Robust Stability of Uncertain Replicator Population Dynamics with Time Delay

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Abstract. In this paper, we generalize the replicator dynamics with time delay of Tao [13] to the time-delayed uncertain population replicator dynamics. Considering the uncertainties of payoff functions, we propose three perturbed models of the traditional time-delayed replicator dynamics. The stability of the evolutionary stable strategy (ESS) is then studied. We prove the ESS of traditional replicator dynamics is also asymptotically stable under some constraints on the time delay in our first model and the result of Tao [13] is a special case of our first model. Then We further study the uniformly perturbed time-delayed replicator dynamics with two delays. In our third model, we show the stability of the ESS is irrelevant to the rate parameter. Some numerical examples are illustrated.

Keywords: Population games · Replicator dynamics · Time delay · Uncertainty · ESS · Stability

1 Introduction and Preliminaries

Evolutionary game theory is an excellent tool for analyzing strategic interactions among populations consisting of numerous individuals. The traditional static game [1] which supposes that players are usually perfectly rational or sometimes even hyper-rational. Evolutionary games however treat individuals as myopic and assume that they dynamically revise their strategies. Early in 1974, the fundamental concept of evolutionarily stable strategy (ESS) was introduced in [2], which has wide applications in the study of animal conflicts. Nevertheless, this fundamental concept neither involves the time aspect on resisting mutants, nor studies the long-term state of populations given an initial configuration. Taylor

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and Jonker [3] proposed a dynamic approach to avoid this drawback. Their replicator dynamics does not rely on any assumption of rationality and describes the evolution of strategies in the way that agents always compare their payoffs to the average payoff of the population they belong to. The replicator equation is one of the most important game dynamics and it is still a hot topic in evolutionary game theory, see [4–9]. The monograph of Weibull [10] provided a thorough description to evolutionary game theory. Sandholm [11] provided a systematic and unified presentation of evolutionary game theory. He extended the evolutionary model to the context of multi-population and considered the nonlinear payoff functions. Recently, Newton [12] excellently surveyed the progress of evolutionary game theory and discussed open topics of importance to economics and the social sciences.

On one hand, classical evolutionary dynamics consider individuals instantaneously react to the system, that is to say, without time delay. It is really unreasonable. Tao and Wang [13] first investigated a two-strategy evolutionary dynamics model with time delay. They analyzed the effect of a time delay on the stability of time-delayed replicator dynamics and showed that the stability of evolutionary stable strategy will lost when the time delay is sufficiently large. The authors [14] considered two models of discrete time-delayed replicator dynamics. In their first model, they showed that large delay made the dynamics unstable just as the result in [13]. Furthermore, it was shown that the introduction of time delay had no effect on the stability in their biological-type model. Ben-Khalifa et al. [15] extended the evolutionary games by introducing two communities. In addition, they defined three different evolutionarily stable strategies with different levels of stability. Moreover, they studied the behavior of the evolutionary game dynamics with different types of time delay. On the other hand, fixed time delay is usually considered in literature. This assumption is too restrictive and demanding. Therefore, it is unrealistic in reality. For instance, it is of course impossible to assume that the eggs hatch simultaneously. Thus considering uncertain time delays is more reasonable. Zhong et al. [16] proposed the replicator dynamics with bounded continuously distributed time delay and discussed the stability conditions of the unique evolutionarily stable strategy in their model.

Our world is full of uncertainties and in some sense nothing in our life is deterministic. When individuals interact in a competition, nobody is sure of the payoffs of him or others. Therefore, the outcome is always unknown or fuzzy to everyone. Thus we should consider the payoffs to be perturbed sometimes.

We first introduce the population game and derive the traditional replicator dynamics as in [14]. Consider a large population of individuals who interact repeatedly through random match and assume that this population is haploid, which means the offspring have the identical strategies as their parents. For simplicity, there are only two different strategies denoted by C_1 and C_2. The interaction outcome is represented by the following matrix

$$G = \begin{array}{c} \\ C_1 \\ C_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \end{array}$$

and we assume that $a_3 > a_1$ and $a_2 > a_4$.

At time t, we denote as $x(t)$ the population share of strategy C_1. Therefore $1-x(t)$ is the share of strategy C_2, which implies $(x(t), 1-x(t))$ always belongs to the population state space $X = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1+x_2 = 1\}$. Let $P_i(t)$ be the fitness or payoff function of C_i, $i = 1, 2$ at the population state $(x(t), 1-x(t))$. Then we have

$$P_1(t) = a_1 x(t) + a_2\big(1 - x(t)\big),$$

$$P_2(t) = a_3 x(t) + a_4\big(1 - x(t)\big).$$

Definition 1 [11]**.** *A population state $x^\star \in X$ is a evolutionarily stable state if*
(1) *$x^\star$ is a Nash equilibrium, that is, $\langle y - x^\star, P(x^\star)\rangle \le 0$, $\forall y \in X$,*
(2) *there is a neighborhood $U_{x^\star}$ of $x^\star$ satisfying*

$$\langle y - x^\star, F(y)\rangle < 0,\ \forall y \in U_{x^\star}.$$

Let $\alpha_1 = a_3 - a_1$, $\alpha_2 = a_2 - a_4$ and $\alpha = \alpha_1 + \alpha_2$. We can easily see that $x^\star = \alpha_2/\alpha$ is the unique ESS in the above population game G.

