Stability Analysis of Quaternion-Valued Neural Network with Non-differentiable Time-Varying Delays and Constant Delays

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Abstract. The main goal of this paper is to investigate the problems of the uniqueness of equilibrium and the global μ-stability for the QVNN (quaternion-valued neural network) with leaky constant delay, non-differentiable discrete time-varying delay, distributed constant delay, which is closer to practical application than the QVNN with differentiable time-varying delay. Firstly, we discuss the QVNN as entirety, and prove the equilibrium of the QVNN is unique by using Homeomorphism mapping theorem and quaternion-valued linear matrix inequality. Then a new Lyapunov-Krasovskii functional is derived from the delayed state. The sufficient condition of the global μ-stability is given, while appraising the derivative of the Lyapunov-Krasovskii functional and quaternion-valued linear matrix inequality, this result is new and different from the approaches in available literatures. A quaternion-valued numerical example is presented to illustrate these results.

Keywords: Quaternion-valued neural network · Non-differentiable delays · Constant delays · Global stability

1 Introduction

In recent years, Neural networks has been becoming as foundation of machine learning technology, and used in diverse fields widely, such as engineering control, economy, transportation, psychology, and so on [1–3]. Applications of real-valued

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neural networks, complex-valued neural networks have been extensively investigated in past decades. W. Hamilton proposed quaternion comprised of one real part as well as three imaginary parts in 1872 [4]. Quaternion has stronger information storage capacity. In high-dimensional data processing, quaternion-valued neural networks (QVNNs) have irreplaceable advantages compare with real-valued neural networks and complex-valued neural networks, by leveraging the benefits of this [5–9]. In practical applications, time delays may reduce the transmission speed of Neural network, and destabilize the overall stable system [10–12], and there are a lot of non-differentiable time-varying delays in reality, it is important to take non-differentiable time-varying delays into neural networks [13–16]. Hence, the stability of neural networks is primary consideration for ensuring its practicality. In 2007, the definition of the μ-stability is proposed firstly [17], that is recapitulative and could be specialized into six stability states, including power stability, exponential stability, log stability, log-log stability, the global asymptotical stability, the Lyapunov stability, via changing the property of the time delay and μ function [18].

In our research, we investigate the global μ-stability of the QVNN with leaky constant delay, non-differentiable discrete time-varying delay, distributed constant delay. Firstly the equilibrium of the QVNN is unique is proved by using Homeomorphism mapping theorem and quaternion-valued linear matrix inequality. Next, based on the delayed state's feature, we constructed a new Lyapunov-Krasovskii function. The global μ-stability of the QVNN is obtained while appraising the derivative of the Lyapunov-Krasovskii function as well as quaternion-valued linear matrix inequality. The vital contributions of this study are summarized as follows:

(1) We discuss the global μ-stability of the QVNN with non-differentiable discrete time-varying delay, which is closer to practical application than the QVNN with differentiable time-varying delay;
(2) We investigate the QVNN as entirety, not decompose the QVNN into two complex-valued neural networks or four real-valued neural networks, the increasing of data dimension caused by decomposition method is avoided.

2 Preliminaries

x is a quaternion, and can be defined as follow:

$$x = x^R + i \cdot x^I + j \cdot x^J + k \cdot x^K,$$

$x^R, x^I, x^J, x^K \in \mathbb{R}$ all are real coefficient, i, j, k all denote the imaginary units. A quaternion satisfies the Hamilton rule:

$$ij = -ji = k, jk = -kj = i, ki = -ik = j,$$

$$i^2 = j^2 = k^2 = -1.$$

Due to the Hamilton rule, quaternion doesn't meet commutative law of multiplication. In the next content, we use following notations: $\mathbb{Q}^{n\times m}$,$\mathbb{C}^{n\times m}$,$\mathbb{R}^{n\times m}$

represent, respectively, the set of $n \times m$ quaternion, complex, real matrices; $SC_n(\mathbb{Q})$ denotes the set of self-conjugate matrices, $SC_n^>(\mathbb{Q})$ denotes the set of positive definite matrices of quaternions [18]; $x \in \mathbb{Q}^n$, x^* is the conjugate transpose of x; $A \in \mathbb{Q}^{n\times m}$, $\bar{A}$, A^*, and $\lambda_{min}(A)$ represent, respectively, the conjugate, the conjugate transpose, and the minimum eigenvalue of A [19].

