Research on Construction of Measurement Matrix Based on Welch Bound

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Abstract. Compressive sensing (CS) is a new theory of data acquisition and reconstruction. It permits the data of interest being sampled at a sub-Nyquist rate, meanwhile still allowing perfect reconstruction of data from highly incomplete measurements. During this process, the construction of measurement matrix is undoubtedly the key point. However, the traditional random measurement matrices, though having good performance, are difficult to implement in hardware and lack the ability of dealing with large signals. In this paper, we construct a series of novel measurement matrices (HWKM and HWCM) based on Welch bound, by sifting the basis matrix based on Hadamard matrix. Therefore, the proposed matrices are deterministic measurement, which can be easily designed in hardware. Specially, it is proved to have low coherence, which can even approach to Welch bound. Experimental results show that the proposed matrices, compared with traditional measurement matrices, not only have considerable reconstruction performance in terms of reconstruction error and the signal-to-noise ratio, but also accelerate recovery time.

Keywords: Compressive sensing · Measurement matrix · Welch bound

1 Introduction

Compressive sensing (CS) is an innovative framework for data acquisition and reconstruction, which uses the sparse structure of most signals typically in the time or transform domain to break through the limitations of the traditional Nyquist sampling band width. Since CS was first proposed by Candès [1], Tao [2] and Donoho [3] in 2004, it has aroused widespread concern in the academic community, which is widely used in various fields, such as information theory, wireless communication and data processing. All the time, CS mainly has three research directions [4]: sparse representation of signals, the design of measurement matrices and recovery (such as orthogonal matching pursuit, OMP) algorithm.

The sparse representation of the signal is the premise of the CS framework. Moreover, measurement matrix plays a dual role in the process of CS, which not only ensures that we can capture and preserve valuable original information

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H. Gao et al. (Eds.): ChinaCom 2020, LNICST 352, pp. 701–712, 2021.
https://doi.org/10.1007/978-3-030-67720-6_48

of the signal with fewer measurements possibly, but also influences the performance of the recovery. Finally, it is also important to design an effective and fast algorithm to recover the signal. Although these three issues are related, it is generally recognized that designing a stable measurement matrix is helpful for several other aspects, so we need pay attention to the construction of a stable measurement matrix. This paper mainly describes the construction of the measurement matrix, which mainly combines the advantages and disadvantages of existing matrices and the properties of measurement matrix itself. Specifically, the restricted isometry property(RIP) is an essential criterion for measurement matrix properties.

Nowadays, measurement matrices are mainly divided into two categories: the first type is random measurement matrices, such as random Gaussian matrix, random Bernoulli matrix [5], partial Hadamard binary matrix, and partial Fourier matrix, which can satisfy RIP with great probability. However, random matrices have the disadvantages of uncertainty and waste of storage resources, which limit their practical applications. For example, the storage space required for random Gaussian matrix elements is large, and its unstructured nature leads to complicated calculations. The second category is deterministic measurement matrices which can overcome these shortcomings. Commonly, there are Toeplitz matrix [6], circulant matrix [7], and polynomial deterministic matrix [8]. However, the construction of the deterministic measurement matrices is hard and is difficult to handle large signals. Many researchers use some techniques to construct deterministic measurement matrices. For example, reference [9] points out that a good low density parity check (LDPC) code for primitive graphs is proposed, LDPC) check matrix can be used as measurement matrix; reference [10] uses m sequence to construct measurement matrix with better performance; reference [11] uses Berlekamp-Justesen code to construct measurement matrix. Therefore, how to combine the advantages and disadvantages of the traditional matrices to construct a novel measurement matrix is the key point of this paper.

The RIP simplifies the investigation of CS reconstruction and provides the best guarantee currently known. However, it is very difficult to judge whether a measurement matrix satisfies RIP [12]. In order to reduce the complexity of the problem, it is another key point to find an alternative method that can easily implement the RIP condition. R Baraniuk [12] pointed out that if the measurement matrix and sparse basis are guaranteed to be irrelevant, the measurement matrix satisfies the RIP condition with a high probability. In this paper, it is found that starting from the lower bound of correlation, some theories have proved that the performance of measurement matrix is better when the correlation bound is close to Welch bound [13]. In order to reduce the randomness of random matrix and simplify the construction of measurement matrix, this paper proposes a set of measurement matrices satisfying the optimal Welch bound. Moreover, theoretical verification and experimental analysis show that the measurement matrices proposed in this paper have good performance.