During a small time interval σ, assume that a σ fraction of the population participates in pairwise competitions, which is to say, they are matched randomly to play the population game G. Assume that $h_i(t)$, $i = 1, 2$, denote the the number of participants at time t using strategies C_1 and C_2 respectively. Thus $h(t) = h_1(t) + h_2(t)$ is the total number of individuals and $x(t) = \frac{h_1(t)}{h(t)}$ is the population share of strategy C_1 at time t. Accordingly, we have

$$h_i(t + \sigma) = (1 - \sigma)h_i(t) + \sigma h_i(t)P_i(t), \quad i = 1, 2. \tag{1}$$

It is not restrictive to assume that the payoff functions are non-negative, which makes sure that h_1 and h_2 are always non-negative.

The total amount of participants at time $t + \sigma$ admits

$$\begin{aligned} h(t+\sigma) &= h_1(t+\sigma) + h_2(t+\sigma) \\ &= (1-\sigma)h(t) + \sigma\big(h_1(t)P_1(t) + h_2(t)P_2(t)\big) \\ &= (1-\sigma)h(t) + \sigma h(t)\big(x(t)P_1(t) + (1-x(t))P_2(t)\big) \\ &= (1-\sigma)h(t) + \sigma h(t)\bar{P}(t), \end{aligned} \tag{2}$$

where $\bar{P}(t) = x(t)P_1(t) + (1 - x(t))P_2(t)$ represents the expected payoff at time t. Let $i = 1$ and divide (1) by (2), we have

$$\begin{aligned} x(t+\sigma) &= \frac{h_1(t+\sigma)}{h(t+\sigma)} \\ &= \frac{(1-\sigma)h_1(t) + \sigma h_1(t)P_1(t)}{(1-\sigma)h(t) + \sigma h(t)\bar{P}(t)} \\ &= \frac{(1-\sigma)x(t) + \sigma x(t)P_1(t)}{1-\sigma+\sigma\bar{P}(t)} \\ &= \frac{\big(1-\sigma+\sigma\bar{P}(t)\big)x(t) + \sigma x(t)\big(P_1(t) - \bar{P}(t)\big)}{1-\sigma+\sigma\bar{P}(t)} \\ &= x(t) + \frac{\sigma x(t)\big(P_1(t) - \bar{P}(t)\big)}{1-\sigma+\sigma\bar{P}(t)}. \end{aligned} \tag{3}$$

Divide Eq. (3) by σ and let $\sigma \to 0$, then we obtain the traditional replicator dynamics proposed by Taylor and Jonker [3] and named by Schuster and Sigmund [17]

$$\frac{dx(t)}{dt} = x(t)\big(P_1(t)) - \bar{P}(t)\big) \tag{4}$$

The remaining part of the paper is organized as follows. In Sect. 2, we propose three modified models of Tao [13] considering the uncertainties and show the stability of the ESS under some constraints. In Sect. 3, we illustrate some simple numerical examples to verify our results. Discussion and conclusion follow in Sect. 4.

2 Models of Payoff Uncertainties

On one hand, in reality, it is impossible for anyone to react to the system immediately, which means there should be a time delay. For instance, a corn field needs some time to become maturity after sowing. Thus time delay should not be ignored. Therefore it is reasonable to modify model (4) to the time-delayed replicator population dynamics. On the other hand, we are living in an uncertain world. Various uncertainties really exist, which make our life colourful. For instance, the number of the offspring is not fixed and the harvest of crops is also unexpected when genes, temperature, climate, geographical position et al. are considered. Therefore in a population game model, the players' uncertainties should take into consideration and the payoffs may be disturbed.

We first assume that at time t individuals replicate according to the payoff obtained by their strategies at time $t - \nu$, $\nu > 0$, that is time delay exists. For example, at time t, only when the child animals born at time $t-\nu$ become mature after some time, they can take part in the competitions. Second, we suppose that the payoff functions are uncertain, which means there exist perturbations. We modify Eq. (1) to the following equations

$$h_i(t+\epsilon) = (1-\epsilon)h_i(t) + \epsilon h_i(t)\big(P_i(t-\nu) + d_i(t-\nu)\big), \quad i = 1, 2, \tag{5}$$

where $d_i(t)$ represents the perturbation of the payoff function $P_i(t)$.

