are meant to incorporate heterogeneous customers, the analysis of multi-class queueing models has been an area of active research. A review of the associated models shows, however, that multi-class queueing systems in which service times depend on the presence of one certain class of customers have not yet been extensively analyzed. To address this research gap, we consider an infinite-capacity single-server discrete-time queueing system with two classes of customers (say *class-1* and *class-2*). We assume that the scheduling discipline in our work is FCFS. We assume that if we have at least one *class-2* customer during an ongoing busy period (until the system becomes empty), the service time distributions of all the customers change to the service time distribution of a *class-2* customer. By further considering the number of customer arrivals of each class to be independent and identically distributed (with a general probability distribution) from slot to slot, we perform the system content analysis by means of a generating function based approach. The results of this analysis reveal that the incorporation of such an interdependency in the service process significantly affects the resulting system content, as compared to a model where the service times are completely attached to the customer classes.

Keywords: Multi-class queueing systems · Dependent service times · System content

1 Introduction

Multi-class queueing models have been widely studied in the queueing theory literature. Such multi-class models allow to take into account nonidentical behaviors of different classes of customers entering a queueing system at the same time [1]. In multi-class queueing systems, various classes of customers can also receive

E. Hyytiä and V. Kavitha (Eds.): VALUETOOLS 2022, LNICST 482, pp. 51–66, 2023.
https://doi.org/10.1007/978-3-031-31234-2_4

different treatments [2]. Some studies on multi-class systems in the literature mainly focus on considering different kinds of class-dependent arrival characteristics for customers (e.g., [3–6]). Other studies focus more on class-dependent service time specifications (e.g., [7–10]) or consider different kinds of queueing disciplines that allow the preferential treatment of a certain class of customers (e.g., [11–15]). In our paper, we focus on a special kind of service time specification where a particular interdependency is introduced between the service time distributions of different customers, namely by letting the service time distribution be dependent on the presence of a certain class of customers in the system during the ongoing busy period. To the best of the authors' knowledge, no analysis of a multi-class queueing system with such interdependency has been reported so far in the literature.

To address the indicated research gap, we work in a discrete-time setting. This means that time is assumed to be divided into time slots of equal length and, therefore, the service time of a customer is defined by the number of slots of service that the customer requires [16]. We consider a two-class system in which the service time distribution varies depending on the presence of *class-2* customers during an ongoing busy period. To be more specific, during the ongoing busy period, if there were no *class-2* customers in our queueing system, then all customers would have a service time distribution of *type-1*. However, as soon as at least one *class-2* customer is or has been present in the system during the busy period, then as long as the system does not become empty, all customers will have a service time distribution of *type-2*. The scheduling discipline in our work is FCFS.

This research is initially motivated by the lack of analytic models considering differences among various types of vehicles in traffic. As a case in point, we know that the presence of a freight vehicle during a busy period leads to a slowdown for all vehicles sharing the same road. To mathematically model such a phenomenon in this paper, we first develop an analytical technique to derive the distribution and moments of the system content in a two-class queueing system with the non-classical service process explained above. Then, in order to evaluate the effect of the particular interdependency between service times, we compare the obtained results in terms of the mean system content for our model to those of a corresponding conventional two-class queueing model represented in [17], where service times are completely attached to the customer classes.

The organization of the paper is as follows. Section 2 explains the specific assumptions of our considered two-class queueing model and states the main system equations describing the behavior of the system. Section 3 presents our performed analysis to obtain an expression for the probability generating function of the system content in steady state. Section 4 illustrates the obtained results through some numerical examples. Section 5 enumerates the main conclusions.

2 Queueing Model and System Equations

As is mentioned earlier, we assume two classes of customers (i.e., *class-1* and *class-2* customers) arriving in an infinite-capacity single-server queueing system (defined in the discrete-time domain) that serves the customers in FCFS order. The random variable $a_{j,k}$ represents the number of arrivals of *class-j customers* ($j = 1, 2$) in the kth slot. The numbers of customer arrivals of both classes are assumed to be independent and identically distributed (with a general probability distribution) from slot to slot and are characterized by the joint probability mass function

$$a(m, n) \triangleq \Pr[a_{1,k} = m,\ a_{2,k} = n], \tag{1}$$

and the joint probability generating function (PGF)

$$A(z_1, z_2) \triangleq \mathrm{E}[z_1^{a_{1,k}} z_2^{a_{2,k}}] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Pr[a_{1,k} = m,\ a_{2,k} = n] z_1^m z_2^n. \tag{2}$$

The marginal PGF of the number of *class-1* (*class-2*) customer arrivals per slot is given by $A_1(z) = A(z, 1)$ ($A_2(z) = A(1, z)$). The PGF of the total number of *class-1* and *class-2* customer arrivals is given by $A_T(z) = A(z, z)$. The arrival rate λ_j for *class-j* is given by $A'_j(1)$ (recall that $j = 1$ or $j = 2$).

Each customer has a service time requirement of a given number of slots. This service time can start (only) at slot boundaries. As explained in Sect. 1, the service process depends on the presence of *class-2* customers during an ongoing busy period. During the ongoing busy period, if there were no *class-2* customers in the queueing system, all customers (of *class-1*) have a service time distribution of *type-1*, characterized by the PGF $S_1(z)$. Otherwise, as soon as at least one *class-2* customer is or has been present in the system during the ongoing busy period, all customers (both those of *class-1* and *class-2*) that are still to be taken into service during the busy period before the system becomes vacant, will have a service time distribution of *type-2*, characterized by the PGF $S_2(z)$. Note that *class-2* customers will therefore always have a service time PGF $S_2(z)$, while the service time PGF for a *class-1* customer can be either $S_1(z)$ or $S_2(z)$ depending on the presence of a *class-2* customer during the ongoing busy period.

