Analyzing the Geometric $\sqrt{\text{Ratios}}$ of Greek Vases

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Abstract. The ancient Greeks possessed a deep understanding of geometry, reflected in their quest for proportion and harmony across various art forms, including Greek Vase design. This study explores the geometric ratios of Greek Vases, uncovering the influence of ancient Greek mathematical principles on aesthetic standards. We have analyzed and quantified the Greek Vases ' parameters based on Caskey's extensive measurements [4], along with additional charts and graphs. This analysis helps group Greek Vases based on their geometric ratios clarifies the relationship between these geometric ratios and vase types, and explores the link between geometric ratios and vase function. Our findings indicate that Greek Vases can be categorized according to geometric ratios, with most Greek Vases of a particular type following these proportional standards. In addition, certain types of vases have specific ratios, and through machine learning, it was found that different types of vases in the same cluster have similar uses, and some also have similar shapes.

Keywords: Greek Vases · Similarity · Clustering · Vase Usage

1 Introduction

The ancient Greek civilization was interested in geometric harmony within art and design, a principle prominently reflected in the Greek vase. These vases are a testament to the Greeks' sophisticated grasp of geometry and aesthetics. While many scholars, such as Boardman [1–3], Walters [18], and Oakley [16] and institutions like the J. Paul Getty Museum [14,15], Matheson [11] in Yale University Art Gallery, and Richter and Milne [17] in Metropolitan Museum have addressed the artistic narratives and historical contexts, a detailed geometric analysis, especially regarding how they reflected their aesthetic to geometric harmony through mathematical principles, remains under-explored. Hambidge [6]

The authors would like to thank Boston University Metropolitan College for their support.

T. Zlateva and G. Tuparov (Eds.): CSECS 2024, LNICST 609, pp. 154–165, 2025.
https://doi.org/10.1007/978-3-031-84312-9_11

has explained the geometric foundations of Greek artistic design, noting symmetry and theories concerning the relationships between vases and rectangles. Caskey, responding to Hambidge's theories, closely examined the use of rectangles, subdivisions, and the resultant ratios in Greek vase analysis, suggesting that their coincidental designs were intentional and that ratios played a significant role in their geometry and design approach.

This paper aims to present mathematical principles and geometric ratios in ancient Greek artistry. By examining the relationship between geometric ratios and vase types, as well as the usage of vases, based on tables and measurements documented by Caskey, we will categorize vases using geometric ratios including the Golden Ratio, which is discussed by Huntley [8], Livio [9] and Meisner [12], presenting the results in tables and charts.

The next sections detail our analytical process, starting with a review of Caskey's data, followed by our classification methods. We then explore the relationships between geometric ratios and vase types, the relationship between vase types, and the usage of ratios in clustering. By analyzing the geometric ratios among different vase types, we can gain insights into the aesthetic and functional preferences of ancient Greek culture.

2 Organizing Caskey's Dataset and Tables

Hambidge introduced the concept of the "rectangle of the whirling squares", where squares labelled 1, 2, 3, 4, 5, 6, etc., rotate around a pole. The fraction 0.618 represents a shape similar to the original shape, or its reciprocal, as shown on the left in Fig. 1 (Hambidge 17). The ratios of the roots outside the square are derived from the diagonals. The right image in Fig. 1, shows that the area of the two-root rectangles inside the square is half that of its external counterpart. Similarly, the areas of the three-root, four-root, and five-root rectangles are proportionately reduced(Hambidge 23). He further pointed out based on these observations that the height and width ratios of the Greek Vases as a whole and the various parts, when measured in terms of the proportions of a particular rectangle, could be represented by the proportions of the same rectangle without recourse to other rectangles of different proportions.