Similar to the procedure in Sect. 1, we have

$$\begin{aligned} h(t+\sigma) &= h_1(t+\sigma) + h_2(t+\sigma) \\ &= (1-\sigma)h(t) + \sigma\big[h_1(t)\big(P_1(t-\nu)+d_1(t-\nu)\big) + h_2(t)\big(P_2(t-\nu)+d_2(t-\nu)\big)\big] \\ &= (1-\sigma)h(t) + \sigma h(t)\big[x(t)\big(P_1(t-\nu)+d_1(t-\nu)\big) + \big(1-x(t)\big)\big(P_2(t-\nu)+d_2(t-\nu)\big)\big] \\ &= (1-\sigma)h(t) + \sigma h(t)\bar{P}(t-\nu), \end{aligned} \tag{6}$$

where $\bar{P}(t-\nu) = x(t)\big(P_1(t-\nu)+d_1(t-\nu)\big) + \big(1-x(t)\big)\big(P_2(t-\nu)+d_2(t-\nu)\big)$. We divide (5) by (6) as $i=1$, thus we have

$$\begin{aligned} x(t+\sigma) &= \frac{h_1(t+\sigma)}{h(t+\sigma)} \\ &= \frac{(1-\sigma)h_1(t) + \sigma h_1(t)\big(P_1(t-\nu)+d_1(t-\nu)\big)}{(1-\sigma)h(t)+\sigma h(t)\bar{P}(t-\nu)} \\ &= \frac{(1-\sigma)x(t) + \sigma x(t)\big(P_1(t-\nu)+d_1(t-\nu)\big)}{1-\sigma+\sigma\bar{P}(t-\nu)} \\ &= \frac{(1-\sigma+\sigma\bar{P}(t-\nu))x(t) + \sigma x(t)\big[P_1(t-\nu)+h_1(t-\nu)-\bar{P}(t-\nu)\big]}{1-\sigma+\sigma\bar{P}(t)} \\ &= x(t) + \frac{\sigma x(t)\big[P_1(t-\nu)+d_1(t-\nu)-\bar{P}(t-\nu)\big]}{1-\sigma+\sigma\bar{P}(t-\nu)}. \end{aligned} \tag{7}$$

Simplifying Eq. (7), we obtain the perturbed time-delayed replicator population dynamics

$$\frac{dx(t)}{dt} = x(t)\big[P_1(t-\nu)+d_1(t-\nu)-\bar{P}(t-\nu)\big] \tag{8}$$

$$= x(t)(1-x(t))\big[P_1(t-\nu)+d_1(t-\nu)-P_2(t-\nu)-d_2(t-\nu)\big]. \tag{9}$$

2.1 Uniform Uncertainty

From now on, we assume that $d_1(t-\nu) - d_2(t-\nu) = \theta\big(P_1(t-\nu) - P_2(t-\nu)\big)$, where θ is a random number satisfying $0 \le \underline{\theta} \le \theta \le \bar{\theta}$. We require that the difference between $d_1(t-\nu)$ and $d_2(t-\nu)$ proportional to that between $P_1(t-\nu)$ and $P_2(t-\nu)$ by a uncertain parameter θ. It is common in many applications. Under this assumption, we deduce the Eq. (9) to the following equation

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(1-x(t))\big[P_1(t-\nu)+d_1(t-\nu)-P_2(t-\nu)-d_2(t-\nu)\big] \\ &= x(t)(1-x(t))\big[P_1(t-\nu)-P_2(t-\nu)+\theta\big(P_1(t-\nu)-P_2(t-\nu)\big)\big] \\ &= (1+\theta)x(t)(1-x(t))\big[P_1(t-\nu)-P_2(t-\nu)\big] \\ &= (1+\theta)x(t)(1-x(t))\big[a_1x(t-\nu)+a_2(1-x(t-\nu)) - \big(a_3x(t-\nu)+a_4(1-x(t-\nu))\big)\big] \\ &= -\alpha(1+\theta)x(t)(1-x(t))(x(t-\nu)-x^\star), \end{aligned} \tag{10}$$

where $\alpha_1 = a_3 - a_1$, $\alpha_2 = a_2 - a_4$, $\alpha = \alpha_1 + \alpha_2$ and $x^\star = \alpha_2/\alpha$.

Definition 2. *For the system (10), the stationary point $x^\star$ is uniformly robust stable with respect to θ if it is asymptotically stable for all $\theta \in [\underline{\theta}\ , \bar{\theta}]$.*

From system (10), we know the unique ESS $x^\star$ is a stationary point of the perturbed replicator population dynamics with time delay. To examine the uniformly robust asymptotically stability of $x^\star$, we transform the rest point $x^\star$ of system (10) into a trivial zero and linearize it. We resort to the variable transformation $y(t) = x(t) - x^\star$ and linearize system (10), we obtain the following linearized system

$$\frac{dy(t)}{dt} = -(1+\theta)\alpha x^\star(1-x^\star)y(t-\nu). \tag{11}$$

It is well known from [18] that the system (10) is stable if and only if all the roots of its characteristic equation have negative real part. We assume that $y(t) = ce^{\lambda}t$ is a solution of the Eq. (11), where c is a real number. Thus we derive the characteristic equation of (11)

$$\lambda = -(1+\theta)\beta e^{-\lambda\nu}, \tag{12}$$

where $\beta = \alpha x^\star(1-x^\star)$.