We now define the random variable X_k as the total number of customers in the system at the beginning of slot k. Furthermore, we introduce R_k as the remaining service time (expressed as a number of slots) of the in-service customer (if any) at the beginning of slot k (we assume $R_k = 0$ if $X_k = 0$). Note that $R_k \geq 1$ if $X_k > 0$. Finally, we define the random variable I_k as the indicator of having had a *class-2* customer in the system during the ongoing busy period. More specifically, $I_k = 1$ if during the ongoing busy period there is or has already been at least one *class-2* customer in the system at the beginning of slot k and $I_k = 0$ otherwise. With these definitions, the vector (I_k, R_k, X_k) is easily seen to constitute a three-dimensional Markovian state description of the system at the beginning of slot k. Indeed, we can establish a set of system equations that describe the relationship between the vectors (I_k, R_k, X_k) and

$(I_{k+1}, R_{k+1}, X_{k+1})$. To do so, depending on the values of I_k, X_k and $a_{2,k}$, we distinguish five different cases:

- If $I_k = 0$ and $a_{2,k} > 0$ then $I_{k+1} = 1$.
- If $I_k = 0$ and $a_{2,k} = 0$ then $I_{k+1} = 0$.
- If $I_k = 1$ and $X_k > 0$ then $I_{k+1} = 1$.
- If $I_k = 1$, $X_k = 0$ and $a_{2,k} > 0$ then $I_{k+1} = 1$.
- If $I_k = 1$, $X_k = 0$ and $a_{2,k} = 0$ then $I_{k+1} = 0$.

If we now introduce the random variables S_1 and S_2 with PGFs $S_1(z)$ and $S_2(z)$ respectively, then the following state equations can be written for these five different cases:

- Case 1: If $I_k = 0$ and $a_{2,k} > 0$ then $I_{k+1} = 1$.
  - If $R_k = 0$ (and, hence, $X_k = 0$):

$$\begin{cases} X_{k+1} = a_{1,k} + a_{2,k}, \\ R_{k+1} = S_2. \end{cases} \tag{3}$$

  - If $R_k = 1$ (and, hence, $X_k > 0$):

$$\begin{cases} X_{k+1} = X_k - 1 + a_{1,k} + a_{2,k}, \\ R_{k+1} = S_2. \end{cases} \tag{4}$$

  - If $R_k > 1$ (and, hence, $X_k > 0$):

$$\begin{cases} X_{k+1} = X_k + a_{1,k} + a_{2,k}, \\ R_{k+1} = R_k - 1. \end{cases} \tag{5}$$

- Case 2: If $I_k = 0$ and $a_{2,k} = 0$ then $I_{k+1} = 0$.
  - If $R_k = 0$ (and, hence, $X_k = 0$):

$$\begin{cases} X_{k+1} = a_{1,k}, \\ R_{k+1} = \begin{cases} 0 & \text{if } a_{1,k} = 0, \\ S_1 & \text{if } a_{1,k} > 0. \end{cases} \end{cases} \tag{6}$$

  - If $R_k = 1$ (and $X_k > 0$):

$$\begin{cases} X_{k+1} = X_k - 1 + a_{1,k}, \\ R_{k+1} = \begin{cases} 0 & \text{if } X_k = 1 \text{ and } a_{1,k} = 0, \\ S_1 & \text{if } X_k > 1 \text{ or } a_{1,k} > 0. \end{cases} \end{cases} \tag{7}$$

  - If $R_k > 1$ (and $X_k > 0$):

$$\begin{cases} X_{k+1} = X_k + a_{1,k}, \\ R_{k+1} = R_k - 1. \end{cases} \tag{8}$$

- Case 3: If $I_k = 1$ and $X_k > 0$ then $I_{k+1} = 1$.
  - If $R_k = 1$:

$$\begin{cases} X_{k+1} = X_k - 1 + a_{1,k} + a_{2,k}, \\ R_{k+1} = \begin{cases} 0 & \text{if } \; X_k = 1 \text{ and } a_{1,k} + a_{2,k} = 0, \\ S_2 & \text{if } \; X_k > 1 \text{ or } a_{1,k} + a_{2,k} > 0. \end{cases} \end{cases} \tag{9}$$

  - If $R_k > 1$:

$$\begin{cases} X_{k+1} = X_k + a_{1,k} + a_{2,k}, \\ R_{k+1} = R_k - 1. \end{cases} \tag{10}$$

- Case 4: If $I_k = 1$, $X_k = 0$ and $a_{2,k} > 0$ then $I_{k+1} = 1$; we only have $R_k = 0$.

$$\begin{cases} X_{k+1} = a_{1,k} + a_{2,k}, \\ R_{k+1} = S_2. \end{cases} \tag{11}$$

- Case 5: If $I_k = 1$, $X_k = 0$ and $a_{2,k} = 0$ then $I_{k+1} = 0$; we only have $R_k = 0$.

$$\begin{cases} X_{k+1} = a_{1,k}, \\ R_{k+1} = \begin{cases} 0 & \text{if } \; a_{1,k} = 0, \\ S_1 & \text{if } \; a_{1,k} > 0. \end{cases} \end{cases} \tag{12}$$

3 Queueing Analysis

The goal of this section is to derive an expression for the PGF of the system content at the beginning of a slot in steady state. To perform the analysis, we first define $P_k(x,z)$ as the joint PGF of the vector (R_k, X_k) by

$$P_k(x,z) = \mathrm{E}[x^{R_k} z^{X_k}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Pr[R_k = i, X_k = j] x^i z^j, \tag{13}$$

where E[.] indicates the expected value of the expression within the square brackets.