Considering the QVNN as follow:

$$\frac{dx(t)}{dt} = -Dx(t-\tau_1) + Ag(x(t)) + Bg\big(x(t-\tau(t))\big) + C\int_{t-\tau_2}^{t} g\big(x(s)\big)\,ds + v \tag{1}$$

$x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{Q}^n$ is the state vector. $D \in \mathbb{R}^{n\times n}$, with $D = diag(d_1, d_2, ..., d_n) \succ 0$ means the self-feedback connection weight matrix. $A, B, C \in \mathbb{Q}^{n\times n}$ mean the connection weight matrix. $\tau_1, \tau_2, \tau(t)$ denote, respectively, the leakage time delay, the distributed constant time delay, and the non-differentiable discrete time-varying delay. $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), ..., g_n(x_n(t)))^T \in \mathbb{Q}^n$ denotes activation function. $v = (v_1, v_2, ..., v_n) \in \mathbb{Q}^n$ means the external input vector. $\bar{\lambda}(A)$ is the minimum of eigenvalues matrix A.

Assumption 1. *There are positive constants δ_l, such that:*

$$\| g_l(x) - g_l(x') \| \le \delta_l \| x - x' \|,$$

where $l = 1, 2, ..., n$. We define matrix:

$$\Gamma = diag(\delta_1, \delta_2, ..., \delta_n),$$

Definition 1. *μ-stability: $\mu(t) \ge 0$ is continuous function, when $t \to \infty$, $\mu(t) \to \infty$. If there exists a constant ω, then*

$$\|x(t)\| \le \frac{\omega}{\mu(t)}.$$

Lemma 1 ([20]). $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbb{Q}^{2n\times 2n}$ *where $G_{11} = G_{11}^*$, $G_{12} = G_{21}^*$, $G_{22} = G_{22}^*$, then $G \prec 0$ is equivalent two conditions:*

(1) $G_{22} \prec 0$, $G_{11} - G_{12}G_{22}^{-1}G_{12}^* \prec 0$;
(2) $G_{11} \prec 0$, $G_{22} - G_{12}^*G_{11}^{-1}G_{12} \prec 0$.

Lemma 2. *$H(x)$ is specified as a homeomorphism of $\mathbb{Q}^n$ onto itself, if $H(x) : \mathbb{Q}^n \to \mathbb{Q}^n$ is a continuous map accord with two qualifications:*

(1) $H(x)$ is a injective on $\mathbb{Q}^n$;
(2) $\lim_{\|x\|\to\infty} H(x) = \infty$.

Lemma 3 ([3]). *For the Hermitian constant matrix $W \in \mathbb{Q}^{n\times n}$, $W \ge 0$, and scalar function $g : [n, m] \to \mathbb{Q}^n$, $n \le m$, then*

$$\left(\int_n^m g(s)\,ds\right)^* W \left(\int_n^m g(s)\,ds\right) \le (m-n)\int_n^m g^*(s)Wg(s)\,ds.$$

Lemma 4 **([3]).** *Let $q, \check{q} \in \mathbb{Q}^n$, $Q \in SC_n^{>}(\mathbb{Q})$, then*

$$q^*\check{q} + \check{q}^*q \leq \frac{1}{\varepsilon}q^*Q^{-1}q + \varepsilon\check{q}^*Q\check{q}.$$

3 Main Result

Theorem 1. *If Assumption 1 holds, the equilibrium of the QVNN (1) is unique, when there are positive matrix U and positive matrices J_1, J_2, J_3, such that*

$$\Sigma = \begin{bmatrix} \Sigma_1 & UA & UB & UC \\ * & -J_1 & 0 & 0 \\ * & * & -J_2 & 0 \\ * & * & * & -J_3 \end{bmatrix} \prec 0 \tag{2}$$

where

$$\Sigma_1 = -UD - DU + \Gamma J_1 \Gamma + \Gamma J_2 \Gamma + \tau^2 \Gamma J_3 \Gamma$$