Theorem 1. *The unique ESS $x^\star$ of the perturbed time-delayed replicator population dynamics (10) is uniformly robust stable if* $0 < \nu < \frac{\pi}{2(1+\bar{\theta})\beta}$.

Proof. Assume that $\nu < \frac{\pi}{2(1+\bar{\theta})\beta}$, we show that each root of the characteristic Eq. (12) has negative real part for any uncertain $\theta \in [\underline{\theta}\ , \bar{\theta}]$.

Suppose that there is a root of (12) $\lambda = \zeta + i\mu$ with ζ, $\mu \in \mathbb{R}$, $i^2 = -1$ and $\zeta \geq 0$. We notice that $\lambda = 0$ is obviously not a root of (12). Furthermore, we assume that $\mu > 0$ since if $\lambda = \zeta + i\mu$ is a solution of (12), so is $\lambda = \zeta - i\mu$.

By separating the real and imaginary parts in Eq. (12), we have

$$\begin{aligned} \zeta &= -(1+\theta)\beta e^{-\zeta\nu}\cos\mu\nu, \\ \mu &= (1+\theta)\beta e^{-\zeta\nu}\sin\mu\nu. \end{aligned}$$

Since $\mu > 0$, this implies

$$\begin{aligned} 0 < \mu\nu &= (1+\theta)\nu\beta e^{-\zeta\nu}\sin\mu\nu \\ &\leq (1+\theta)\nu\beta \\ &\leq (1+\bar{\theta})\nu\beta \\ &< (1+\bar{\theta}) \cdot \frac{\pi}{2(1+\bar{\theta})\beta} \cdot \beta \\ &= \frac{\pi}{2}. \end{aligned}$$

The second inequality holds since $\zeta > 0$ and $\sin\mu\nu \leq 1$, the third and forth inequalities hold due to the assumption that $\theta \in [\underline{\theta}\ , \bar{\theta}]$ and $\nu < \frac{\pi}{2(1+\bar{\theta})\beta}$. Accordingly, we obtain

$$\zeta = -(1+\theta)\beta e^{-\zeta\nu}\cos\mu\nu < 0,$$

since $\beta = \alpha x^\star(1 - x^\star) > 0$. This contradicts with our assumption $\zeta \geq 0$, which prove the roots of the characteristic Eq. (12) have negative real parts.

According to Theorem 1, the stability of the perturbed time-delayed replicator population dynamics relies on the parameter θ. If θ equals to 0, we derive a special case discussed in [13], which means that we develop their consequence.

Corollary 1. *For the time-delayed replicator population dynamics of Tao [13]*

$$\frac{dx(t)}{dt} = x(t)\big(F_1(t-\nu)) - \bar{F}(t-\nu)\big),$$

the mixed ESS $x^\star$ is asymptotically stable if the delay $\nu < \frac{\alpha\pi}{2\alpha_1\alpha_2}$.

2.2 Uniform Uncertainty with Two Delays

In this section, we further study the perturbed time-delayed replicator dynamics discussed in the previous subsection. We assume that there are two time delays for each of the two strategies C_1 and C_2. Moreover, for strategy C_1, if players consider no delay with probability p_1, a time delay ν_1 with probability p_2, and a time delay ν_2 with probability p_3, where $p_1 + p_2 + p_3 = 1$, then the expected payoff of players who select strategy C_1 becomes

$$\begin{aligned} P_1(t) &= a_1[p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2)] \\ &\quad + a_2[p_1(1-x(t)) + p_2(1-x(t-\nu_1)) + p_3(1-x(t-\nu_2))] \\ &= a_1[p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2)] \\ &\quad + a_2[1 - p_1x(t) - p_2x(t-\nu_1) - p_3x(t-\nu_2)]. \end{aligned}$$

In the same way, we have

$$\begin{aligned} P_2(t) &= a_3[p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2)] \\ &\quad + a_4[p_1(1-x(t)) + p_2(1-x(t-\nu_1)) + p_3(1-x(t-\nu_2))] \\ &= a_3[p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2)] \\ &\quad + a_4[1 - p_1x(t) - p_2x(t-\nu_1) - p_3x(t-\nu_2)]. \end{aligned}$$

Accordingly, under the assumption of the uniform uncertainty, the perturbed replicator dynamics with two time delays is expressed as

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(1-x(t))\big[P_1(t) + d_1(t) - P_2(t) - d_2(t)\big] \\ &= x(t)(1-x(t))\big[P_1(t) - P_2(t) + \theta\big(P_1(t) - P_2(t)\big)\big] \\ &= (1+\theta)x(t)(1-x(t))\big(P_1(t) - P_2(t)\big) \\ &= (1+\theta)x(t)(1-x(t))\big[(a_1 - a_3 + a_4 + a_2)\big(p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2)\big) + a_2 - a_4\big] \\ &= -\alpha(1+\theta)x(t)(1-x(t))\big(p_1x(t) + p_2x(t-\nu_1) + p_3x(t-\nu_2) - x^\star\big), \end{aligned} \tag{13}$$

where $\alpha_1 = a_3 - a_1$, $\alpha_2 = a_2 - a_4$, $\alpha = \alpha_1 + \alpha_2$, $x^\star = \alpha_2/\alpha$ and θ is the uncertain parameter as defined in the previous subsection.