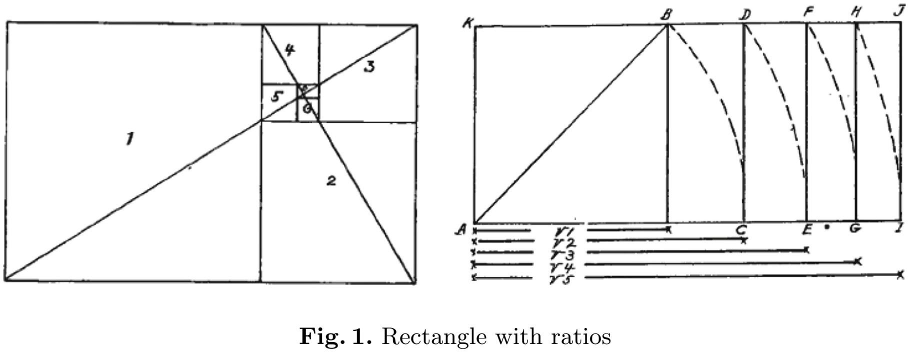

Fig. 1. Rectangle with ratios

Building on Hambidge's observations regarding the relationship between vases and rectangles, Caskey conducted extensive measurements on Greek Vases. He proposed that both vases and their elements could fit within rectangles maintaining a uniform ratio, denoted as k, which represents the side lengths' proportion. These k includes simple proportions like $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ as Hambidge mentioned, as well as the golden ratio, and other geometric ratios.

We present 10 ratios, with each row comprising equations, formulas, numerical values, and the names of the ratios. The inclusion of two values per ratio accounts for both the original value and its reciprocal, as both represent the same ratio for the vases, following their corresponding equations, formulas, and names.

To be more specific, we will focus on the 10 most common ratios k found by Caskey with some adjustments as below:

Table 1. The 10 Most Common Ratios with Adjustment

Index	Equations	Formulas	Values	Ratio Names
1	$x^2 - 2 = 0$	$\sqrt{2}$	1.4142	root-two
2	$x^2 - 8 = 0$	$2\sqrt{2}$	2.8284	two root-two
3	$x^2 - 4 = 0$	$\sqrt{2}\sqrt{2}$	2.0000	two
4	$x^2 - 18 = 0$	$3\sqrt{2}$	4.2426	three root-two
5	$2x^2 - 3 = 0$	$\sqrt{3/2}$	1.2247	root-three-over-two
6	$x^2 - 3 = 0$	$\sqrt{3}$	1.7321	root-three
7	$x^2 - 12 = 0$	$2\sqrt{3}$	3.4641	two root-two
8	$2x^2 - 5 = 0$	$\sqrt{5/2}$	1.5811	root-five-over-two
9	$x^2 - 5 = 0$	$\sqrt{5}$	2.2361	root-five
10	$x^2 - x - 1 = 0$	$\phi = (\sqrt{5} + 1)/2$	1.6180	golden ratio ϕ

Here is the one vase with a specific shape, indicating the specific data collected from Caskey in Fig. 2.

We will use two datasets. One contains data in occurrence tables, which include most vases but lack detailed specific data, and the other contains vases with specific shapes that have specific data from Caskey. Based on these two datasets, we will present the relationship between the ratio and occurrence of vases and analyze the relationship between the ratios and types of vases.

3 Ratios and Frequencies

3.1 The Vases in the Occurrence Table

The vases in the occurrence Table include most of the vases observed by Caskey, but the table only provides data on occurrences and ratios.