2 CS Theoretical Basis

2.1 CS Model

For an original finite-length signal $x \in R^N$, assuming $\{\psi_i\}_{i=1}^N$ is a set of orthogonal basis vectors of R^N, the finite-length signal x can be linearly expressed as:

$$x = \sum_{i=1}^{N} s_i \psi_i \tag{1}$$

where $s_i = \langle x, \psi_i \rangle = \psi^T x$, Eq. 1 can also be written as the following matrix form:

$$x = \Psi s \tag{2}$$

where $\Psi = [\psi_1, \ \ \psi_2, \cdots, \psi_N] \in R^{N \times N}$, and $\Psi \Psi^T = \Psi^T \Psi = I$, $s = [s_1, s_2, \cdots, s_N]^T$, assuming that the column vector s has only $K(K \leq N)$ non-zero coefficients, the signal x is said to be K sparse under the base matrix Ψ. At this time, the measurement matrix $\Phi \in R^{M \times N}(M \leq N)$ that is not related to the orthogonal basis matrix Ψ can be used for projection compression:

$$y = \Phi x \tag{3}$$

It can be obtained that there are M linear observations $y \in R^M(M \leq N)$. From the signal sparsity, it can be known that the information contained in the small linear projection is sufficient to reconstruct the signal x. Substituting Eq. 2 into Eq. 3 gives

$$y = \Phi \Psi s = As \tag{4}$$

where $\Phi \in R^{M \times N}$ is called measurement matrix or observation matrix, $\Psi \in R^{N \times N}$ is called sparse matrix or transform matrix, $A = \Phi \Psi$, $A \in R^{M \times N}$ is called sensing matrix or information operator.

For formula 3 , since $M \leq N$, recovering x from y is obviously a ill-conditioned problem. After being converted to formula 4 , although it is still ill-conditioned, it becomes a reality to recover the signal x from the sensing matrix A, because s is K sparse and $K \leq N$. The focus of this paper is to construct measurement matrix. Since we assumed that the original signal x itself is sparse and the sparse matrix Ψ is a identity matrix, then we only need to ensure that the measurement matrix Φ satisfies the conditions. On top of this, we can reconstruct x accurately by using the l_0 optimization:

$$\begin{cases} \min_x \|x\|_0 \\ s.t. y = \Phi x \end{cases} \tag{5}$$

2.2 Restricted Isometry Property and Correlation Property

For the case where the original signal x is a sparse signal, the measurement matrix Φ needs to meet the Restricted Isometry Property (RIP).

Definition 1 [14]. *The RIP parameter δ_k of the measurement matrix Φ is defined as the minimum value δ satisfying the following formula:*

$$(1-\delta)\left\| x|_2^2 \leq | \Phi x\right\|_2^2 \leq (1+\delta)\|x\|_2^2 \tag{6}$$

where x is a K sparse signal. If $\delta_k < 1$, the measurement matrix Φ is said to satisfy the K order RIP.

However, judging whether a given measurement matrix has RIP properties is a combination complexity problem. In order to reduce the complexity of the problem, the judgment of the RIP property of the matrix can be converted into correlation discrimination [15], Spark discrimination, etc. [16]. This article mainly analyzes and constructs the measurement matrix based on the correlation. From [16], it can be known that the measurement matrix with low correlation satisfies the RIP property, so the following description will be made on the correlation.

Definition 2. *For a matrix $\Phi = (\phi_1, \phi_2, \ldots, \phi_N) \in R^{M\times N}$, its correlation $\mu(\Phi)$ is defined as follows:*

$$\mu(\Phi) = \max_{1\leq i\neq j\leq N} \frac{|\langle \phi_i, \phi_j\rangle|}{\|\phi_i\|_2 \bullet \|\phi_j\|_2} \tag{7}$$

where the inner product of the vector is represented $\langle \phi_i, \phi_j\rangle = \phi_i^T \phi_j$.