Similar to the procedure in the previous subsection, we can derive the linearized equation of (13) around $x^\star$ as follows

$$\frac{dy(t)}{dt} = -\alpha(1+\theta)x^\star(1-x^\star)\big(p_1 y(t) + p_2 y(t-\nu_1) + p_3 y(t-\nu_2)\big). \tag{14}$$

To investigate the stability of the unique ESS $x^\star$, we resort to the following lemma in [19].

Lemma 1. *Let $\dot{y} = -a_1 y(t) - a_2 y(t-h_1) - a_3 y(t-h_2)$, where $h_1 > 0$, $h_2 > 0$. Suppose at least one of the following conditions holds*

(1) $a_1 > 0$, $|a_2| + |a_3| < a_1$,

(2) $0 < a_1 + a_2 + a_3$, $|a_2|h_1 + |a_3|h_2 < \dfrac{a_1 + a_2 + a_3}{|a_1| + |a_2| + |a_3|}$,

(3) $0 < a_1 + a_2$, $|a_2|h_1 < \dfrac{a_1 + a_2 - |a_3|}{|a_1| + |a_2| + |a_3|}$,

(4) $0 < a_1 + a_3$, $|a_3|h_2 < \dfrac{a_1 + a_3 - |a_2|}{|a_1| + |a_2| + |a_3|}$,

then the above equation is asymptotically stable.

From this lemma, we get the corresponding result for the Eq. (14).

Theorem 2. *If at least one of the following conditions are satisfied*

(1) $p_2 + p_3 < p_1$,

(2) $p_2\nu_1 + p_3\nu_2 < \dfrac{1}{\alpha(1+\theta)x^\star(1-x^\star)}$,

(3) $p_2\nu_1 < \dfrac{p_1 + p_2 - p_3}{\alpha(1+\theta)x^\star(1-x^\star)}$,

(4) $p_3\nu_2 < \dfrac{p_1 + p_3 - p_2}{\alpha(1+\theta)x^\star(1-x^\star)}$,

then the unique ESS $x^\star$ is asymptotically stable for the perturbed replicator dynamics with two time delays (13).

2.3 Exponential Uncertainty

In this part, we continue to investigate the perturbed time-delayed replicator population dynamics of form (9). We now assume that at time $t-\nu$, the difference between the two perturbations $d_1(t-\nu)$ and $d_2(t-\nu)$ satisfies

$$d_1(t-\nu) - d_2(t-\nu) = \int_\nu^\infty \xi e^{-\xi\phi}\big(P_1(t-\phi) - P_2(t-\phi)\big)d\phi,$$

where ξ is called the rate parameter. Now the difference between the perturbations is relevant to the differences between the payoffs at every moment. Moreover, $e^{-\xi\phi}$ indicates the effect on the perturbation. Accordingly, the perturbed time-delayed replicator population dynamics (9) becomes

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)(1-x(t))\big[P_1(t-\nu)+d_1(t-\nu)-P_2(t-\nu)-d_2(t-\nu)\big]\\ &= x(t)(1-x(t))\big[P_1(t-\nu)-P_2(t-\nu)+\int_\nu^\infty \xi e^{-\xi\phi}\big(P_1(t-\phi)-P_2(t-\phi)\big)d\phi\big]\\ &= -\alpha x(t)(1-x(t))\big[x(t-\nu)-x^\star+\int_\nu^\infty \xi e^{-\xi\phi}\big(x(t-\phi)-x^\star\big)d\phi\big]. \end{aligned} \tag{15}$$

Compared with the normal time delay replicator dynamics in Tao[13], the system (15) is perturbed in a more complicated but realistic way since we consider every moment before time $t-\nu$. At the same time, we see that the unique ESS $x^\star$ is a also a stationary point of the exponentially perturbed time-delayed replicator population dynamics (15).

To investigate the stability of $x^\star$, we transform the stationary point $x^\star$ of the system (15) by the variable transformation $y(t)=x(t)-x^\star$ and linearize it, we obtain

$$\begin{aligned}\frac{dy(t)}{dt} &= -\alpha x^\star(1-x^\star)\big(y(t-\nu)+\int_\nu^\infty \xi e^{-\xi\phi}y(t-\phi)d\phi\big)\\ &= -\beta\big(y(t-\nu)+\int_\nu^\infty \xi e^{-\xi\phi}y(t-\phi)d\phi\big),\end{aligned} \tag{16}$$

where $\beta=\alpha x^\star(1-x^\star)$. Same as the procedure in the last subsection, we easily derive the characteristic equation of (16) as follows

$$\lambda = -\beta\big(e^{-\lambda\nu}+\xi\int_\nu^\infty e^{-(\xi+\lambda)\phi}d\phi\big). \tag{17}$$

The next theorem shows that the ESS $x^\star$ is still asymptotically stable under perturbation for any rate parameter, which implies $x^\star$ is nicely perfect in some sense.