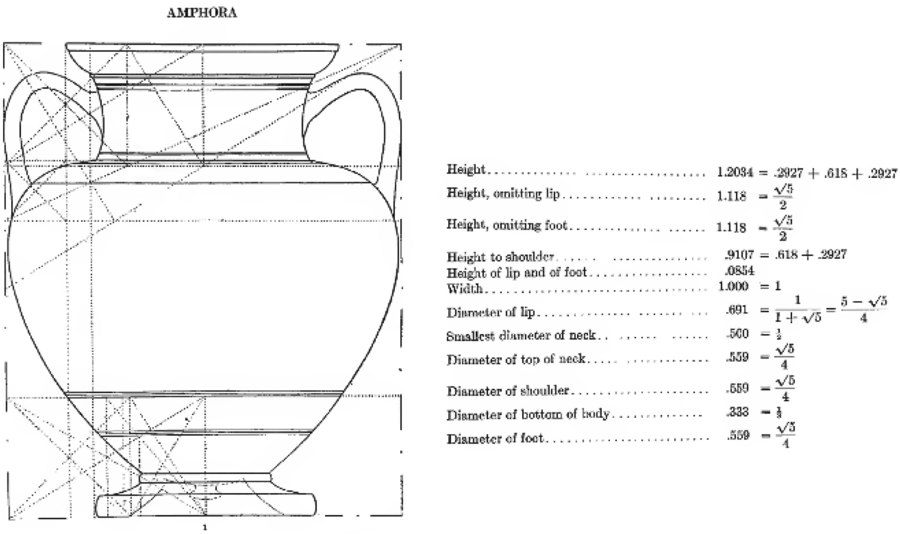

Fig. 2. Specific Vase's data from Caskey

Here is the summary of these vases related to their occurrence. We count the occurrences of vases and put them into Table 1. Additionally, we make them into a histogram with some colours. The table and the histogram are shown below in Table 2 and Fig. 3:

Let us modify the numbers to represent ratios and add similar color bars to highlight the 5 different groups which are $\sqrt{2}$, $\sqrt{3/2}$, $\sqrt{3}$, $\sqrt{5/2}$, and $\sqrt{5}$. The histogram is shown below in Fig. 4:

Table 2. Most 10 Adjusted Ratios with Occurrences

Index	Equation	Formula	Value	Occurrences
1	$x^2 - 2 = 0$	$\sqrt{2}$	1.4142	42
2	$x^2 - 8 = 0$	$2\sqrt{2}$	2.8284	65
3	$x^2 - 4 = 0$	$\sqrt{2}\sqrt{2}$	2.0000	46
4	$x^2 - 18 = 0$	$3\sqrt{2}$	4.2426	60
5	$2x^2 - 3 = 0$	$\sqrt{3/2}$	1.2247	31
6	$x^2 - 3 = 0$	$\sqrt{3}$	1.7321	51
7	$x^2 - 12 = 0$	$2\sqrt{3}$	3.4641	64
8	$2x^2 - 5 = 0$	$\sqrt{5/2}$	1.5811	29
9	$x^2 - 5 = 0$	$\sqrt{5}$	2.2361	55
10	$x^2 - x - 1 = 0$	$\phi = (\sqrt{5} + 1)/2$	1.6180	30

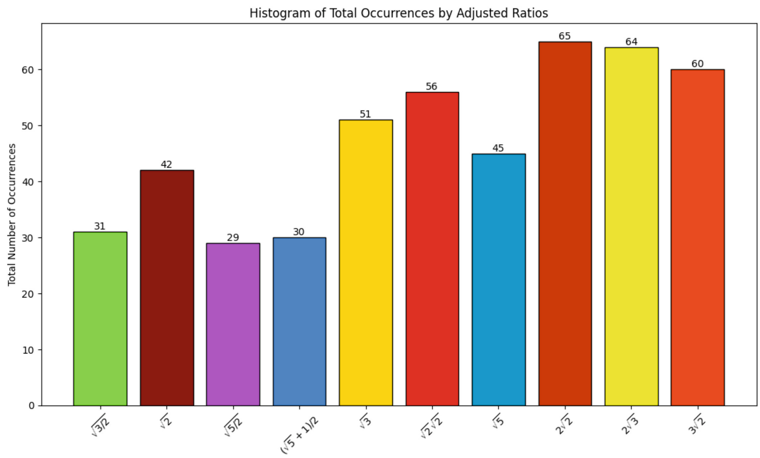

Fig. 3. Occurrences of Most 10 Ratios in Ascending Order

As observed in Fig. 4, there are three bars representing numbers above 60: two in the $\sqrt{2}$ group, one in the $\sqrt{3}$ group, followed by two in the 50–59 group, another in the $\sqrt{3}$ group, and one in the $\sqrt{5}$ group. In the 40–49 group, both numbers fall within the $\sqrt{2}$ group. The less than 40 group contains three numbers which are in the $\sqrt{5}$, $\sqrt{3/2}$, and $\sqrt{5/2}$ groups.