Proof. Establishing a continuous map

$$H(x) = -Dx + Ag(x) + Bg(x) + \tau Cg(x) + V \tag{3}$$

we assume there are two different vectors $x_1, x_2 \in \mathbb{Q}^n$, such that $H(x_1) = H(x_2)$,

$$\begin{aligned} &-(x_1 - x_2)^*(UD + DU)(x_1 - x_2) + (x_1 - x_2)^*U(A + B)\big(g(x_1) - g(x_2)\big)^*J)_1 \\ &\cdot \big(g(x_1) - g(x_2)\big) + \tau(x_1 - x_2)^*UC\big(g(x_1) - g(x_2)\big) + \tau\big(g(x_1) - g(x_2)\big)^*C^*U \\ &\cdot (x_1 - x_2) = 0 \end{aligned} \tag{4}$$

Based on Lemma 4,

$$\begin{aligned} &(x_1 - x_2)^*U(A + B)\big(g(x_1) - g(x_2)\big)^*J)_1\big(g(x_1) - g(x_2)\big) \\ \leq &(x_1 - x_2)^*UAJ_1^{-1}A^*U(x_1 - x_2) + \big(g(x_1) - g(x_2)\big)^*J_1\big(g(x_1) - g(x_2)\big) + (x_1 \\ &- x_2)^*UBJ_2^{-1}B^*U(x_1 - x_2) + \big(g(x_1) - g(x_2)\big)^*J_2\big(g(x_1) - g(x_2)\big) \end{aligned} \tag{5}$$

$$\begin{aligned} &\tau(x_1 - x_2)^*UC\big(g(x_1) - g(x_2)\big) + \tau\big(g(x_1) - g(x_2)\big)^*C^*U(x_1 - x_2) \\ \leq &(x_1 - x_2)^*UCJ_3^{-1}C^*U(x_1 - x_2) + \tau^2\big(g(x_1) - g(x_2)\big)^*J_3\big(g(x_1) - g(x_2)\big) \end{aligned} \tag{6}$$

$$\big(g(x_1) - g(x_2)\big)^*J_l\big(g(x_1) - g(x_2)\big) \leq (x_1 - x_2)^*\Gamma J_l \Gamma(x_1 - x_2) \tag{7}$$

where $l = 1, 2, 3$.

$$\begin{aligned} &(x_1 - x_2)^*(-UD - DU + UAJ_1^{-1}A^*U + UBJ_2^{-1}B^*U \\ &+ UCJ_3^{-1}C^*U + \Gamma J_1\Gamma + \Gamma J_2\Gamma + \tau^2\Gamma J_3\Gamma)(x_1 - x_2) \geq 0 \end{aligned} \tag{8}$$

According to Lemma 1 and $\varSigma \prec 0$, the following inequality can be obtained

$$\begin{aligned} &-UD - DU + UAJ_1^{-1}A^*U + UBJ_2^{-1}B^*U \\ &+ UCJ_3^{-1}C^*U + \varGamma J_1 \varGamma + \varGamma J_2 \varGamma + \tau^2 \varGamma J_3 \varGamma \prec 0, \end{aligned} \tag{9}$$

therefore, $H(x)$ is an injective on $\mathbb{Q}$. Besides,

$$\begin{aligned} & x^*U\big(H(x) - H(0)\big) + \big(H(x) - H(0)\big)^* Ux \\ \leq & x^*(-UD - DU + UAJ_1^{-1}A^*U + UBJ_2^{-1}B^*U + UCJ_3^{-1}C^*U + \varGamma J_1 \varGamma + \varGamma J_2 \varGamma \\ & + \tau^2 \varGamma J_3 \varGamma) x \\ \leq & -\bar{\lambda}(-UD - DU + UAJ_1^{-1}A^*U + UBJ_2^{-1}B^*U + UCJ_3^{-1}C^*U + \varGamma J_1 \varGamma + \varGamma J_2 \\ & \cdot \varGamma + \tau^2 \varGamma J_3 \varGamma) x^* x \\ = & \lambda x^* x \end{aligned} \tag{10}$$

$$\lambda \parallel x \parallel^2 \leq 2 \parallel x \parallel \parallel U \parallel \big(\parallel H(x) \parallel + \parallel H(0) \parallel \big) \tag{11}$$

$$\lambda \parallel x \parallel \leq 2 \parallel U \parallel \big(\parallel H(x) \parallel + \parallel H(0) \parallel \big) \tag{12}$$

Consequently, $\parallel H(x) \parallel \to \infty$, as $\parallel x \parallel \to \infty$, an unique equilibrium point of the QVNN (1) is proved.