Theorem 3. *The evolutionary stable strategy $x^\star$ is asymptotically stable under the exponentially perturbed time-delayed replicator population dynamics for any $\xi>0$ if $0<\nu<\frac{\pi}{4\beta}$.*

Proof. Suppose that $\nu<\frac{\pi}{4\beta}$, we prove none of the roots for the characteristic Eq. (17) have nonnegative real part for any $\xi>0$.

Let $\lambda=\zeta+i\mu$ be a root of (17) with $\zeta\geq 0$. Without loss of generality, we assume that $\mu>0$. Substituting $\lambda=\zeta+i\mu$ into (17) and separating the real and imaginary parts, we get

$$\begin{aligned}\zeta &= -\beta e^{-\zeta\nu}\cos\mu\nu-\beta\xi\int_\nu^\infty e^{-(\xi+\zeta)\phi}\cos\mu\phi d\phi,\\ \mu &= \beta e^{-\zeta\nu}\sin\mu\nu+\beta\xi\int_\nu^\infty e^{-(\xi+\zeta)\phi}\sin\mu\phi d\phi.\end{aligned}$$

Consequently, we have

$$\begin{aligned} 0 < \mu\nu &= \nu\beta e^{-\zeta\nu}\sin\mu\nu + \nu\beta\xi\int_{\nu}^{\infty} e^{-(\xi+\zeta)\phi}\sin\mu\phi d\phi \\ &\le \nu\beta + \nu\beta\xi\int_{\nu}^{\infty} e^{-(\xi+\zeta)\phi}d\phi \\ &= \nu\beta + \nu\beta\frac{\xi}{\xi+\zeta}e^{-(\xi+\zeta)\nu} \\ &< 2\nu\beta \\ &< 2\cdot\frac{\pi}{4\beta}\cdot\beta \\ &= \frac{\pi}{2}. \end{aligned}$$

Furthermore, since $0 < \mu\nu < \frac{\pi}{2}$, we have

$$\begin{aligned} 0 \le \zeta &= -\beta e^{-\zeta\nu}\cos\mu\nu - \beta\xi\int_{\nu}^{\infty} e^{-(\xi+\zeta)\phi}\cos\mu\phi d\phi \\ &< -\beta + \beta\xi\int_{\nu}^{\infty} e^{-(\xi+\zeta)\phi}d\phi \\ &= -\beta\big(1 - \frac{\xi}{\xi+\zeta}e^{-(\xi+\zeta)\nu}\big) \\ &< 0. \end{aligned}$$

The second inequality holds since $\cos\mu\phi \ge -1$.

This contradiction prove that each of the roots of the characteristic Eq. (17) has negative real part, which implies the ESS $x^{\star}$ is also asymptotically stable even in the exponentially perturbed time-delayed replicator population dynamics.

3 Numerical Examples

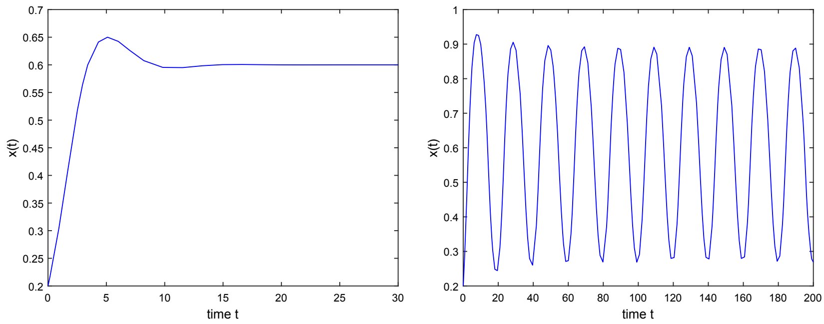
In hawk and dove games, two individual compete for a resource. They share a same strategy set with two strategies, namely, Hawk (H) and Dove (D). The individuals choosing (H) strategy always want to monopolize the resource and never share. The strategy (D) represents a gentle attitude on the resource and never choose to fight. The payoff matrix is given as follows

$$G = \begin{array}{c} \\ H \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} \;\;H & \quad D \end{array} \\ \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix} \end{array},$$

where C and V are positive. V represents the resource value and C stands for the cost of fight. The cost is usually assumed to be high if agents fight, therefore we assume that $V < C$.

3.1 Uniform Uncertainty

In the case of interval perturbation model of Eq. (10), the ESS $x^\star$ is uniformly robust asymptotically if the time delay is small than $\frac{\pi}{2(1+\bar{\theta})\beta}$, according to Theorem 1. In the context of Hawk and Dove games, this value is given as $\frac{C\pi}{(1+\bar{\theta}V(V-C)}$. We depict in Fig. 1(a) the trajectory of numerical solution of the perturbed time-delayed replicator population dynamics with parameter $C = 5$, $V = 3$, the perturbation upper bound $\bar{\theta} = 0.5$ and $\nu_1 = 1.7$. We observe the convergence to the ESS. In Fig. 1(b), as we increase the value of ν to 5, the system obviously becomes to oscillate.