In terms of the number of bars, there are four bars in the $\sqrt{2}$ group; two bars are in the $\sqrt{3}$ and $\sqrt{5}$ groups; one bar is in the $\sqrt{3/2}$ and $\sqrt{5/2}$ groups. That means most of the vases are concentrated in the $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ groups, and more than half of these vases belong to the $\sqrt{2}$ group. By calculation, 44.80% of the total vases are in the $\sqrt{2}$ group, 24.42% of total vases are in the $\sqrt{3}$ group, 18.05% of total vases are in the $\sqrt{5}$ group, and 12.73% of total vases are in other groups.

Suppose a vase is given at random, then it will come from the $\sqrt{2}$ group with almost half the probability. By calculation from $\sqrt{2}$ group, 59.24% of these belonging to either the $2\sqrt{2}$ or $3\sqrt{2}$ categories. On the other hand, suppose we know a vase from any of the five groups. If it was told that it is to belong to the $\sqrt{3}$ group, there is a greater than 50% chance it belongs to the $2\sqrt{3}$ category. By applying this method, we can similarly deduce the characteristics of the $\sqrt{5}$ group.

3.2 The Vases with the Specific Shapes

According to the 10 most common ratios with adjustments identified by Caskey, the ratios of these ten types of vases were also classified, and we obtained the following two graphs in Figs. 6 and 7. In one graph, the ratio is on the x-axis and the specific number of vases is on the y-axis. The other graph uses the vase type as the x-axis and the percentage of each vase type as the y-axis. From these two figures, we discovered some very interesting facts.