Choosing the variable substitution $\widetilde{x}(t) = x(t) - \hat{x}$, $\hat{x}$ is the equilibrium point of the QVNN (1), the following system is obtained from the QVNN (1):

$$\frac{d\widetilde{x}(t)}{dt} = -D\widetilde{x}(t - \tau_1) + Ag\big(\widetilde{x}(t)\big) + Bg\big(\widetilde{x}(t - \tau(t))\big) + C \int_{t-\tau_2}^{t} g(\widetilde{x}(s))\, ds,$$

Theorem 2. *The QVNN (1) is global $\mu-$stability, if $t \geq T$, $max\{\tau_1, \tau_2, \tau(t)\} \leq \tau$, there are constants α, β, such that $\frac{\dot{\mu}(t)}{\mu(t)} \leq \beta$, $\eta \leq \frac{\mu(t-\tau)}{\mu(t)} \leq \alpha$, and matrices $R_1, R_2, R_3, R_4, R_5 \in SC_n^{>}(Q)$, the following matrix $\varPi$ exists:*

$$\varPi = \begin{bmatrix} \varPi_1 & R_1 & R_3 & \eta^2 R_2 & 0 & 0 & 0 \\ * & \tau_1 R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & 2\beta R_3 & -R_3 & 0 & 0 & 0 \\ * & * & * & -\eta^2 R_2 & 0 & 0 & 0 \\ * & * & * & * & R_4 & 0 & 0 \\ * & * & * & * & * & -\eta^2 R_4 & 0 \\ * & * & * & * & * & * & -\eta^2 R_5 \end{bmatrix} \prec 0 \tag{13}$$

$$\varPi_1 = 2\beta R_1 - \eta^2 R_2 + \tau_2 R_5$$

Proof. Using Lyapunov-Krasovskii function as below,

$$V(t) = \sum_{q=1}^{5} V_q(t) \tag{14}$$

$$\begin{aligned}
V_1(t) =&\mu^2(t)\widetilde{x}^*(t)R_1\widetilde{x}(t),\\
V_2(t) =&\tau_1 \int_0^{\tau_1}\int_{t-\theta}^{t} \mu^2(s)\dot{\widetilde{x}}^*(s)R_2\dot{\widetilde{x}}(s)dsd\theta,\\
V_3(t) =&\mu^2(t)\big(\int_{t-\tau_1}^{t} \widetilde{x}(s)ds\big)^* R_3\big(\int_{t-\tau_1}^{t} \widetilde{x}(s)ds\big),\\
V_4(t) =& \int_{t-\tau}^{t} \mu^2(s)g^*\big(\widetilde{x}(s)\big)R_4 g\big(\widetilde{x}(s)\big)ds,\\
V_5(t) =&\tau_2 \int_0^{\tau_2}\int_{t-\theta}^{t} \mu^2(s)\widetilde{x}^*(s)R_5\widetilde{x}(s)dsd\theta.
\end{aligned}$$

The derivative of $V(t)$ along the trajectories of the QVNN (1) is:

$$\begin{aligned}
\dot{V}_1(t) \;&= 2\mu(t)\dot{\mu}(t)\widetilde{x}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\dot{\widetilde{x}}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\widetilde{x}^*(t)R_1\dot{\widetilde{x}}(t)\\
&\le 2\beta\mu^2(t)\widetilde{x}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\dot{\widetilde{x}}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\widetilde{x}^*(t)R_1\dot{\widetilde{x}}(t)
\end{aligned} \tag{15}$$

$$\begin{aligned}
\dot{V}_2(t) =& \tau_1 \int_0^{\tau_1} \mu^2(t)\dot{\widetilde{x}}^*(t)R_2\dot{\widetilde{x}}(t) - \mu^2(t-\theta)\cdot\dot{\widetilde{x}}^*(t-\theta)R_2\dot{\widetilde{x}}(t-\theta)d\theta\\
\le&\tau_1^2\mu^2(t)\dot{\widetilde{x}}^*(t)R_2\dot{\widetilde{x}}(t) - \eta^2\tau_1\mu^2(t)\int_0^{\tau_1}\dot{\widetilde{x}}^*(t-\theta)R_2\dot{\widetilde{x}}(t-\theta)d\theta\\
\le&\tau_1^2\mu^2(t)\dot{\widetilde{x}}^*(t)R_2\dot{\widetilde{x}}(t) - \eta^2\mu^2(t)\big(\widetilde{x}^*(t)R_2\widetilde{x}(t) - \widetilde{x}^*(t)R_2\widetilde{x}(t-\tau_1) - \widetilde{x}^*(t\\
&- \tau_1)R_2\widetilde{x}(t) + \widetilde{x}^*(t-\tau_1)R_2\widetilde{x}(t-\tau_1)\big)
\end{aligned} \tag{16}$$

