An SIR Epidemic Model with Birth Pulse and Pulse Vaccination on the Newborn

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Abstract. Pulse vaccination is an important strategy to eradicate an infectious disease. In this paper, we investigate an SIR epidemic model with birth pulse and pulse vaccination on the newborn. By using the discrete dynamical system determined by stroboscopic map, we obtain the condition for the global asymptotical stability of the disease-free periodic solution of the studied system. The permanent condition of the investigated system is also given. Numerical simulation is employed to illustrate our results. The result indicates that pulse vaccination rate on the newborn plays an important role in eradicating the disease. It provides a reliable tactic basis for preventing the disease from spreading.

Keywords: SIR epidemic model · Birth pulse · Global asymptotical stability · Permanence · Pulse vaccination

1 Introduction

Since last century, there has been a great deal of work in the mathematical theory of epidemics; for example, we can refer to the books, [1–4]. SIR (susceptible, infective, recovered) model is suitable for describing the transmission of infectious diseases with life long immunity, which is one of the most important epidemic models in epidemiology. In 1927, a classical SIR model was initially presented by Kermack and Mckendrick [5]. After that, lots of continuous SIR models with various transmission rates have been purposed, which have been investigated extensively and many threshold conditions have been obtained [6,7]. However, these models do not consider pulse vaccination, neither do they contain birth pulse, which is the novelty of our model in this present paper.

Supported by National Natural Science Foundation of China (11761019), Guizhou Science and Technology Platform Talents ([2017] 5736-019), Science and Technology Foundation of Guizhou Province ([2020]1Y001), and Guizhou Education Department Youth Science and Technology Talents Growth Project (Guizhou Jiaohe KY[2018]157), Guizhou Team of Scientific and Technological Innovation Talents (No. 20175658), Guizhou University of Finance and Economics Project Funding (No. 2019XYB11), Guizhou University of Finance and Economics Introduced Talent Research Project (2017).

D. Jiang and H. Song (Eds.): SIMUtools 2021, LNICST 424, pp. 555–563, 2022.
https://doi.org/10.1007/978-3-030-97124-3_43

Recently, pulse vaccination strategy, a new vaccination strategy against measles, has been proposed. Its theoretical study was started by Agur *et al.* in [8]. As far as pulse vaccination strategy are concerned, a lot of original work has been done in [9–13].

In the real world, individual members of many species experience two stages of life, immature and mature ones. Stage-structured population models have attracted great attention, and many stage-structured models have been studied in recent years [14–16].

Theories of impulsive differential equations have been introduced into population dynamics lately. Impulsive equations are found in almost every domain of applied science and have been studied in many investigations [17,18]. They generally describe phenomena which are subject to steep or instantaneous changes.

Motivated by the above studies, our study is to investigate transmission dynamics of an SIR epidemic model with birth pulse and pulse vaccination. We assume full immunity of recovered individuals; that is to say, those individuals are no longer susceptible after they have recovered.

The present paper is to introduce birth pulse of the population and pulse vaccination into SIR epidemic model, and obtain some important qualitative properties for the investigated system. As a matter of fact, pulse birth and pulse vaccination on the newborn are used in an epidemic model. To the best of our knowledge, few research has been conducted.

2 The Model

In this paper, we consider an SIR epidemic model with birth pulse and pulse vaccination on the newborn:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t) - \beta S_1(t)I(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t), \\ \frac{dI(t)}{dt} &= \beta S_1(t)I(t) - (r+d_3)I(t), \\ \frac{dR(t)}{dt} &= rI(t) - d_4R(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} \Delta S_1(t) &= S_2(t)(a - bS_2(t)), \\ \Delta S_2(t) &= 0, \\ \Delta I(t) &= 0, \\ \Delta R(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_1(t) &= -\mu S_1(t), \\ \Delta S_2(t) &= 0, \\ \Delta I(t) &= 0, \\ \Delta R(t) &= \mu S_1(t), \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots, \end{cases} \tag{1}$$