Obviously, when the column vectors of the matrix are unitized, $\mu(\Phi) = \max_{1\leq i\neq j\leq N} |\langle \phi_i, \phi_j\rangle|$, the correlation of the matrix Φ is the maximum value of the inner product of any two columns. Since the low correlation measurement matrix satisfies the RIP property, our task comes to find a inner product with a suitable lower bound. At this point, Welch bound provides us with a good solution.

Theorem 1. *For a matrix $\Phi \in R^{M\times N}(M \leq N)$, after its column vectors being unitized, and its correlation satisfies $\mu(\Phi) \geq \sqrt{\frac{N-M}{M(N-1)}}$, that is, the inner product of any two columns satisfies this lower bound. This theorem gives a lower bound of correlation function, also called Welch bound.*

In order to build a measurement matrix, the premise, from the RIP to the correlation and eventually to the lower bound, is simplified step by step. Next, we will focus on the construction of low-randomness measurement matrices that satisfie Welch bound.

3 Construction of Low-Randomness Measurement Matrices Satisfying Welch Bound

3.1 Generation of Measurement Matrices

Based on Hadamard matrix, and screened by Welch bound, we can get a measurement matrix of size $M \times N (N = 2^n)$, which is the Hadamard Welch Matrix

(HWKM) we need. However, when the number of matrix rows or columns is too large, the correlation of this matrix may not satisfy Welch bound. In this case, we can solve a small matrix satisfying the condition by Kronecker integral, then construct the measurement matrix HWKM of size $\Phi \in R^{M\times N}(M = m^p, N = n^p, p > 1)$. The concrete steps to realize are as follows:

Step 1: first, we select a size of $N \times N$ Hadamard matrix, and randomly select M rows from it to generate a partial Hadamard matrix. Second, replace the -1 value in the matrix with 0 to generate a 0–1 matrix, and unit the column vector to obtain the matrix $\Phi = (\phi_1, \phi_2, \ldots, \phi_N) \in R^{M\times N}$, then determine whether its correlation satisfies Welch bound for screen. If the correlation is satisfied, the matrix is required to be output, otherwise, step 2 is performed.

Step 2: If the value of M or N is too large, we may not find the matrix that satisfies the Welch bound condition. In this case, we can get the basis matrix by Kronecker decomposition for the matrix $\Phi \in R^{M\times N}$ of known size [17]. The basis matrix $S \in R^{m\times n}(M = m^p, N = n^p)$ that satisfies Welch bound is smaller and easier to find.

Step 3: If the matrix $S = (s_1, s_2, \ldots, s_n) \in R^{m\times n}$ satisfies the Welch bound condition and its column vectors are unitized, $||s_i|| = 1(i = 1, 2, \ldots, n)$, then the correlation of any two columns $s_i, s_j(i, j = 1, 2, \ldots, n)$ of the matrix S $\mu = \max|\langle s_i, s_j\rangle| \geq \sqrt{\frac{n-m}{m(n-1)}}$. Through mathematical verification, the matrix Φ expanded by Kronecker product of matrix S also satisfies the Welch bound, which is the measurement matrix HWKM we seek.

From the above construction process, it can be seen that the matrix filtered by Welch bound meets the requirements of the measurement matrix, reduces randomness and improves efficiency. At the same time, due to the characteristics of the family of low correlation sequences, the matrix HWKM $\Phi \in R^{M\times N}(M \leq N)$ that meets the Welch bound has been verified to meet the low correlation requirement of the measurement matrix. The matrix Hadamard Welch Circulant Matrix (HWCM) $\Phi \in R^{M\times N}(M \leq N)$ constructed by cyclic shift has also been verified to meet the low correlation requirement of the measurement matrix. The two kinds of measurement matrices constructed above are filtered by the Welch bound to reduce the randomness, and the matrices are verified to meet the low correlation characteristic of the measurement matrix, which guarantee the reconstruction probability, reduce the calculation time, and also have some improvement in hardware.