$$\begin{aligned}
\dot{V}_3(t) =& \mu^2(t)\big(\widetilde{x}(t)-\widetilde{x}(t-\tau_1)\big)^* R_3\int_{t-\tau_1}^{t}\widetilde{x}(s)ds + 2\mu(t)\dot{\mu}(t)\big(\int_{t-\tau_1}^{t}\widetilde{x}(s)ds\big)^* R_3\\
&\cdot\int_{t-\tau_1}^{t}\widetilde{x}(s)ds + \mu^2(t)\big(\int_{t-\tau_1}^{t}\widetilde{x}(s)ds\big)^* R_3\big(\widetilde{x}(t)-\widetilde{x}(t-\tau_1)\big)
\end{aligned} \tag{17}$$

$$\dot{V}_4(t) = \mu^2(t)g^*\big(\widetilde{x}(t)\big)R_4 g\big(\widetilde{x}(t)\big) - \mu^2(t-\tau)g^*\big(\widetilde{x}(t-\tau)\big)R_4 g\big(\widetilde{x}(t-\tau)\big) \tag{18}$$

$$\begin{aligned}
\dot{V}_5(t) =& \tau_2\int_0^{\tau_2}\mu^2(t)\widetilde{x}^*(t)R_5\widetilde{x}(t) - \mu^2(t-\theta)\widetilde{x}^*(t-\theta)R_5\widetilde{x}(t-\theta)d\theta\\
\le&\mu^2(t)\big(\tau_2^2\widetilde{x}^*(t)R_5\widetilde{x}(t) - \eta^2\tau_2\int_{t-\tau_2}^{t}\mu^2(s)\widetilde{x}^*(s)R_5\widetilde{x}(s)ds\big)\\
\le&\mu^2(t)\Big(\tau_2\widetilde{x}^*(t)R_5\widetilde{x}(t) - \eta^2\big(\int_{t-\tau_2}^{t}\widetilde{x}(s)ds\big)^* R_5\int_{t-\tau_2}^{t}\widetilde{x}(s)ds\Big)
\end{aligned} \tag{19}$$