where $S_1(t), S_2(t)$ represent the numbers of the immature and the mature of the susceptible. $I(t), R(t)$ represent the numbers of the infectious, and the recovered, respectively. c is called the rate of the immature susceptible turning into the mature susceptible. d_1, d_2, d_3, d_4, respectively denote the natural death rate of the immature susceptible, the mature susceptible, the infectious and the recovered. β is the average number of adequate contacts of an immature infectious individual per unit time. r stands for the recovery rate of the immature infectious individual. The mature susceptible is birth pulse with intrinsic rate of natural increase and density dependence rate of the mature susceptible denoted by a, b, respectively. The pulse birth and pulse vaccination occurs every τ period (τ is a positive constant). $\Delta S_1(t) = S_1(t^+) - S_1(t)$. $\mu(0 < \mu < 1)$ is the proportion of the successful vaccination which is called pulse vaccination rate, at $t = (n+l)\tau$, $0 < l < 1$, $n \in Z_+$. $S_2(t)(a - bS_2(t))$ represents the birth effort of the mature susceptible at $t = n\tau, n \in Z_+$.

In this work, we assume:

(i) The infection is not fully susceptible; that is to say, the disease is spread by the immature individual, the recovery from the disease will confer long lasting immunity.
(ii) The mature susceptible is immune to the disease; that is to say, the mature susceptible achieves lifetime immunity.

Since the first, second, and third equations do not include $R(t)$, we can simplify system (1) as follows:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t) - \beta S_1(t)I(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t), \\ \frac{dI(t)}{dt} &= \beta S_1(t)I(t) - (r+d_3)I(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} \Delta S_1(t) &= S_2(t)(a - bS_2(t)), \\ \Delta S_2(t) &= 0, \\ \Delta I(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_1(t) &= -\mu S_1(t), \\ \Delta S_2(t) &= 0, \\ \Delta I(t) &= 0, \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots. \end{cases} \tag{2}$$

This is equivalent to system (1).

3 Some Lemmas

Before discussing the main results, we will introduce some definitions, notations, and lemmas. Denote by $f = (f_1, f_2, f_3, f_4)^T$ the map defined by

the right-hand side of system (1), the solution of (1), denoted by $z(t) = (S_1(t), S_2(t), I(t), R(t))^T$, is a piecewise continuous function $z : R_+ \to R_+^4$, where $R_+ = [0, \infty)$, $R_+^4 = \{z \in R^4 : z > 0\}$. $z(t)$ is continuous on $(n\tau, (n + l)\tau] \times R_+^4$ and $((n + l)\tau, (n + 1)\tau] \times R_+^4$ $(n \in Z_+, 0 < l < 1)$. According to [17,18], the global existence and uniqueness of solutions of system (1) is guaranteed by the smoothness properties of f, the mapping defined by the right-hand side of system (1).

Let $V : R_+ \times R_+^4 \to R_+$. Then V is said to be belonged to class V_0 if:

(*i*) V is continuous in $(n\tau, (n + l)\tau] \times R_+^4$ and $((n + l)\tau, (n + 1)\tau] \times R^4$, for all $z \in R_+^4$, $n \in Z_+$, and $\lim_{(t,y)\to((n+l)\tau^+,z)} V(t, y) = V((n + l)\tau^+, z)$ and $\lim_{(t,y)\to((n+1)\tau^+,z)} V(t, y) = V((n + 1)\tau^+, z)$ exist.
(*ii*) V is locally lipschitzian in z.

Definition 3.1. If $V \in V_0$, then, for $(t, z) \in (n\tau, (n + l)\tau] \times R_+^4$ and $((n + l)\tau, (n + 1)\tau) \times R_+^4$, the upper right derivative of $V(t, z)$ with respect to the impulsive differential system (1) is defined as

$$D^+V(t, z) = \lim_{h\to 0} \sup \frac{1}{h}[V(t + h, z + hf(t, z)) - V(t, z)].$$

Lemma 3.2. (see [17], Theorem 1.4.1) Let the function $m \in PC'[R_+, R]$ satisfy the inequalities

$$\begin{cases} m'(t) \le p(t)m(t) + q(t),\ t \ne t_k,\ k = 1, 2, \ldots, \\ m(t_k^+) \le d_k m(t_k) + b_k,\ t = t_k,\ t \ge t_0, \end{cases} \tag{3}$$

where $p, q \in C[R_+, R]$ and $d_k \ge 0$ and b_k are constants. Then

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t p(s)ds\right)\right) b_k$$

$$+ \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)d\sigma\right) q(s)ds, \quad t \ge t_0.$$

Lemma 3.3. There exists a constant $M > 0$ such that $S_1(t) \le M$, $S_2(t) \le M$, $I(t) \le M$, $R(t) \le M$ for each solution $(S_1(t), S_2(t), I(t), R(t))$ of system (1) with t large enough.