3.2 Theoretical Analysis

In this section, we mainly carry out theoretical analysis. First, the 0–1 matrix $\Phi = (\phi_1, \phi_2, \ldots, \phi_N) \in R^{M\times N}$ of size $M \times N$, after column vectors ϕ_i being unitized, it can be known from Theorem 1 that its correlation must satisfy the Welch bound:

$$\mu(\Phi) \geq \sqrt{\frac{N-M}{M(N-1)}} \tag{8}$$

Judging the Welch bound of the constructed basis matrix, if Eq. 8 is satisfied, the measurement matrix that meets our need can be obtained. Of course, if the value of M or N is too large, the basis matrix that satisfies the condition may not be found. At this time, through the Kronecker product decomposition of the matrix $\Phi \in R^{M\times N}$ of known size [16], a smaller basis matrix $S \in R^{m\times n}(M = m^p, N = n^p)$ that satisfies Welch bound is easier to find. If the matrix S satisfies Welch bound, the matrix Φ expanded by its Kronecker product also satisfies Welch bound, which is the measurement matrix we seek, and it is named HWKM matrix.

It is assumed that the basis matrix $S = (s_1, s_2, \ldots, s_n) \in R^{m\times n}$ satisfies the Welch bound and its column vectors $s_i = (x_1, x_2, \ldots, x_m)(i = 1, 2, \ldots, n)$ have been unitized, $\|s_i\|_2^2 = \sum_{j=1}^{m} x_j^2 = 1 \quad (i = 1, 2, \cdots, n)$, then the correlation function of any two column vectors $s_i, s_j(i, j = 1, 2, \ldots, n)$ of the matrix S satisfies $\mu_{i,j} = \max|\langle s_i, s_j\rangle| \geq \sqrt{\frac{n-m}{m(n-1)}}$. The matrix $\Phi \in R^{M\times N}(M = m^p, N = n^p, p \geq 1)$ is extended by the Kronecker product, and the correlation function between any two columns $i + qn, j + qn(q = 0, 1, \ldots, p-1)$ is:

$$\mu_{i+gn,j+gn} = \left(\sum_{l=1}^{m} x_l^2\right)^p \cdot \mu_{i,j} = \mu_{i,j} \geq \sqrt{\frac{n-m}{m(n-1)}} \geq \sqrt{\frac{n^p - m^p}{m^p\,(n^p - 1)}} \tag{9}$$

It can be proved that the matrix $\Phi \in R^{M\times N}(M = m^p, N = n^p, p > 1)$ also meets Welch bound.

$$\sqrt{\frac{n-m}{m(n-1)}} \geq \sqrt{\frac{n^p - m^p}{m^p\,(n^p - 1)}} \tag{10}$$

Using Mathematical Induction to prove as follows:

1. When $p = 1$, inequality 10 was equal, which obviously holds.
2. Assuming when $p = k$, inequality 10 holds, then $\sqrt{\frac{n-m}{m(n-1)}} \geq \sqrt{\frac{n^k-m^k}{m^k(n^k-1)}}$.
3. Well, when $p = k + 1$,

$$\sqrt{\frac{n^{k+1}-m^{k+1}}{m^{k+1}(n^{k+1}-1)}} = \sqrt{\frac{(n-m)(n^k+mn^{k-1}+\cdots m^k)}{m^{k+1}(n-1)(n^k+n^{k-1}+\cdots 1)}}$$
$$= \sqrt{\frac{n-m}{m(n-1)}} \cdot \sqrt{\frac{n^k+mn^{k-1}+\cdots m^k}{m^k(n^k+n^{k-1}+\cdots 1)}} = \sqrt{\frac{n-m}{m(n-1)}} \cdot \sqrt{\frac{n^k+mn^{k-1}+\cdots m^k}{m^kn^k+m^kn^{k-1}+\cdots m^k}}$$

Because $n^k + mn^{k-1} + \cdots + m^k < m^kn^k + m^kn^{k-1} + \cdots + m^k$, $\sqrt{\frac{n^k+mn^{k-1}+\cdots m^k}{m^kn^k+m^kn^{k-1}+\cdots m^k}} < 1$, which is $\sqrt{\frac{n^{k+1}-m^{k+1}}{m^{k+1}(n^{k+1}-1)}} < \sqrt{\frac{n-m}{m(n-1)}}$.

Therefore, when $p = k + 1$, inequality 10 also holds, and the proposition is proved. Therefore, formula 9 holds, that is, the matrix $\Phi \in R^{M\times N}(M = m^p, N = n^p, p > 1)$ obtained by the Kronecker product expansion also meets the

Welch bound. So, the measurement matrix Φ is what we seek, which is named the HWKM matrix.