With (14)–(19),

$$\begin{aligned}
D^+V(t) \leq& 2\beta\mu^2(t)\widetilde{x}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\dot{\widetilde{x}}^*(t)R_1\widetilde{x}(t) + \mu^2(t)\widetilde{x}^*(t)R_1\dot{\widetilde{x}}(t) + \tau_1^2\mu^2(t) \\
&\cdot \dot{\widetilde{x}}^*(t)R_2\dot{\widetilde{x}}(t) - \eta^2\mu^2(t)\big(\widetilde{x}^*(t)R_2\widetilde{x}(t) - \widetilde{x}^*(t)R_2\widetilde{x}(t-\tau_1) - \widetilde{x}^*(t-\tau_1) \\
&\cdot R_2\widetilde{x}(t) + \widetilde{x}^*(t-\tau_1)R_2\widetilde{x}(t-\tau_1)\big)\mu^2(t)\big(\widetilde{x}(t) - \widetilde{x}(t-\tau_1)\big)^*R_3 \\
&\cdot \int_{t-\tau_1}^t \widetilde{x}(s)ds + 2\mu(t)\dot{\mu}(t)\big(\int_{t-\tau_1}^t \widetilde{x}(s)ds\big)^*R_3\int_{t-\tau_1}^t \widetilde{x}(s)ds + \mu^2(t) \\
&\cdot \big(\int_{t-\tau_1}^t \widetilde{x}(s)ds\big)^*R_3\big(\widetilde{x}(t) - \widetilde{x}(t-\tau_1)\big)\mu^2(t)g^*\big(\widetilde{x}(t)\big)R_4g\big(\widetilde{x}(t)\big) - \mu^2(t \\
&- \tau)g^*\big(\widetilde{x}(t-\tau)\big)R_4g\big(\widetilde{x}(t-\tau)\big) + \mu^2(t)\Big(\tau_2\widetilde{x}^*(t)R_5\widetilde{x}(t) - \eta^2 \\
&\cdot \big(\int_{t-\tau_2}^t \widetilde{x}(s)ds\big)^*R_5\int_{t-\tau_2}^t \widetilde{x}(s)ds\Big) \\
\leq& \mu^2(t)\Big(2\beta\widetilde{x}^*(t)R_1\widetilde{x}(t) + \dot{\widetilde{x}}^*(t)R_1\widetilde{x}(t) + \widetilde{x}^*(t)R_1\dot{\widetilde{x}}(t) + \tau_1^2\dot{\widetilde{x}}^*(t)R_2\dot{\widetilde{x}}(t) \\
&- \eta^2\big(\widetilde{x}^*(t)R_2\widetilde{x}(t) - \widetilde{x}^*(t)R_2\widetilde{x}(t-\tau_1) - \widetilde{x}^*(t-\tau_1)R_2\widetilde{x}(t) + \widetilde{x}^*(t-\tau_1) \\
&\cdot R_2\widetilde{x}(t-\tau_1)\big)\big(\widetilde{x}(t) - \widetilde{x}(t-\tau_1)\big)^*R_3\int_{t-\tau_1}^t \widetilde{x}(s)ds + 2\beta\big(\int_{t-\tau_1}^t \widetilde{x}(s)ds\big)^* \\
&\cdot R_3\int_{t-\tau_1}^t \widetilde{x}(s)ds + \big(\int_{t-\tau_1}^t \widetilde{x}(s)ds\big)^*R_3\big(\widetilde{x}(t) - \widetilde{x}(t-\tau_1)\big)g^*\big(\widetilde{x}(t)\big)R_4 \\
&\cdot g\big(\widetilde{x}(t)\big) - \eta^2g^*\big(\widetilde{x}(t-\tau)\big)R_4g\big(\widetilde{x}(t-\tau)\big) + \Big(\tau_2\widetilde{x}^*(t)R_5\widetilde{x}(t) - \eta^2 \\
&\cdot \big(\int_{t-\tau_2}^t \widetilde{x}(s)ds\big)^*R_5\int_{t-\tau_2}^t \widetilde{x}(s)ds\Big)\Big) \\
=& \mu^2(t)q(t)\varPi q^*(t)
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
q(t) =& \Big(\widetilde{x}^*(t), \dot{\widetilde{x}}(t), \big(\int_{t-\tau_1}^t \widetilde{x}(s)ds\big)^*, \widetilde{x}^*(t-\tau_1), \\
& g^*\big(\widetilde{x}(t)\big), g^*\big(\widetilde{x}(t-\tau)\big), \big(\int_{t-\tau_2}^t \widetilde{x}(s)ds\big)^*\Big).
\end{aligned}$$

With (13) and (20),

$$D^+V(t) \leq 0. \tag{21}$$

For $t \in [T, +\infty)$,

$$\mu^2(t)\bar{\lambda}(R_1)\|\widetilde{x}(t)\|^2 \leq V(t) \leq V_0 = \max_{0<s<T} V(s) \tag{22}$$

Hence,

$$\| \widetilde{x}(t) \| \leq \frac{\omega}{\mu(t)}, \tag{23}$$

with

$$\omega = \sqrt{\frac{V_0}{\overline{\lambda}(R_1)}}.$$

Therefore, the QVNN (1) is global μ-stability.

4 Simulation Example

A numerical simulation example is presented to strengthen the new conclusions above.

Considering the following QVNN:

$$\frac{dx(t)}{dt} = - Dx(t-\tau_1) + Ag(x(t)) + Bg\big(x(t-\tau(t))\big) + C\int_{t-\tau_2}^{t} g\big(x(s)\big)\, ds + v \tag{24}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

we randomly define quaternion matrices of the QVNN: $a_{11} = 0.27 - 0.35i + 0.043j - 0.18k, a_{12} = -0.15 - 0.22i + 0.28j - 0.04k, a_{21} = -0.31 - 0.293i + 0.2j - 0.06k, a_{22} = 0.175 - 0.2i + 0.18j - 0.165k; b_{11} = -0.38 + 0.223i - 0.57j - 0.139k, b_{12} = 0.16 - 0.121i + 0.031j - 0.09k, b_{21} = 0.93 - 0.701i + 0.02j - 0.27k, b_{22} = 0.0907 - 0.24i + 0.104j + 0.08k; c_{11} = -0.231 + 0.208i - 0.179j + 0.05k, c_{12} = 0.77 + 0.01i + 0.04j - 1.23k, c_{21} = 0.092 - 0.113i + 0.25j - 0.7k, c_{22} = -0.336 + 0.134i - 0.22j + 0.4k; v_1 = 0.052 - 0.14i + 0.08j - 0.12k, v_2 = -0.58 + 0.304i - 0.19j + 0.145k.$