We choose the following notation:

$$\mu^* = \frac{ace^{-d_2\tau}(1 - e^{-(c+d_1-d_2)\tau}) - (c + d_1 - d_2)(1 - e^{-d_2\tau})(1 - e^{-(c+d_1)\tau})}{ace^{-d_2\tau}(e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}) + (c + d_1 - d_2)(e^{-(c+d_1)\tau} - e^{-(c+d_1+d_2)\tau})}.$$

If $I(t) = 0$, then we have the following subsystem of (2):

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} \Delta S_1(t) &= S_2(t)(a - bS_2(t)), \\ \Delta S_2(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_1(t) &= -\mu S_1(t), \\ \Delta S_2(t) &= 0, \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots. \end{cases} \tag{4}$$

We easily obtain the analytic solution of system (4) between pulses as follows:

$$\begin{cases} S_1(t) = \begin{cases} S_1(n\tau^+)e^{-(c+d_1)(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ (1-\mu)S_1(n\tau^+)e^{-(c+d_1)(t-n\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ S_2(t) = \begin{cases} e^{-d_2(t-n\tau)}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right], \quad t \in (n\tau, (n+l)\tau], \\ \frac{ce^{-d_2(t-n\tau)}S_1(n\tau^+)}{c+d_1-d_2}\left[1-\mu e^{-(c+d_1-d_2)l\tau} - (1-\mu)e^{-(c+d_1-d_2)(t-n\tau)}\right] \\ \qquad +e^{-d_2(t-n\tau)}S_2(n\tau^+), t \in ((n+l)\tau, (n+1)\tau]. \end{cases} \end{cases} \tag{5}$$

Considering the fourth, fifth, seventh, and eighth equations of system (2), we have the stroboscopic map of (2)

$$\begin{cases} S_1((n+1)\tau^+) = \left[(1-\mu)e^{-(c+d_1)\tau} + \frac{ac\zeta}{c+d_1-d_2}\right]S_1(n\tau^+) + ae^{-d_2\tau}S_2(n\tau^+) \\ \qquad\qquad -b\left[\frac{c\zeta}{c+d_1-d_2}S_1(n\tau^+) + e^{-d_2\tau}S_2(n\tau^+)\right]^2, \\ S_2((n+1)\tau^+) = \frac{c\zeta}{c+d_1-d_2}S_1(n\tau^+) + e^{-d_2\tau}S_2(n\tau^+), \end{cases} \tag{6}$$

where $\zeta = e^{-d_2\tau}[(1-e^{-(c+d_1-d_2)l\tau}) + (1-\mu)e^{-(c+d_1-d_2)l\tau} - (1-\mu)e^{-(c+d_1-d_2)\tau}] > 0$. If we choose $A = (1-\mu)e^{-(c+d_1)\tau} + \frac{ac\zeta}{c+d_1-d_2} > 0$, $B = ae^{-d_2\tau} > 0$, $C = \frac{c\zeta}{c+d_1-d_2}$, $D = e^{-d_2\tau}$, $A < 1$, and $0 < D < 1$, the following two equivalence relations are found by calculation

$$\begin{aligned} \mu < \mu^* &\Leftrightarrow 1 - A - D + AD - BC < 0, \\ \mu > \mu^* &\Leftrightarrow 1 - A - D + AD - BC > 0. \end{aligned}$$

The two fixed points of (6) are obtained as $G_1(0,0)$ and $G_2(S_1^*, S_2^*)$, where

$$\begin{cases} S_1^* = \frac{(1-D-A+AD-BC)(-1+D)}{bC^2}, \mu < \mu^*, \\ S_2^* = \frac{-(1-D-A+AD-BC)}{bC}, \qquad \mu < \mu^*. \end{cases} \tag{7}$$

Lemma 3.4. (i) If $\mu > \mu^*$, then the fixed point $G_1(0,0)$ is globally asymptotically stable. (ii) If $\mu < \mu^*$, then the fixed point $G_2(S_1^*, S_2^*)$ is globally asymptotically stable (The proof can refer to [19]).

Lemma 3.5. (i) If $\mu > \mu^*$, then the trivial periodic solution $(0,0)$ of system (4) is globally asymptotically stable.