Furthermore, we introduce the partial PGFs $P_{i,k}(x,z)$ as

$$P_{i,k}(x,z) = \mathrm{E}[x^{R_k} z^{X_k} | I_k = i] \Pr[I_k = i] \quad \text{, i=0,1.} \tag{14}$$

Clearly, the joint PGF $P_k(x,z)$ can then be expressed as $P_k(x,z) = P_{0,k}(x,z) + P_{1,k}(x,z)$. Since $R_k = 0$ if and only if $X_k = 0$, we have also

$$P_{i,k}(x,0) = P_{i,k}(0,0), \quad \text{for all } \; x, \tag{15}$$

which will be used later in the analysis.

The next step in our analysis is now to derive relationships between the functions $P_{0,k}(x,z)$ and $P_{1,k}(x,z)$ for slot k and the functions $P_{0,k+1}$ and $P_{1,k+1}$ for slot $k+1$ by using the system equations shown in Eqs. (3) to (12). We note that case 2 and case 5 of the system equations lead to $I_{k+1} = 0$. Accordingly, we can express $P_{0,k+1}(x, z)$ in terms of $P_{0,k}(x, z)$ and $P_{1,k}(x, z)$ based on cases 2 and 5. We proceed as follows:

$$\begin{aligned} P_{0,k+1}(x,z) =& \mathrm{E}[x^{R_{k+1}} z^{X_{k+1}} | I_{k+1} = 0] \Pr[I_{k+1} = 0] \\ =& \Pr[I_k = R_k = X_k = 0] \Pr[a_{1,k} = a_{2,k} = 0] \\ &+ \Pr[I_k = R_k = X_k = 0] \Pr[a_{1,k} > 0, a_{2,k} = 0] \\ &\ \mathrm{E}[x^{S_1} z^{a_{1,k}} | a_{2,k} = 0, a_{1,k} > 0] \\ &+ \Pr[I_k = 0, R_k = X_k = 1] \Pr[a_{1,k} = a_{2,k} = 0] \\ &+ \Pr[I_k = 0, R_k = 1, X_k - 1 + a_{1,k} > 0, a_{2,k} = 0] \\ &\ \mathrm{E}[x^{S_1} z^{X_k - 1 + a_{1,k}} | I_k = 0, R_k = 1, X_k - 1 + a_{1,k} > 0, a_{2,k} = 0] \\ &+ \Pr[I_k = 0, R_k > 1, X_k > 0] \Pr[a_{2,k} = 0] \\ &\ \mathrm{E}[x^{R_k - 1} z^{X_k + a_{1,k}} | I_k = 0, R_k > 1, X_k > 0, a_{2,k} = 0] \\ &+ \Pr[I_k = 1, R_k = X_k = 0] \Pr[a_{1,k} = a_{2,k} = 0] \\ &+ \Pr[I_k = 1, R_k = X_k = 0] \Pr[a_{1,k} > 0, a_{2,k} = 0] \\ &\ \mathrm{E}[x^{S_1} z^{a_{1,k}} | a_{2,k} = 0, a_{1,k} > 0], \end{aligned} \tag{16}$$

where we also used the independence of (I_k, R_k, X_k) and $(a_{1,k}, a_{2,k})$ at places.

We can further work out Eq. (16) by using the definitions (2) and (14). This leads to

$$\begin{aligned} P_{0,k+1}(x,z) =& P_{0,k}(0,0)A(0,0) \\ &+ P_{0,k}(0,0)\big[A(z,0) - A(0,0)\big]S_1(x) \\ &+ F_{0,k}(0)A(0,0) \\ &+ \big[F_{0,k}(z)A(z,0) - F_{0,k}(0)A(0,0)\big]S_1(x) \\ &+ \frac{A(z,0)}{x}\big[P_{0,k}(x,z) - xzF_{0,k}(z) - P_{0,k}(0,0)\big] \\ &+ P_{1,k}(0,0)A(0,0) \\ &+ P_{1,k}(0,0)\big[A(z,0) - A(0,0)\big]S_1(x), \end{aligned} \tag{17}$$

where the function $F_{i,k}(z)$ is defined as

$$F_{i,k}(z) = \mathrm{E}[z^{X_k - 1} | I_k = i, R_k = 1, X_k > 0] \Pr[I_k = i, R_k = 1, X_k > 0]. \tag{18}$$

In a similar way, we can also express $P_{1,k+1}(x, z)$ in terms of $P_{0,k}(x, z)$ and $P_{1,k}(x, z)$, by noting that cases 1, 3 and 4 of the system equations all lead to

$I_{k+1} = 1$. As a result, we then obtain

$$\begin{aligned}
P_{1,k+1}(x,z) =& \mathrm{E}[x^{R_{k+1}} z^{X_{k+1}} | I_{k+1} = 1] \Pr[I_{k+1} = 1] \\
=& \Pr[I_k = R_k = X_k = 0] \Pr[a_{2,k} > 0] \mathrm{E}[x^{S_2} z^{a_{1,k}+a_{2,k}} | a_{2,k} > 0] \\
&+ \Pr[I_k = 0, R_k = 1, X_k > 0] \Pr[a_{2,k} > 0] \\
&\quad \mathrm{E}[x^{S_2} z^{X_k - 1 + a_{1,k} + a_{2,k}} | I_k = 0, R_k = 1, X_k > 0, a_{2,k} > 0] \\
&+ \Pr[I_k = 0, R_k > 1, X_k > 0] \Pr[a_{2,k} > 0] \\
&\quad \mathrm{E}[x^{R_k - 1} z^{X_k + a_{1,k} + a_{2,k}} | I_k = 0, R_k > 1, X_k > 0, a_{2,k} > 0] \\
&+ \Pr[I_k = R_k = X_k = 1] \Pr[a_{1,k} = a_{2,k} = 0] \\
&+ \Pr[I_k = R_k = 1, X_k - 1 + a_{1,k} + a_{2,k} > 0] \\
&\quad \mathrm{E}[x^{S_2} z^{X_k - 1 + a_{1,k} + a_{2,k}} | I_k = R_k = 1, X_k - 1 + a_{1,k} + a_{2,k} > 0] \\
&+ \Pr[I_k = 1, R_k > 1, X_k > 0] \\
&\quad \mathrm{E}[x^{R_k - 1} z^{X_k + a_{1,k} + a_{2,k}} | I_k = 1, R_k > 1, X_k > 0] \\
&+ \Pr[I_k = 1, R_k = X_k = 0] \Pr[a_{2,k} > 0] \mathrm{E}[x^{S_2} z^{a_{1,k}+a_{2,k}} | a_{2,k} > 0].
\end{aligned} \tag{19}$$