The matrix HWKM $\Phi \in R^{M\times N}(M \leq N)$ meets the Welch bound condition and has the characteristics of a low correlation sequence family. The matrix HWCM $A \in R^{N\times MN}(M \leq N)$ constructed by cyclic shift has been verified to still meet the low correlation requirement of the measurement matrix.

The matrix HWKM can be regarded as a sequence set of $(N, M, \mu_{\max})$, where $\mu_{\max}$ is the maximum correlation value of the sequence set. Along with Welch bound [13], the maximum value of the correlation function for a sequence set S consisting of M sequences with period N satisfies $\mu_{\max} \geq \sqrt{\frac{N-M}{M(N-1)}}$.

Through cyclic shift, we can construct a measurement matrix from a sequence set. Specifically, for the above mentioned sequence set $S = \left\{s_i = \{s_i(t)\}_{t=1}^{t=N} : 1 \leq i \leq M\right\}$, with $s_1^1 = (s_1(1), s_1(2), \ldots, s_1(N))^T$, where T represents the transpose of the vector, after one cyclic shift, with $s_1^2 = (s_1(N), s_1(1), \ldots, s_1(N-1))^T$, and so on, after k cyclic shifts $(2 \leq k \leq N)$, we can get:

$$s_1^k = (s_1(N-k), s_1(N-k+1), \cdots, s_1(N), s_1(1), s_1(2), \cdots, s_1(N-k-1))^{\mathrm{T}} \tag{11}$$

From that we can see that by cyclic shift, we can construct a matrix A of size $N \times NM$:

$$A = \left[s_1^{\mathrm{T}}, s_2^{\mathrm{T}}, \cdots, s_M^{\mathrm{T}}, L\left(s_1^{\mathrm{T}}\right), L\left(s_2^{\mathrm{T}}\right), \cdots, L\left(s_M^{\mathrm{T}}\right), \cdots, L^N\left(s_1^{\mathrm{T}}\right), L^N\left(s_2^{\mathrm{T}}\right), \cdots, L^N\left(s_M^{\mathrm{T}}\right)\right] \tag{12}$$

At this time, the correlation function of the matrix A is:

$$\mu(A) = \mu_{\max} \geq \sqrt{\frac{N-M}{M(N-1)}} > \sqrt{\frac{NM-N}{N(MN-1)}} \tag{13}$$

which meets Welch bound condition. So the matrix $A \in R^{N\times MN}(M \leq N)$ is the measurement matrix HWCM we seek. Among them,

$$\sqrt{\frac{N-M}{M(N-1)}} > \sqrt{\frac{NM-N}{N(MN-1)}} \tag{14}$$

Inequality (14) is proved as follows:

Because $M \leq N$, $N - M > M - 1$,that is $\sqrt{\frac{N-M}{M(N-1)}} > \sqrt{\frac{M-1}{N(N-1)}}$. Also because $MN - M < MN - 1$, $\sqrt{\frac{N-M}{M(N-1)}} > \sqrt{\frac{M-1}{N(N-1)}}$. So $\sqrt{\frac{N-M}{M(N-1)}} > \sqrt{\frac{M-1}{MN-1}} = \sqrt{\frac{MN-N}{N(MN-1)}}$, that is, inequality (14) is proved, and formula (13) holds.

It is verified that the measurement matrices that constructed above all meet the requirement of low correlation. The measurement matrix HWKM constructed by Welch bound screening effectively reduce the randomness, and the measurement matrix HWCM are easier to implement in hardware.

4 Simulation Results

In this section, firstly, we analyze the RIP of random Gaussian matrix, partial Hadamard matrix and measurement matrices HWKM and HWCM constructed in this paper.

As seen in Fig. 1, we can clearly find that the measurement matrices HWKM and HWCM have similar properties. From the last two subgraphs of each row, we can find that the diagonal entries of the Gram matrix converge to 1, and the nondiagonal entries float around 0, which means that the measurement matrices HWKM and HWCM satisfy the RIP [18].

Then we compare traditional random Gaussian matrix, partial Hadamard matrix, low-randomness measurement matrices HWKM and HWCM constructed in this paper, and analyze the results both in noiseless and noisy environments.