(ii) If $\mu < \mu^*$, then the periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)})$ of system (4) is globally asymptotically stable, where

$$\begin{cases} \widetilde{S_1(t)} = \begin{cases} S_1^* e^{-(c+d_1)(t-n\tau)}, & t \in (n\tau, (n+l)\tau], \\ (1-\mu) S_1^* e^{-(c+d_1)(t-n\tau)}, & t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ \widetilde{S_2(t)} = \begin{cases} e^{-d_2(t-n\tau)} \left[S_2^* + \dfrac{cS_1^*(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2} \right], & t \in (n\tau, (n+l)\tau], \\ \dfrac{ce^{-d_2(t-n\tau)} S_1^*}{c+d_1-d_2} \left[1-\mu e^{-(c+d_1-d_2)l\tau} - (1-\mu) e^{-(c+d_1-d_2)(t-n\tau)} \right] \\ \qquad + e^{-d_2(t-n\tau)} S_2^*, \quad t \in ((n+l)\tau, (n+1)\tau], \end{cases} \end{cases} \tag{8}$$

in which S_1^*, S_2^* are determined as in (7).

4 The Dynamics

In this section, for system (2) there obviously exists a disease-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$. First, we prove that the disease-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (2) is globally asymptotically stable. After that, we prove that system (2) is permanent.

Theorem 4.1. If

$$\mu < \mu^*$$

and

$$\mu > \frac{1-e^{-(c+d_1)\tau}}{e^{-(c+d_1)l\tau} - e^{-(c+d_1)\tau}} - \frac{(c+d_1)(r+d_3)\tau}{\beta S_1^* (e^{-(c+d_1)l\tau} - e^{(c+d_1)\tau})},$$

then the disease-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (2) is globally asymptotically stable, where S_1^*, S_2^* are defined by (7).

Definition 4.2. System (2) is said to be permanent if there are constants $m, M > 0$ (independent of initial value) and a finite time T_0, such that for all solutions $(S_1(t), S_2(t), I(t))$ with any initial values $S_1(0^+) > 0, S_2(0^+) > 0, I(0^+) > 0$, we have $m \le S_1(t) \le M, m \le S_2(t) \le M, m \le I(t) \le M$ for all $t \ge T_0$. Here T_0 may depend on the initial values $(S_1(0^+), S_2(0^+), I(0^+))$.

Theorem 4.3. If

$$\mu < \mu^*$$

and

$$\mu < \frac{1-e^{-(c+d_1)\tau}}{e^{-(c+d_1)l\tau} - e^{-(c+d_1)\tau}} - \frac{(c+d_1)(r+d_3)\tau}{\beta S_1^* (e^{-(c+d_1)l\tau} - e^{(c+d_1)\tau})}, \tag{9}$$

then system (2) is permanent, where S_1^*, S_2^* are defined by (7).

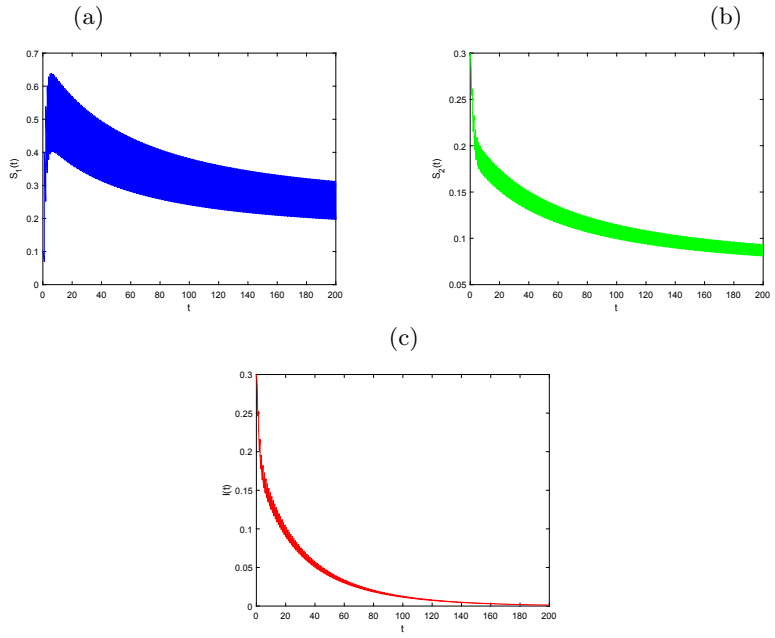

Fig. 1. Globally asymptotically stable disease-free periodic solution of System (1) with $S_1(0) = 0.1, S_2(0) = 0.3, I(0) = 0.3, a = 0.1, b = 0.2, c = 0.1, \beta = 0.2, \mu = 0.37, d_1 = 0.02, d_2 = 0.018, d_3 = 0.016, r = 0.15, \tau = 1, l = 0.25$. (a) Time-series of $S_1(t)$; (b) Time-series of $S_2(t)$; (c) Time-series of $I(t)$.