Define $g(x(t)) = tanh(x(t))$, $\tau_1 = 0.4$, $\tau_2 = 0.3$, $\tau(t) = 0.37|sint|$, $\mu = e^{0.8t}$, obviously, $\tau = 0.4$, $\beta = 0.8, \eta = 0.6, \alpha = 0.8$.

The solutions of $\Sigma \prec 0$ and $\Pi \prec 0$ can be resolved as:

$$\Sigma = \begin{bmatrix} \Sigma_1 & UA & UB & UC \\ * & -J_1 & 0 & 0 \\ * & * & -J_2 & 0 \\ * & * & * & -J_3 \end{bmatrix} \prec 0$$

$$\Pi =$$
$$\begin{bmatrix} \Pi_1 & R_1 & R_3 & \eta^2 R_2 & 0 & 0 & 0 \\ * & \tau_1 R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & 2\beta R_3 & -R_3 & 0 & 0 & 0 \\ * & * & * & -\eta^2 R_2 & 0 & 0 & 0 \\ * & * & * & * & R_4 & 0 & 0 \\ * & * & * & * & * & -\eta^2 R_4 & 0 \\ * & * & * & * & * & * & -\eta^2 R_5 \end{bmatrix}$$
$$\prec 0$$

$$\Pi_1 = 2\beta R_1 - \eta^2 R_2 + \tau_2 R_5$$

where

$$\Sigma_1 = -UD - DU + \Gamma J_1 \Gamma + \Gamma J_2 \Gamma + \tau^2 \Gamma J_3 \Gamma$$
$$U = 10^{-9} diag[0.1335, 0.1155, 0.1357, 0.0649]$$

$$J_1 = 10^{-10} \begin{bmatrix} j_1^{11} & j_1^{12} \\ j_1^{21} & j_1^{22} \end{bmatrix}, J_2 = 10^{-10} \begin{bmatrix} j_2^{11} & j_2^{12} \\ j_2^{21} & j_2^{22} \end{bmatrix}, J_3 = 10^{-14} \begin{bmatrix} j_3^{11} & j_3^{12} \\ j_3^{21} & j_3^{22} \end{bmatrix},$$
$$R_1 = 10^{-11} \begin{bmatrix} r_1^{11} & 0 \\ 0 & r_1^{22} \end{bmatrix}, R_2 = 10^{-10} \begin{bmatrix} r_2^{11} & 0 \\ 0 & r_2^{22} \end{bmatrix}, R_3 = 10^{-11} \begin{bmatrix} r_3^{11} & 0 \\ 0 & r_3^{22} \end{bmatrix},$$
$$R_4 = 10^{-11} \begin{bmatrix} r_4^{11} & 0 \\ 0 & r_4^{22} \end{bmatrix}, R_5 = 10^{-10} \begin{bmatrix} r_5^{11} & 0 \\ 0^{21} & r_5^{22} \end{bmatrix}.$$

$j_1^{11} = -0.1771 - 0.1771j; j_1^{12} = 0.0193 + 0.0111i + 0.0193j + 0.0111k; j_1^{21} = 0.0193 - 0.0111i + 0.0193j - 0.0111k; j_1^{22} = -0.1801 - 0.1801j; j_2^{11} = 0.03846 - 0.4789j; j_2^{12} = 0.00341 + 0.2058i + 0.0035j + 0.0023k; j_2^{21} = 0.00341 - 0.2058i + 0.0035j - 0.0023k; j_2^{22} = -0.7801 - 0.4674j; j_3^{11} = 0.189 + 0.189j; j_3^{12} = 0.0012 + 0.0032i + 0.0012j + 0.0012k; j_3^{21} = 0.0012 - 0.0032i + 0.0012j - 0.0032k; j_3^{22} = 0.2105 + 0.2105j; r_1^{11} = -0.5063 - 0.5063j; r_1^{22} = -0.5063 - 0.5063j; r_2^{11} = 0.3272 + 0.3272j; r_2^{22} = 0.3272 + 0.3272j; r_3^{11} = -0.2932 - 0.2932j; r_3^{22} = -0.2932 - 0.2932j; r_4^{11} = 0.4514 + 0.451j; r_4^{22} = 0.4514 + 0.451j; r_5^{11} = 0.8523 + 0.8523j; r_5^{22} = 0.8523 + 0.8523j.$

Hence the QVNN (24) has unique equilibrium and is global μ-stability under Theorem 1 and Theorem 2.