With the definitions (2), (14) and (18), Eq. (19) then simplifies into

$$\begin{aligned}
P_{1,k+1}(x,z) =& P_{0,k}(0,0)\big[A(z,z) - A(z,0)\big] S_2(x) \\
&+ S_2(x) F_{0,k}(z)\big[A(z,z) - A(z,0)\big] \\
&+ \frac{A(z,z) - A(z,0)}{x}\big[P_{0,k}(x,z) - xzF_{0,k}(z) - P_{0,k}(0,0)\big] \\
&+ F_{1,k}(0)A(0,0) \\
&+ S_2(x)\big[F_{1,k}(z)A(z,z) - F_{1,k}(0)A(0,0)\big] \\
&+ \frac{A(z,z)}{x}\big[P_{1,k}(x,z) - xzF_{1,k}(z) - P_{1,k}(0,0)\big] \\
&+ P_{1,k}(0,0)\big[A(z,z) - A(z,0)\big] S_2(x).
\end{aligned} \tag{20}$$

Given the above calculations, Eq. (17) provides a relationship between the functions $P_{0,k}$, $P_{1,k}$ and $P_{0,k+1}$ and Eq. (20) provides a relationship between $P_{0,k}$, $P_{1,k}$ and $P_{1,k+1}$. Once a steady state is reached, the functions $P_{i,k}$ and $F_{i,k}$ $(i = 0, 1)$ converge to limiting functions (for k going to infinity):

- $P_0(x,z) = \lim_{k\to\infty} P_{0,k}(x,z) = \lim_{k\to\infty} P_{0,k+1}(x,z)$,
- $P_1(x,z) = \lim_{k\to\infty} P_{1,k}(x,z) = \lim_{k\to\infty} P_{1,k+1}(x,z)$,
- $F_0(z) = \lim_{k\to\infty} F_{0,k}(z) = \lim_{k\to\infty} F_{0,k+1}(x,z)$,
- $F_1(z) = \lim_{k\to\infty} F_{1,k}(z) = \lim_{k\to\infty} F_{1,k+1}(x,z)$.

Under the assumption of stationary distributions and after some simplifications, Eq. (17) can be rewritten as

$$\begin{aligned}
P_0(x,z) = \frac{1}{x - A(z,0)} \Big\{ & \big[\big(xS_1(x) - 1\big)A(z,0) - xA(0,0)\big(S_1(x) - 1\big)\big] P_0(0,0) \\
& + \big[xS_1(x)A(z,0) - xA(0,0)\big(S_1(x) - 1\big)\big] P_1(0,0) \\
& + \big[xA(0,0)\big(1 - S_1(x)\big)\big] F_0(0) \\
& + \big[xA(z,0)\big(S_1(x) - z\big)\big] F_0(z) \Big\}.
\end{aligned} \tag{21}$$

The same approach holds for Eq. (20) that after some simplifications can be rewritten as

$$\begin{aligned}
P_1(x,z) = {} & \frac{1}{(x - A(z,z))(x - A(z,0))} \bigg\{ \Big[\big(\big(S_1(x) - S_2(x)\big)A(z,0) \\
& + \big(1 - S_1(x)\big)A(0,0) + xS_2(x) - 1\big)x\big(A(z,z) - A(z,0)\big)\Big] P_0(0,0) \\
& + \bigg[- x\big(S_1(x) - S_2(x)\big)A(z,0)^2 + \bigg(\big(xS_1(x) - xS_2(x) + 1\big)A(z,z) \\
& - x\big(\big(1 - S_1(x)\big)A(0,0) + xS_2(x)\big)\bigg)A(z,0) \\
& + \bigg(\big(1 - S_2(x)\big)A(0,0) + xS_2(x) - 1\bigg)xA(z,z)\bigg] P_1(0,0) \\
& + \Big[xA(0,0)\big(1 - S_1(x)\big)\big(A(z,z) - A(z,0)\big)\Big] F_0(0) \\
& + \bigg[\bigg(\big(S_2(x) - S_1(x)\big)A(z,0) + x\big(z - S_2(x)\big)\bigg)\big(A(z,0) - A(z,z)\big)x\bigg] F_0(z) \\
& + \Big[xA(0,0)\big(1 - S_2(x)\big)\big(A(z,0) + x\big)\Big] F_1(0) \\
& + \Big[xA(z,z)\big(z - S_2(x)\big)\big(A(z,0) - x\big)\Big] F_1(z) \bigg\}.
\end{aligned} \tag{22}$$

In Eqs. (21) and (22), $F_0(z)$ and $F_1(z)$ are unknown functions and $P_0(0,0)$, $P_1(0,0)$, $F_0(0)$, and $F_1(0)$ are the unknown constants that should still be determined. We now explain the derivation of these unknowns in three subsequent steps. In the first step, to find $P_0(0,0)$ and $P_1(0,0)$, we use Eq. (15). Imposing this condition in Eq. (21) for $z = 0$, we obtain

$$P_0(0,0)\big[1 - A(0,0)\big] = A(0,0)\big[F_0(0) + P_1(0,0)\big]. \tag{23}$$

Similarly, by applying Eq. (15) in Eq. (22) for $z = 0$, we find

$$P_1(0,0) = F_1(0)A(0,0). \tag{24}$$

Notice that in Eqs. (23) and (24), $P_0(0,0)$ and $P_1(0,0)$ are derived based on $F_0(0)$ and $F_1(0)$.