Assuming that the original signal $x \in R^{256}$, the measurement matrix Φ with size of 64×256 and sparsity k, where $k \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65\}$, use the above four matrices to perform signal reconstruction separately, and use Matlab 2018b to generate 2000 averages for each sparsity k. This section aims to analysis and contrast the results from three aspects: recovery error, signal-to-noise ratio (SNR) and recovery time.

This experiment uses the basic tracking algorithm (BP) to obtain the recovery signal x^* and other results. The recovery error is defined by $||x - x^*||^2$, and the SNR is defined by:

$$SNR = 10 \log_{10} \left(\frac{\|x\|^2}{\|x - x^*\|^2} \right) dB \tag{15}$$

For a signal x, we say x^* is a perfect recovery if $SNR(x) \geq 100$.

4.1 Signal Reconstruction in Noiseless Environment

For random Gaussian matrix, partial Hadamard matrix, low-randomness measurement matrices HWKM and HWCM which constructed in this paper, assuming that all with a size of 64×256 and under noiseless condition, we mainly compare the recovery error, the SNR and the recovery time.

As seen in Fig. 2, when the sparsity $k \leq 40$, the measurement matrices HWKM and HWCM constructed in this paper have smaller recovery error, higher SNR, shorter recovery time than random Gaussian matrix and partial Hadamard matrix.

4.2 Signal Reconstruction in Noisy Environment

For random Gaussian matrix, partial Hadamard matrix, low-randomness measurement matrix HWKM and HWCM which constructed in this paper, assuming that all with a size of 64×256 and under noisy condition, we mainly compare the recovery error, SNR and the recovery time.

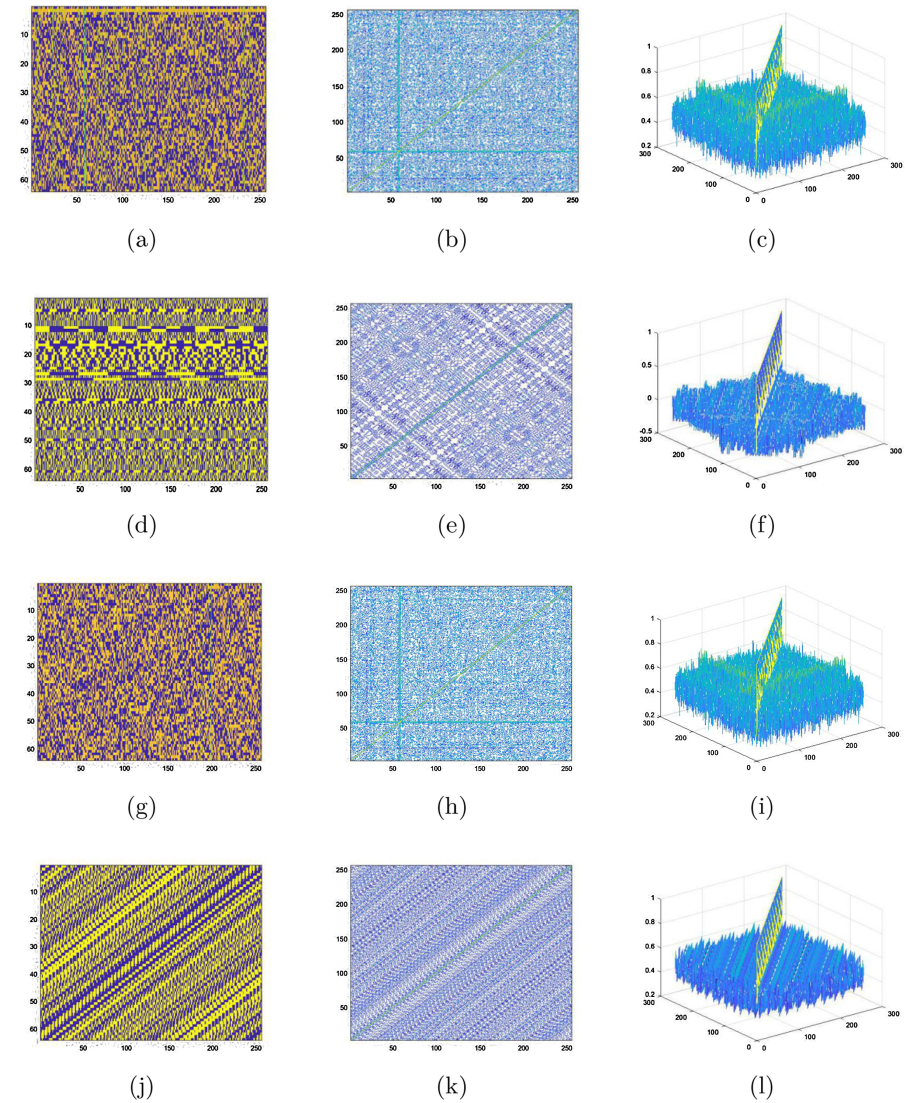
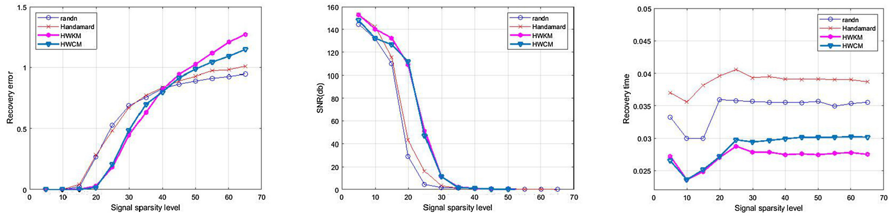
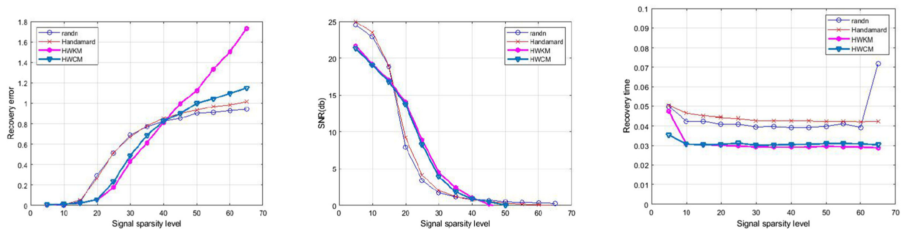