Figure 1, 2, 3 and 4 respectively are the trajectories of $x^R(t)$, $x^I(t)$, $x^J(t)$, $x^K(t)$ in QVNN (4.1) with original value $x(0) = (-0.5 - 0.32i + 0.28j - 0.541k, 0.69 + 1.05i - 1.1j + 1.2k)$.

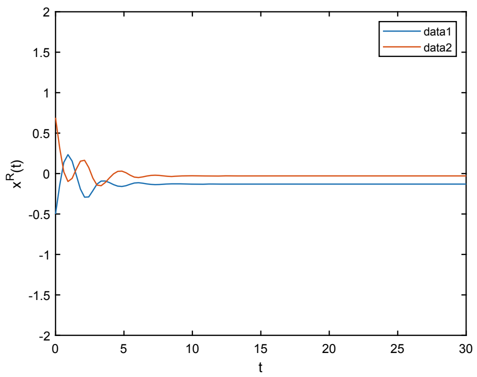

Fig. 1. The trajectories of $x^R(t)$ in the QVNN (24)

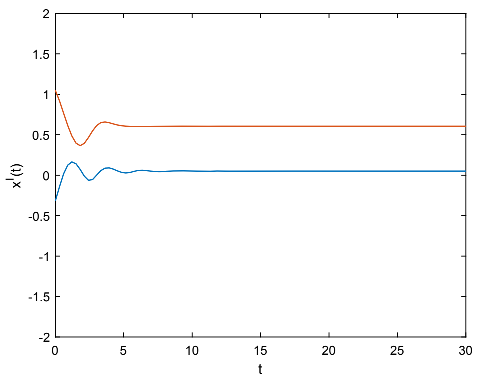

Fig. 2. The trajectories of $x^I(t)$ in the QVNN (24)

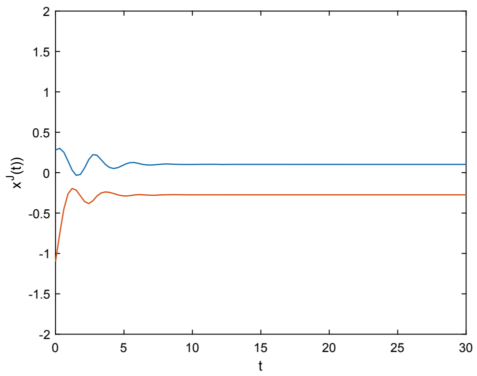

Fig. 3. The trajectories of $x^J(t)$ in the QVNN (24)

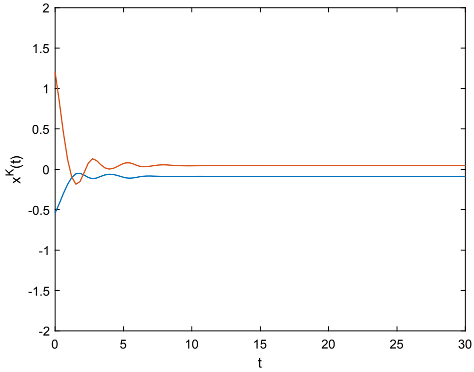

Fig. 4. The trajectories of $x^K(t)$ in the QVNN (24)

5 Conclusion

This paper discuss the equilibrium's uniqueness and the global μ-stability of the QVNN (quaternion-valued neural network) with leaky constant delay, non-differentiable discrete time-varying delay, distributed constant delay. Firstly, considering the QVNN as entirety, the uniqueness of the QVNN's equilibrium is obtained, via Homeomorphism mapping theorem and quaternion-valued linear matrix inequality. Next a novel Lyapunov-Krasovskii functional is derived according to the delayed state, and the sufficient condition of the global μ-stability is presented. Finally, we give a quaternion-valued numerical example to illustrate these new results.

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