In the second step, we find the functions $F_0(z)$ and $F_1(z)$. Given definition (14), we know that the partial PGF $P_0(x,z)$ must be bounded for all possible values of x and z, such that $|x| < 1$ and $|z| < 1$. Specifically, this should hold for $x = A(z, 0)$ and $|z| < 1$, since $|A(z, 0)| < 1$ for all such z. Therefore, if we choose $x = A(z, 0)$ in Eq. (21), the denominator vanishes. Then the same must be true for the numerator of Eq. (21) (in which we have considered both Eqs. (23) and (24)). This way, we can find $F_0(z)$ in terms of $F_0(0)$ and $F_1(0)$:

$$\begin{aligned} F_0(z) = {} & \frac{S_1(A(z,0))A(0,0)}{A(z,0)\big(1 - A(0,0)\big)\big(S_1(A(z,0) - z\big)} \\ & \Big[\big(1 - A(z,0)\big)F_0(0) + \big(A(0,0) - A(z,0)\big)F_1(0)\Big]. \end{aligned} \tag{25}$$

Similarly, in Eq. (22), we can see that if $x = A(z, 0)$ and/or $x = A(z, z)$, the denominator of Eq. (22) becomes zero. If we substitute x with $A(z, 0)$ in the numerator of Eq. (22), we see that the numerator becomes zero as well. However, if we choose $x = A(z, z)$, the denominator of Eq. (22) becomes zero but the numerator does not automatically become zero. Again, we know that with respect to Eq. (14), $P_1(x, z)$ must be bounded for all possible values of x and z such that $|x| < 1$ and $|z| < 1$. Therefore, the numerator of Eq. (22) should become zero if we choose $x = A(z, z)$. After some calculations and using the Eqs. (21)-(25), we can find $F_1(z)$ in terms of $F_0(0)$ and $F_1(0)$ as

$$\begin{aligned} & F_1(z) = \\ & \quad \frac{1}{A(z,z)A(z,0)(-1 + A(0,0))(z - S_2(A(z,z))(z - S_1(A(z,0)))} \\ & \quad \Bigg\{\Bigg[- \Big(- (A(z,0) - A(z,z))(zA(z,0) - S_1(A(z,0)))S_2(A(z,z)) \\ & \quad + z(A(z,0) - 1)(A(z,0)S_1(A(z,z)) - A(z,z)S_1(A(z,0))))A(0,0)\Bigg]F_0(0) \\ & \quad \Bigg[A(0,0)(((-A(z,0) + A(z,z)A(0,0))S_1(A(z,0)) - \Big(- A(z,0) \\ & \quad + A(z,z) - 1 + A(0,0)\Big)A(z,0)z)S_2(A(z,z)) \\ & \quad + z(A(0,0) - A(z,0))(A(z,0)S_1(A(z,z)) - A(z,z)S_1(A(z,0))))\Bigg]F_1(0)\Bigg\}. \end{aligned} \tag{26}$$

In the last step, we determine the two remaining unknown constants $F_0(0)$ and $F_1(0)$. To this end, we first utilize the normalization condition, which should be met for $P_0(x, z) + P_1(x, z)$. Accordingly, $P_0(1, 1) + P_1(1, 1)$ should be equal to 1 in order to have a normalized distribution. After applying $P_0(1, 1) + P_1(1, 1) = 1$ and using L'Hôp ital's rule (twice), we find

$$\begin{aligned}
&P_0(1,1)+P_1(1,1)= \\
&\frac{1}{\big[\big(-1+S_2'(1)\big(\lambda_1+\lambda_2\big)\big(-1+S_1(A(1,0))\big)\big(-1+A(0,0)\big)\big]} \\
&\cdot\Big\{\big[F_0(0)+F_1(0)\big]\big[S_1(A(1,0))-\big(S_1'(1)-S_2'(1)\big)A(1,0)\big] \\
&F_1(0)\big[\big(S_1'(1)-S_2'(1)\big)A(0,0)-1\big] \\
&+F_0(0)\big[S_1'(1)-S_2'(1)-1\big]\Big\}A(0,0)=1,
\end{aligned} \tag{27}$$

in which λ_1 is the arrival rate of *class-1* customers and λ_2 is the arrival rate of *class-2* customers. Eq. (27) provides a first relationship between $F_0(0)$ and $F_1(0)$. To arrive at a second relation between these unknowns, we now focus our attention on the PGF $X(z)$ of the total number of customers in the system at the beginning of a slot in steady state. According to the equations developed above, $X(z)$ can easily be found as $X(z)=P(1,z)=P_0(1,z)+P_1(1,z)$.

Setting $x=1$ in Eqs. (21) and (22) and using the results shown in Eqs. (23)–(26), we obtain

$$\begin{aligned}
&X(z)= \\
&\frac{A(0,0)(z-1)}{\big[\big(z-S_2(A(z,z))\big)\big(z-S_1(A(z,0))\big)(-1+A(0,0))(A(z,z)-1)\big]} \\
&\Big\{\big[F_0(0)+F_1(0)\big]\Big[\Big(\big(A(z,z)-1\big)S_1(A(z,0))-z\big(A(z,z)-A(z,0)\big)\Big) \\
&S_2(A(z,z))-zS_1(A(z,z))A(z,0)\Big]+F_1(0)\Big[\big(-z\big(-1+A(0,0)\big) \\
&S_2(A(z,z))+zS_1(A(z,z))A(0,0)\Big]+F_0(0)\Big[zS_1(A(z,z))\Big]\Big\}.
\end{aligned} \tag{28}$$