(a) (b)

Fig. 1. (a) The numerical solution of the interval perturbed system, where $V = 3$, $C = 5$, $\bar{\theta} = 0.5$ and $\nu_1 = 1.7$. (b) The oscillation of the solution of the interval perturbed system, where $\nu_2 = 5$.

3.2 Uniform Uncertainty with Two Delays

Under the perturbed time-delayed replicator dynamics with two time delays, we derived that the unique ESS $x^\star$ is asymptotically stable as long as the probability distribution satisfies $p_2 + p_3 < p_1$, no matter how large the delays are. We see the truth from Fig. 2(a). In addition, if $p_2 + p_3 > p_1$, the solution always oscillates even if the time delays are small. From Fig. 3, we know that values of perturbation parameter θ affect only the rate of convergence of $x^\star$.

3.3 Exponential Uncertainty

When we consider the exponentially perturbed time-delayed replicator population dynamics, we proved in Theorem 3 the stability of the ESS $x^\star$ is not related to rate parameter ξ. In fact, the parameter ξ only affects the convergent rate.

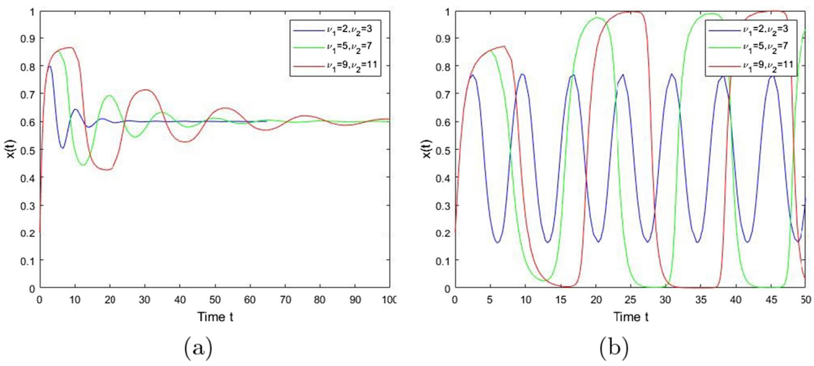


Fig. 2. (a) The numerical solution of the uniformly perturbed system with two time delays, where $V = 3$, $C = 5$, $p_1 = 0.6$, $p_2 = 0.1$ and $p_3 = 0.3$. (b) The oscillation of solutions of the uniformly perturbed systems with two time delays, where $p_1 = 0.2$, $p_2 = 0.5$ and $p_3 = 0.3$.

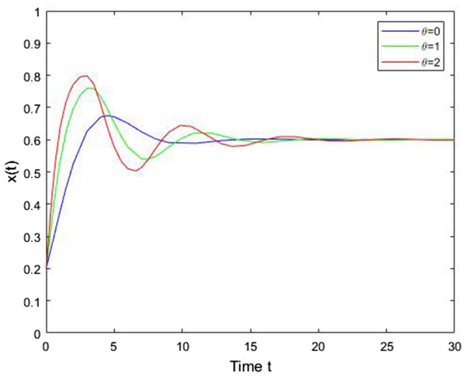
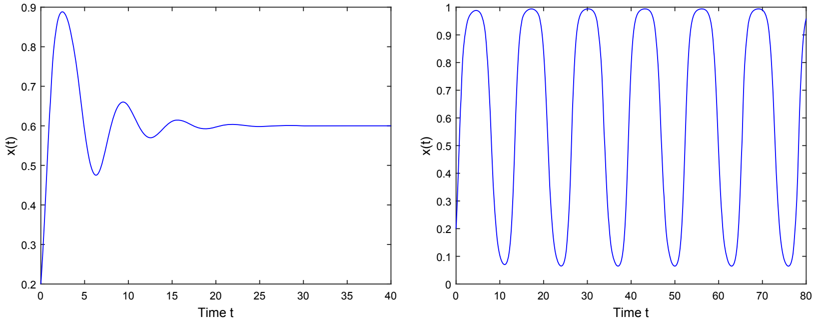


Fig. 3. The convergent rate of the numerical solutions for different perturbation parameter θ, where $V = 3$, $C = 5$.

Now The value of V and C are the same as that in the last subsection and we set $\nu = 1$ and $\xi = 0.4$. We see from Fig. 4, the ESS $x^{\star} = 1.7$ is asymptotically stable and becomes unstable when $\nu = 3.8$. From Fig. 5, we conclude that the rate parameter ξ only has effects on the convergent rate.



(a) (b)

Fig. 4. (a) The numerical solution of the exponentially perturbed system, where $V = 3$, $C = 5$, $\xi = 0.4$ and $\nu_1 = 1.7$. (b) The oscillation of the solution of the exponentially perturbed system, where $\nu_2 = 3.8$.