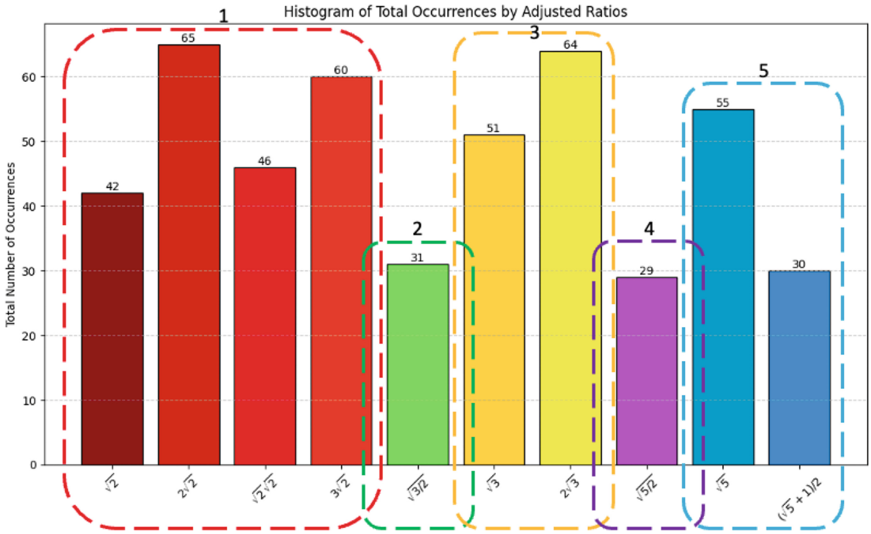

Fig. 4. Occurrences of Most 10 Adjusted Ratios in 5 Groups

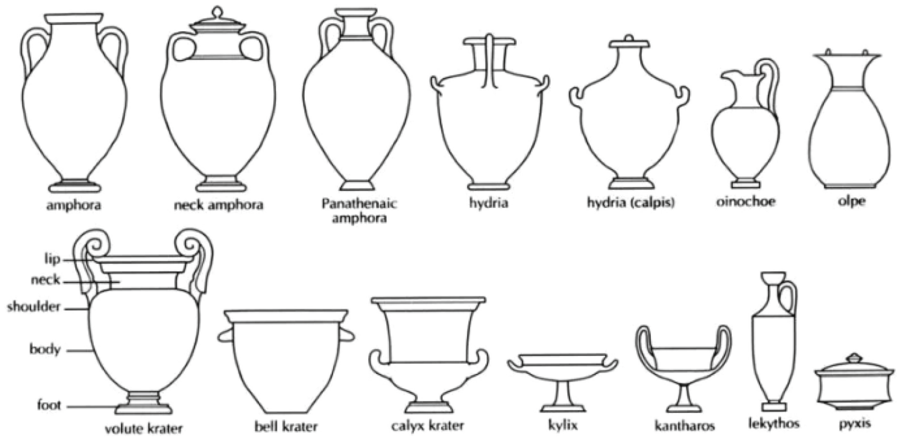

Fig. 5. Different Types of Greek Vases

Before we discuss the two figures, let us introduce the different types of vases with the following picture[1] so that people can get an idea of what the vases look like in reality in Fig. 5:

Now, let's discuss Fig. 6 first. Among the vases with specific ratios, the ones with the largest numbers are $\sqrt{3/2}$ and $\sqrt{3}$, both are 30 vases. These are followed by $2\sqrt{3}$ (28 vases), $\sqrt{5/2}$ (21 vases), and $\sqrt{2}$ (17 vases). Among the five ratios

[1] See https://thedelphiguide.com/ancient-greek-pottery-shapes/ for further description and discussion.

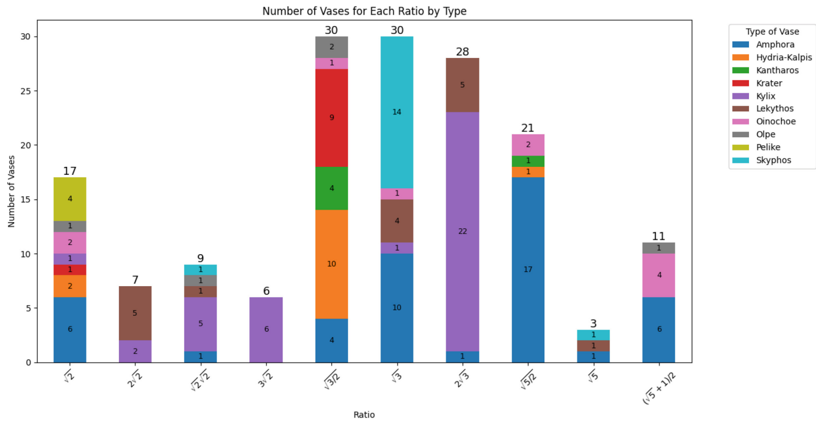

Fig. 6. Number of Vases for Each Ratio by Types

with the most vases, we find that among the 30 vases with $\sqrt{3/2}$, there are 10 Hydria-Kalpis and 9 Krater. Among the 30 vases with $\sqrt{3}$, there are 14 Skyphos and 10 Amphora. Furthermore, there are 22 Lekythos out of 28 in $2\sqrt{3}$, and 17 Amphora out of 21 in $\sqrt{5/2}$. The distribution in $\sqrt{2}$ is relatively even across different vases.