We know that the PGF $X(z)$ must be bounded for all possible values of z with $|z|\leq 1$. Note that the denominator of the right-hand side of Eq. (28) has a unique zero inside the complex unit circle; this can be proved by Rouché's theorem. Let us define this unique zero by z_r; it is characterized by the following equation:

$$z_r-S_1(A(z_r,0))=0, \quad \text{with} \quad |z_r|<1. \tag{29}$$

Since $X(z)$ is analytic inside the complex unit circle, the numerator of $X(z)$ should also be zero for $z=z_r$. This property leads to

$$\begin{aligned}
&\Big[\Big(-F_0(0)-F_1(0)\Big)A(z_r,0)+F_1(0)A(0,0)+F_0(0)\Big]A(0,0)z_r\big(z_r-1\big) \\
&\Big(S_1\big(A(z_r,z_r)\big)-S_2\big(A(z_r,z_r)\big)\Big)=0,
\end{aligned} \tag{30}$$

which is the second relationship between $F_0(0)$ and $F_1(0)$ we were looking for. The two last unknown constants $F_0(0)$ and $F_1(0)$ can then be found by solving

Eqs. (27) and (30) (two equations and two unknowns). Accordingly,

$$F_0(0) = \frac{-\Big(A(0,0) - A(z_r,0)\Big)\Big(-1 + S_1(A(1,0))\Big)\Big(S_1'(1)\lambda_1 + S_2'(1)\lambda_2 - 1\Big)}{A(0,0)\Big(\big(S_1'(1) - S_2'(1)\big)\Big(A(1,0) - A(z_r,0)\Big) + 1 - S_1(A(1,0))\Big)} \tag{31}$$

and

$$F_1(0) = \frac{\Big(A(z_r,0) - 1\Big)\Big(-1 + S_1(A(1,0))\Big)\Big(S_1'(1)\lambda_1 + S_2'(1)\lambda_2 - 1\Big)}{A(0,0)\Big(\big(S_1'(1) - S_2'(1)\big)\Big(A(1,0) - A(z_r,0)\Big) + 1 - S_1(A(1,0))\Big)}. \tag{32}$$

With the results for $F_0(0)$ and $F_1(0)$, the expressions for $P_0(x,z)$, $P_1(x,z)$ and $X(z)$ are now completely determined. Based on the PGF $X(z)$, we can then also calculate moments of the system content. In particular, we can calculate the average number of customers in the system $\mathrm{E}[X]$ by

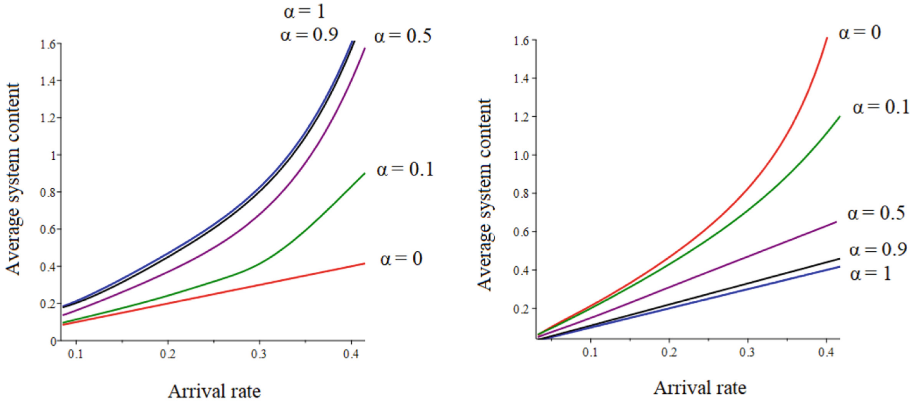
$$\mathrm{E}[X] = \left.\frac{dX(z)}{dz}\right|_{z=1}. \tag{33}$$

4 Numerical Results

In this section, we present some numerical examples to illustrate how the considered interdependency in the service process impacts the average system content. To do so, we assume here that the total number of customer arrivals is independent and identically distributed from slot to slot and follows a Bernoulli distribution, i.e., $A_T(z) = 1 - \lambda + \lambda z$, where λ is the total arrival rate. Moreover, we assume that each arrival belongs to one of the two classes with a given probability, independently from customer to customer. Let α be the probability that an arriving customer is a *class-2* customer; $1-\alpha$, then, indicates the probability that an arriving customer belongs to *class-1*. Under these assumptions, the joint PGF $A(z_1, z_2)$ of the arrival process becomes

$$A(z_1, z_2) = 1 - \lambda + \lambda(\alpha z_2 + (1-\alpha)z_1), \tag{34}$$

In Sect. 3, we obtained Eqs. (28), (31), (32), and (33) to calculate the PGF and the average value of the total number of customers in the system at the beginning of a slot in steady state. Based on these equations, the average number of customers in the system depends on (1) the arrival rate of each type of customers (i.e., α and λ in Eq. (34)) and (2) the two types of service time distributions (i.e., the PGFs $S_1(z)$ and $S_2(z)$). To study the basic effects of these parameters, we further focus on the case where all service times have a fixed length of (dependent on their type) either S_1 or S_2 slots, i.e., we consider $S_1(z) = z^{S_1}$ and $S_2(z) = z^{S_2}$. In order to show the effects of these parameters on the average number of customers in the system, Figs. 1 and 2 are developed. Figure 1 illustrates the change of the average system content with respect to the parameter λ by assuming in (a) $S_1 = 1$, $S_2 = 2$ and in (b) $S_1 = 2$, $S_2 = 1$. Figure 2 presents the change of the average system content with respect to the parameter α by assuming in (a) $S_1 = 1$, $S_2 = 2$, and in (b) $S_1 = 2$, $S_2 = 1$.