Fig. 1. First column is the imagec functions of random Gaussian matrix, partial Hadamard matrix, HWKM and HWCM, as well as the last two subgraphs of each row are the contour functions and mesh functions of their Gram matrix.



(a) (b) (c)

Fig. 2. Performance comparisons for recovering real singal using random Gaussian matrix, partial Hadamard matrix, HWKM and HWCM in noiseless environment: (a) Recovery errors with different sparsity, (b) SNR with different sparsity, (c) Recovery time with different sparsity.



(a) (b) (c)

Fig. 3. Performance comparisons for recovering real singal using random Gaussian matrix, partial Hadamard matrix, HWKM and HWCM in noisy environment: (a) Recovery errors with different sparsity, (b) SNR with different sparsity, (c) Recovery time with different sparsity.

As seen in Fig. 3, in the case of sparsity $15 \leq k \leq 40$, which is just within a relatively more suitable sparsity range, the measurement matrices HWKM and HWCM constructed in this paper have smaller recovery error, higher SNR, and significant reduction in time than random Gaussian matrix and partial Hadamard matrix.

Through the above simulation, it is found that the measurement matrices HWKM and HWCM constructed in this paper have better reconstruction performance than the random Gaussian matrix and partial Hadamard matrix in the appropriate sparsity range under the same conditions, no matter whether it is noisy or noiseless.

5 Conclusion

In this paper, we propose a series of novel measurement matrices(HWKM and HWCM) based on Welch bound, which use the Hadamard matrix as the basis matrix and perform a series of screening and construction. It is easy to come to the conclusion that, under the same experimental situation, the measurement matrices HWKM and HWCM have preferable performance than random Gaussian matrix and partial Hadamard matrix, whether the background is noisy or not. In addition, within the appropriate sparsity range, compared to typical constructions, the proposed measurement matrices in this paper can facilitate the reestablishment of signals and could perform more practically and effectively. Although CS has been implemented in many areas, many algorithms still have room for improvement. Next step, test will be carried out in its application in Wireless Sensor Networks to optimize the process of data collection.

Acknowledgements. This paper is supported by NSFC (No. 61372069).

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