Additionally, let's discuss Fig. 7. The distribution of ratios varies among different vase types. For instance, the most numerous vases are Amphora (46 vases), 37% belonging to $\sqrt{5/2}$ and 22% to $\sqrt{3}$. Additionally, 59% of Kylix (37 vases) belong to $2\sqrt{3}$. In the case of Skyphos (16 vases), up to 88% of the vases belong to $\sqrt{3}$, and in Lekythos (16 vases), 31% belong to $2\sqrt{3}$ and another 31% to $2\sqrt{2}$. For Hydria-Kalpis (13 vases), as many as 77% of the ratios are $\sqrt{3/2}$, and in Krater, up to 90% of the ratios are $\sqrt{3/2}$. Also, 40% of Oinochoe ratios are the golden ratio which is $(\sqrt{5}+1)/2$.

In summary, we found that for certain types of vases such as Skyphos, Kylix, Krater, and Hydria-Kalpis, there are relatively fixed ratios,$\sqrt{3}$, $2\sqrt{3}$, $\sqrt{3/2}$, and these three ratios all incorporate the number $\sqrt{3}$, with both $\sqrt{3}$ and $2\sqrt{3}$ belonging to the $\sqrt{3}$ group. In vases like Amphora, Lekythos, and Oinochoe, ratios are widely distributed because they incorporate multiple ratios.

We have summarized the main proportions shown in Fig. 7 and listed them in Table 3. These proportions, along with the usage of different types of vases, were discussed by Horner [7], Moignard [13], and Clark et al. [5] in their books.

From the usage of these different vases, we further found that most vases with relatively fixed ratios are mainly used for drinking and mixing wine, while those with a wide distribution of ratios are mostly used for storing and pouring wine.

While Mackay et al. [10] used K-Means to measure the shape of Greek vases, we will use a similar machine-learning algorithm to analyze the ratios of Greek vases. First, we apply a quantile transformation to these ratios, perform K-Means

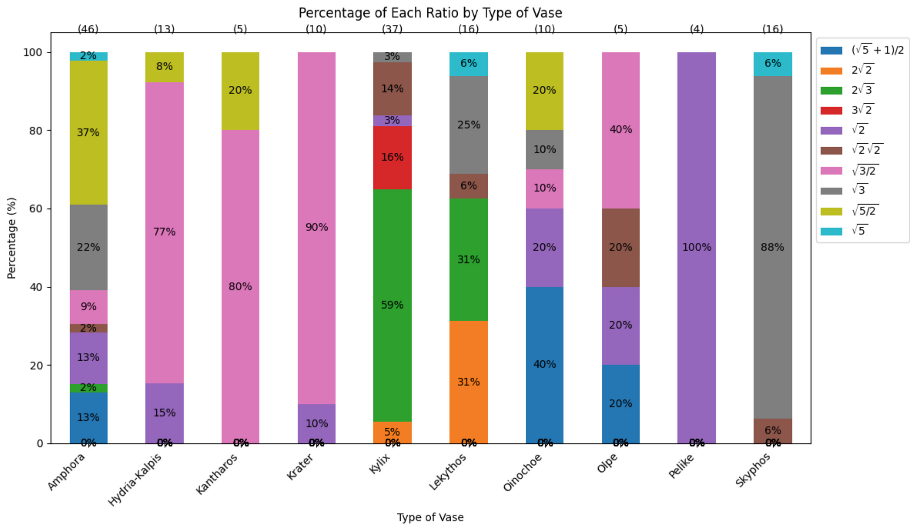

Fig. 7. Percentage for Each Ratio by Type of Vase

Table 3. 10 Most Frequent Vases with Ratios and their Usage

Index	Type	Descriptions of the Main	Primary Usage
1	Amphora	37% in $\sqrt{5/2}$	Storing wine, olive oil
2	Hydria-Kalpis	77% in $\sqrt{3/2}$	Carrying water
3	Kantharos	80% in $\sqrt{3/2}$	Drinking Cup
4	Krater	90% in $\sqrt{3/2}$	Mixing wine with water
5	Kylix	59% in $2\sqrt{3}$	Drinking Cup
6	Lekythos	31% in $2\sqrt{3}$, 31% in $2\sqrt{2}$	Storing olive oil
7	Oinochoe	40% in $(\sqrt{5}+1)/2$	Pouring wine
8	Olpe	40% in $\sqrt{3/2}$	Pouring liquids
9	Pelike	100% in $2\sqrt{2}$	Storing wine and oil
10	Skyphos	88% in $\sqrt{3}$	Drinking Cup