(a) $S_1 = 1$ and $S_2 = 2$

(b) $S_1 = 2$ and $S_2 = 1$

Fig. 1. Average system content versus arrival rate λ in case of (a) $S_1 = 1$, $S_2 = 2$ and (b) $S_1 = 2$, $S_2 = 1$

First, it is trivial to see in all the above figures that the average number of customers in the system increases in case of higher arrival rates (as expected). In Fig. 1(a) and Fig. 2(a), for the case of $S_1 = 1$ and $S_2 = 2$ (i.e., in case service times are twice as long when at least one *class-2* customer has been present in the system during the ongoing busy period), we see that a higher α as well as a higher λ results in an increase in the average number of customers in the system. Notice that a higher α means a higher probability of having a *class-2* customer in the system during the ongoing busy period and relatively more *class-2* customers on average. Given the imposed interdependency between service times and since $S_2 > S_1$, this result is also expected. On the contrary, in Fig. 1(b) and Fig. 2(b), we evaluate the case where $S_1 = 2$ and $S_2 = 1$, meaning that the service times are longer when no *class-2* customers have been present in the system. Accordingly, a higher α leads typically to a shorter service time for more customers. Therefore, we see a lower average system content for higher α in this case.

To compare our results with a conventional two-class model in which service times of different classes of customers are independent and completely attached to the customer classes, we use [17] in which the authors presented a multi-class

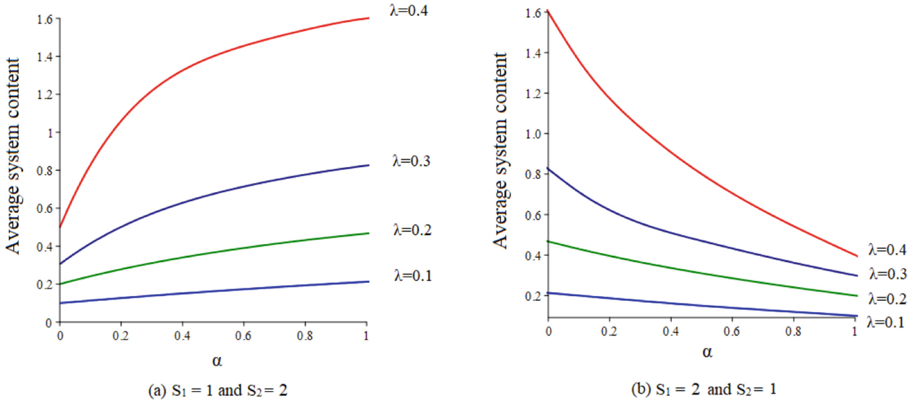
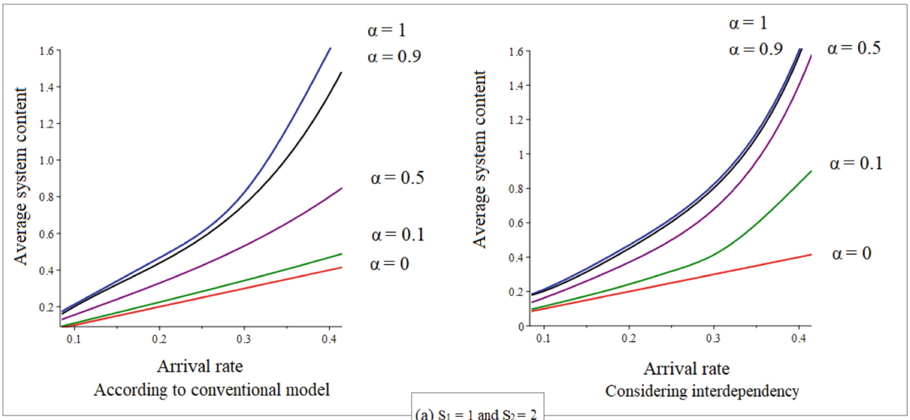


Fig. 2. Average system content versus α in case of (a) $S_1 = 1$, $S_2 = 2$ and (b) $S_1 = 2$, $S_2 = 1$

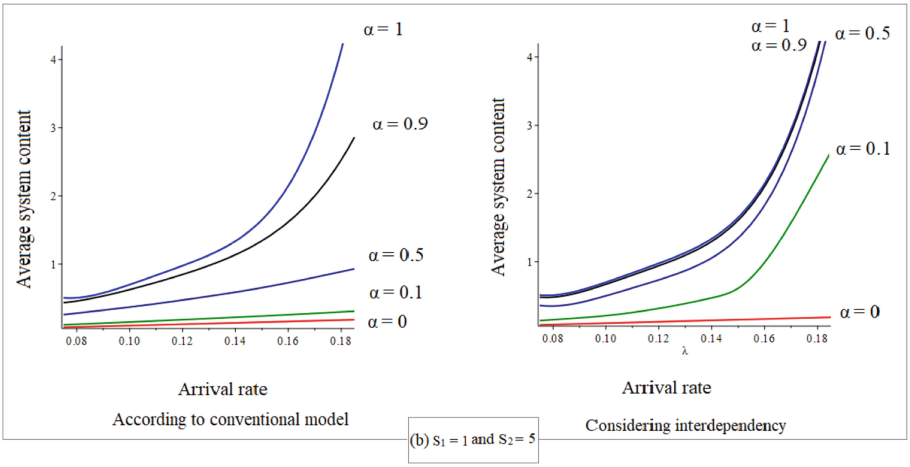
queueing model with a general arrival distribution and general service times. In Fig. 3, we compare the results of the model presented in [17] to those from our model, again considering the Bernoulli arrival distribution of Eq. (34) and fixed-length service times. For the conventional model, the latter means that all *class-1* customers receive a service time of S_1 slots, while the service time of *class-2* customers equals S_2 slots. Figure 3 shows that the conventional model cannot, in general, provide accurate estimations of the average system content in case there exists interdependency between the service times of different classes.