5 Conclusion and Simulation

In this work, we consider an SIR epidemic model with birth pulse and pulse vaccination on the newborn at different fixed moments. All solutions of system (1) are uniformly ultimately bounded. The condition for the global asymptotic stability of the disease-free periodic solution of system (1) is given, and the permanence of system (1) is also obtained.

According to the relevant statistical data of the National Health and Family Planning Commission, it is assumed that $S_1(0) = 0.1, S_2(0) = 0.3, I(0) = 0.3, a = 0.1, b = 0.2, c = 0.1, \beta = 0.2, \mu = 0.37, d_1 = 0.02, d_2 = 0.018, d_3 = 0.016, r = 0.15, \tau = 1, l = 0.25$, the conditions of the Theorem 4.1 are obviously satisfied, then the disease-free periodic solution of system (1) is globally asymptotically stable. (see Fig. 1). It is also assumed that $S_1(0) = 0.1, S_2(0) = 0.3, I(0) = 0.3, a = 0.1, b = 0.2, c = 0.1, \beta = 0.2, \mu = 0.1, d_1 = 0.02, d_2 = 0.018, d_3 = 0.016, r = 0.15, \tau = 1, l = 0.25$, the conditions of the Theorem 4.3 are obviously satisfied, then system (1) is permanent (see Fig. 2). The threshold dynamics about parameters l, τ can be also analyzed. If birth pulse and pulse vaccination are not considered in the traditional method, the disease of the system as a whole tends to be in a pandemic state. The results show that

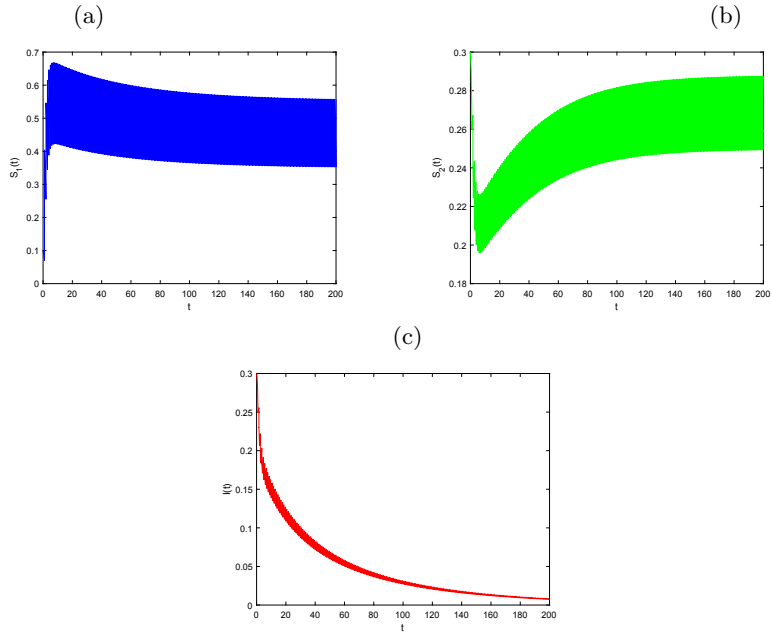

Fig. 2. The permanence for System (1) with $S_1(0) = 0.1, S_2(0) = 0.3, I(0) = 0.3, a = 0.1, b = 0.2, c = 0.1, \beta = 0.2, \mu = 0.1, d_1 = 0.02, d_2 = 0.018, d_3 = 0.016, r = 0.15, \tau = 1, l = 0.25$. (a) Time-series of $S_1(t)$; (b) Time-series of $S_2(t)$; (c) Time-series of $I(t)$.

the factors considered in this paper are more consistent with the actual situation. Our results indicate that pulse vaccination rate on the newborn plays an important role in eradicating the disease. It also provides a reliable tactic basis for preventing the disease from spreading.

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