clustering on the quantile-transformed ratios, and specify 5 clusters through the scatter plot as shown in Fig. 8. The appropriate clustering k=5 is obtained through the elbow method. We group data according to similarity and calculate the distribution of vase types within each cluster. After that, we convert the counts of these distributions into percentages. These data are then visualized through a bar chart, as shown in Fig. 9.

We discuss the scatter plot first. We can see clusters 4 and 3 are highly concentrated, with cluster 0 following closely behind. By analyzing these clusters, we find that cluster 0 is the most concentrated, with a mean of 0.6605 and a standard deviation of 0.0323. Clusters 3 and 4 are affected by outliers. The mean for cluster 3 is 0.8759 with a standard deviation of 0.0393, while the mean for cluster 4 is 0.1074 with a standard deviation of 0.0481. The data points in clusters 1 and 2 are widely spread, so they will not be discussed further.

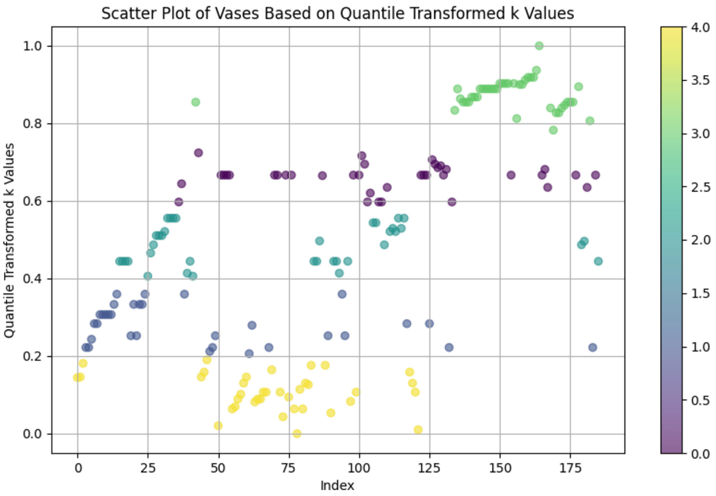

Fig. 8. Clustering the Greek Vases in Scatter Plot

Next, we examine clusters 0, 3, and 4 using the bar chart shown in Fig. 9. In cluster 3, the proportion containing Kylix is as high as 71%, and the proportion containing Lekythos is 26%. These two vase types account for 97% of cluster 3, with the remaining 3% belonging to Amphora. This indicates that Kylix, Lekythos, and Amphora are similar in some aspects. Clusters 0 and 4 contain multiple types of vases. In cluster 0, Skyphos and Kylix each account for 24%. In cluster 4, Krater and Hydria-Kalpis each occupy a large proportion, both at 29%.

Let us discuss their similarities in more detail. Cluster 3 contains only three types of vases: Kylix, Lekythos, and Amphora. These are versatile vases, used both in daily life and ceremonial occasions. Amphorae were used to store and transport goods but also appeared as prizes in ceremonial occasions such as sporting competitions. Kylixes are the centerpiece of the symposium, blending social, recreational, and ritual drinking. Lekythos were used to store oils and perfumes, common in daily grooming rituals, and also important in funerary events. Moreover, each of these three vase types played a significant role in the rituals and practices of Greek life beyond their practical use. For example, lekythos were often placed in tombs as offerings to the dead, amphorae could be offerings or trophies, and kylixes participated in the social ritual of public drinking. Cluster 0 contains a larger proportion of Skyphos and Kylix. They are similar in shape and use, both having shallow bowls and handles for easy gripping, and they are used as wine glasses. Krater and Hydria-Kalpis, which