There are two different cases considered in Fig. 3. The first case compares the average system contents of the two models when $S_1 = 1$ and $S_2 = 2$. The same comparison is also carried out for $S_1 = 1$ and $S_2 = 5$ in the second case. Our comprehensive evaluation takes into account a range of α values. From Fig. 3, the following conclusions can be drawn:

- The effect of considering interdependency between service times of different classes of customers will vanish when α (the percentage of *class-2* customers) equals 0 or 1. When $\alpha = 0$, the model simplifies to a conventional queueing system where all customers have service time S_1. When $\alpha = 1$, the model simplifies to a conventional queueing system with service time S_2 for all customers.
- When $0 < \alpha < 1$, (see, for example, $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 0.9$ in Fig. 3), our developed model shows higher average system contents compared to the conventional model.
- When $0 < \alpha < 1$, as the ratio of S_2 to S_1 increases, the mentioned differences become more tangible.



(a) $S_1 = 1$ and $S_2 = 2$



(b) $S_1 = 1$ and $S_2 = 5$

Fig. 3. Comparison between average system contents versus arrival rate λ calculated from a conventional model and from our developed formulas, for (a)$S_1 = 1$, $S_2 = 2$ and (b)$S_1 = 1$, $S_2 = 5$

5 Conclusions

In this paper, we have presented a discrete-time queueing model for evaluating the behavior of a multi-class queueing system in which service times are affected by the presence of a certain class of customers during an ongoing busy period. In this respect, we defined two different classes of customers (labeled by *class-1* customers and *class-2* customers) and two different types of service time distributions (with PGFs $S_1(z)$ and $S_2(z)$). When a *class-2* customer is or has been present during the current busy period, all service times for all customers will change to PGF $S_2(z)$. Otherwise, customer service times will have PGF $S_1(z)$. Under these assumptions, we developed an analytical formulation describing the average system content using a generating function technique. The carried out analysis confirms that the incorporation of heterogeneity of customers along with their associated service times whose values depend on the presence of a certain class of customers leads to considerable accuracy improvements over a conventional model where interdependency between the service times is not considered.

We are currently in the process of investigating the delay (average delay) a given class of customer (using this model) may tolerate.

Acknowledgments. This work was supported in part by an EOS grant from the Belgian Research Councils FWO and FNRS (grant number 30452698) and in part by a research project from the Flemish Research Council FWO (grant number 3G051118).

References

1. De Clercq, S., Steyaert, B., Bruneel, H.: Queue content analysis in a 2-class discrete-time queueing system under the slot-bound priority service rule. Math. Prob. Eng. **2012** (2012)
2. De Clercq, S., Laevens, K., Steyaert, B., Bruneel, H.: A multi-class discrete-time queueing system under the FCFS service discipline. Ann. Oper. Res. **202**(1), 59–73 (2013)
3. Baetens, J., Steyaert, B., Claeys, D., Bruneel, H.: Analysis of a batch-service queue with variable service capacity, correlated customer types and generally distributed class-dependent service times. Perform. Eval. **135**, 102012 (2019)
4. Walraevens, J., Bruneel, H., Fiems, D., Wittevrongel, S.: Delay analysis of multiclass queues with correlated train arrivals and a hybrid priority/FIFO scheduling discipline. Appl. Math. Model. **45**, 823–839 (2017)
5. Van Houdt, B., Blondia, C.: The delay distribution of a type k customer in a first-come-first-served MMAP [K]/PH[K]/1 queue. J. Appl. Probab. **39**, 213–223 (2002)
6. De Clercq, S., Walraevens, J.: Delay analysis of a two-class priority queue with external arrivals and correlated arrivals from another node. Ann. Oper. Res. **293**(1), 57–72 (2020)
7. Takine, T.: Single-server queues with Markov-modulated arrivals and service speed. Queueing Syst. **49**(1), 7–22 (2005)
8. Wittevrongel, S., Feyaerts, B., Bruneel, H., De Vuyst, S.: Delay characteristics in place-reservation queues with class-dependent service times. J. Ind. Manage. Optim. **15**(1), 37–58 (2019)
9. Boxma, O.J., Takine, T.: The M/G/1 FIFO queue with several customer classes. Queueing Syst. **45**(3), 185 (2003)
10. Kim, B., Kim, J.: Stability of a multi-class multi-server retrial queueing system with service times depending on classes and servers. Queueing Syst. **94**(1), 129–146 (2020)
11. Miller, R.G., Jr.: Priority queues. Ann. Math. Stat. **31**(1), 86–103 (1960)
12. Maertens, T., Walraevens, J., Bruneel, H.: Priority queueing systems: from probability generating functions to tail probabilities. Queueing Syst. **55**(1), 27–39 (2007)
13. Walraevens, J., Steyaert, B., Bruneel, H.: A preemptive repeat priority queue with resampling: performance analysis. Ann. Oper. Res. **146**(1), 189–202 (2006)

14. Fiems, D., Walraevens, J., Bruneel, H.: Performance of a partially shared priority buffer with correlated arrivals. In: Mason, L., Drwiega, T., Yan, J. (eds.) ITC 2007. LNCS, vol. 4516, pp. 582–593. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-72990-7_52
15. Maertens, T., Walraevens, J., Bruneel, H.: On priority queues with priority jumps. Perform. Eval. **63**(12), 1235–1252 (2006)
16. De Muynck, M., Bruneel, H., Wittevrongel, S.: Analysis of a queue with general service demands and correlated service capacities. Ann. Oper. Res. **293**(1), 73–99 (2020)
17. Bruneel, H., Kim, B.G.: Discrete-time models for communication systems including ATM. Kluwer Academic Publishers, Boston (1993)