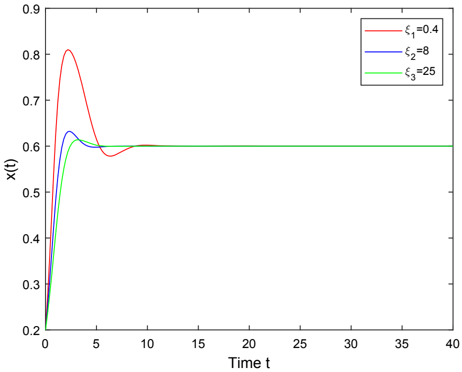


Fig. 5. The convergent rate of the numerical solutions for different values of ξ, where $V = 3$, $C = 5$.

4 Conclusion

In this paper, considering the uncertainty, we developed the traditional population replicator dynamics with time delay to time-delayed population dynamics with uncertainties. The first model treated the perturbation parameter as uncertain in a interval and we derived the uniformly perturbed replicator dynamics. The second model further investigated the uniformly perturbed replicator dynamics considering two time delays exist. The last model discussed the parameter as a stochastic variable with exponential distribution. We showed that under some constraints, the $x^{\star}$ of the traditional replicator dynamics is always asymptotically stable in our three models and the result in [13] is a special case of our result. We verified our results with some simple examples.

As a future work, we plan to investigate the population games with more than two strategies and more than one population, which means we will derive equations other than a single equation. We also plan to study the problem Hopf bifurcation.

References

1. Aumann, R.J.: Rationality and bounded rationality. In: Hart, S., Mas-Colell, A. (eds.) Cooperation: Game-Theoretic Approaches. NATO ASI Series (Series F: Computer and Systems Sciences), vol. 155, pp. 219–231. Springer, Heidelberg (1997). https://doi.org/10.1007/978-3-642-60454-6_15
2. Maynard, J.: The theory of games and the evolution of animal conflicts. J. Theor. Biol. **47**(1), 209–221 (1974)
3. Taylor, P., Jonker, L.: Evolutionary stable strategies and game dynamics. Math. Biosci. **40**(1), 145–156 (1978)
4. Gutierrez, S.M., Adeli, H.: Many-objective control optimization of high-rise building structures using replicator dynamics and neural dynamics model. Struct. Multidiscip. Optim. **56**(6), 1521–1537 (2017). https://doi.org/10.1007/s00158-017-1835-9
5. Wang, Q., He, N.R., Chen, X.J.: Replicator dynamics for public goods game with resource allocation in large populations. Appl. Math. Comput. **328**, 162–170 (2018)
6. Argasinski, K., Broom, M.: Evolutionary stability under limited population growth: eco-evolutionary feedbacks and replicator dynamics. Ecol. Complex. **34**, 198–212 (2018)
7. Requejo, R.J., Díaz-Guilera, A.: Replicator dynamics with diffusion on multiplex networks. Phys. Rev. E **94**(2), 022301 (2018). Article ID: 022301
8. Tan, S., Wang, Y.: Graphical Nash equilibria and replicator dynamics on complex networks. IEEE Trans. Neural Netw. Learn. Syst. **31**(6), 1831–1842 (2019). https://doi.org/10.1109/TNNLS.2019.2927233
9. Ramazi, P., Cao, M.: Global convergence for replicator dynamics of repeated snowdrift games. IEEE Trans. Autom. Control (2020). https://doi.org/10.1109/TAC.2020.2975811
10. Weibull, J.: Evolutionary Game Theory, 2nd edn. MIT Press, Cambridge (1994)
11. Sandholm, J.: Population Games and Evolutionary Dynamics. MIT Press, Cambridge (2010). Journal of Theoretical Biology
12. Newton, J.: Evolutionary game theory: a renaissance. Games **9**, 31 (2018)
13. Tao, Y., Wang, Z.: Effect of time delay and evolutionarily stable strategy. J. Theor. Biol. **187**(1), 111–116 (1997)
14. Alboszta, J., Miękisz, J.: Stability and evolutionary stable strategies in discrete replicator dynamics with delay. J. Theor. Biol. **231**(2), 175–179 (2004)
15. Ben Khalifa, N., El-Azouzi, R., Hayel, Y., et al.: Evolutionary games in interacting communities. Dyn. Games Appl. **7**(2), 131–156 (2017)
16. Zhong, C., Yang, H., Liu, Z., et al.: Stability of replicator dynamics with bounded continuously distributed time delay. Mathematics **8**(3), 431 (2020)
17. Schuster, P., Sigmund, K.: Replicator dynamics. J. Theor. Biol. **100**(3), 533–538 (1983)
18. Gopalsamy, K.: Stability and oscillations in dealy differential equations of population dynamics. Kluwer Academic Publishers, Kluwer, Dordrecht, The Netherlands (1992)
19. Berezansky, L., Braverman, E.: On stability of some linear and nonlinear delay differential equations. J. Math. Anal. Appl. **314**(2), 391–411 (2006)