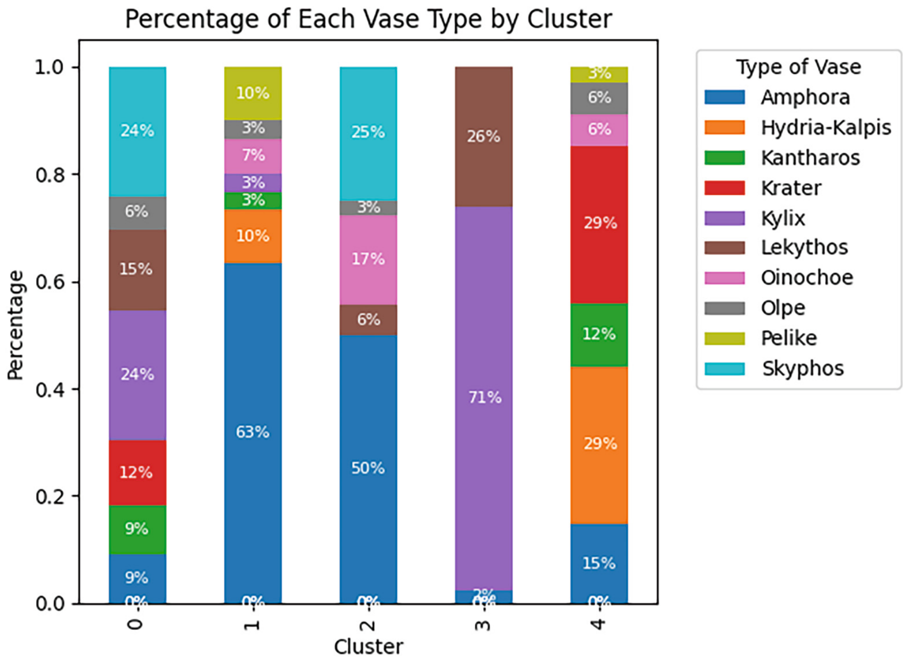

Fig. 9. Clustering the Greek Vases in Percentages

make up a large proportion of cluster 4, have wide bodies and therefore larger capacities.

To summarize, Kylix, Lekythos, and Amphora are used in diverse and complex scenes. Skyphos and Kylix share similar shapes and uses, and Krater and Hydria-Kalpis have wide body shapes.

4 Discussion

This paper seeks to determine the relationship between geometric ratios, vase types and the usage of vases. The results show that when vases are grouped by ratio, Greek vases prefer certain ratios, which can help predict the random grouping of vases. Different vase types also have different ratio preferences. Through K-Means clustering, which performs machine learning on these ratios, the study found that different types of vases in the same cluster have similar usages, and some also have similar shapes.

However, this study also has limitations. The sample size was relatively small, and the vase types studied did not represent the diversity of all vases. Furthermore, the research is mainly based on Caskey's book and is therefore limited in scope. Errors in the data records of some vases documented in the book could affect the reliability and accuracy of the research results.

Future research will focus on adding more vases from additional sources to address these limitations. Additionally, we will study whether there is a correlation between the ratio and the volume of the vase, and further explore how mathematical geometry was reflected in the volume of ancient Greek vases.

In summary, this study provides evidence to support the important relationship between ratios and Greek vases, including vase type and usage. The research results can help us better understand the artistic aesthetics and social etiquette of ancient Greek culture, and strengthen the discussion of geometric and mathematical principles.

5 Conclusion

In conclusion, our research shows that by classifying Greek vases by ratios, we can not only predict which group a Greek vase belongs to by knowing its ratio information but also use machine learning to analyze whether different vases with the same ratio share similar functionality. These insights give us a deeper understanding of visual harmony, values, and religious beliefs in ancient Greek culture, and enhance the sharing of mathematical principles in the field of geometric analysis.

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