Numerical Solution of Robin-Dirichlet Problem for a Nonlinear Wave Equation with Memory Term

Le Thi Mai Thanh[1,2,3], Tran Trinh Manh Dung[2,3], and Nguyen Huu Nhan[1](✉)

[1] Nguyen Tat Thanh University, 300A Nguyen Tat Thanh Street, Dist. 4, Ho Chi Minh City, Vietnam
{ltmthanh,nhnhan}@ntt.edu.vn
[2] Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, 227 Nguyen Van Cu Street, Dist. 5, Ho Chi Minh City, Vietnam
[3] Vietnam National University, Ho Chi Minh City, Vo Truong Toan Street, Linh Trung Ward, Thu Duc City, Vietnam

Abstract. In this paper, we propose a numerical procedure to find the approximate solution of initial-boundary value problem for a nonlinear wave equation with memory term. First, the local existence and uniqueness is proved by the linear approximating method and the Faedo-Galerkin method. Next, a numerical scheme is constructed by the finite-difference method and the standard arguments of ordinary differential equation. Finally, an example is given to simulate the numerical results of the proposed algorithm.

Keywords: Wave equation · Faedo-Galerkin method · Finite-difference method · Robin-Dirichlet condition · Memory term

AMS subject classification: 35L05 · 35L15 · 35L20 · 35L55 · 35L70

1 Introduction

In the present paper, we are concerned with the following initial-boundary value problem with memory term

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + \int_0^t g(t-s)\frac{\partial}{\partial x}\left(\bar{\mu}(x,s)u_x(x,s)\right)ds = f(x,t,u,u_t),$$
$$0 < x < 1, 0 < t < T, \tag{1}$$

$$u_x(0,t) - u(0,t) = u(1,t) = 0, \tag{2}$$

$$u(x,0) = \tilde{u}_0(x), u_t(x,0) = \tilde{u}_1(x), \tag{3}$$

where μ, $\bar{\mu}$, g, f, $\tilde{u}_0$ and $\tilde{u}_1$ are given functions which satisfy conditions specified later.

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As is well known, the integral term in the Eq. (1) is the memory term responsible for the viscoelastic property. The wave equation with memory term (or so called viscoelastic term) is arised in studies about viscoelastic materials, which stands for a capacity of storage and dissipation of mechanical energy. The dynamic properties of viscoelastic materials are very importance and interest because they appear in many applied sciences, for more literature on this topic, one can see [1,3,4] and references therein.

During few past decades, the mathematical models similar to the problem (1)–(3) have been extensively studied by many researchers. Indeed, for multi-dimension spatial problems, numerous works are dedicated to the investigation of the general form of viscoelastic wave equation obtained by

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \lambda \Delta u_t + \gamma h\left(u_t\right) = \mathcal{F}(u), \tag{4}$$

see [2,5,6,9,12,17] for details and their references. When $\lambda = 0,\ \gamma = 1,\ h \equiv u_t, \mathcal{F} \equiv u\left|u\right|^{p-1}$, Kafin and Messaoudi [5] investigated the following Cauchy problem on $\mathbb{R}^n$, in which some conditions have been put on the kernel g to get the finite-time blow up of solution of the corresponding problem. Recently, Li and He [6] have studied the Eq. (4) for a bounded domain $\Omega \subset \mathbb{R}^n$ and $\lambda = \gamma = 1,\ h \equiv u_t, \mathcal{F} \equiv u\left|u\right|^{p-2}$, i.e., they considered the following nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \Delta u_t + u_t = u\left|u\right|^{p-2}, \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Then the global existence, general decay and blow-up properties of solutions for the above model were proved. For one-dimension spatial problems, some interesting results of existence, exponential decay and asymptotic expansion have been investigated in different contexts. For example, in [19], Ngoc et al. proved a local existence of the wave equation with nonlinear convolution as follows

$$\begin{aligned} u_{tt} - \lambda u_{xxt} &- \frac{\partial}{\partial x}\left[\mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right)u_x\right] \\ &+ \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2\left(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2\right)u_x(x,s)\right]ds \\ &= F\left(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2\right), 0<x<1, 0<t<T, \end{aligned} \tag{6}$$

associated with Robin-Dirichlet boundary conditions (2) and initial conditions (3), where $\lambda > 0$ is a constant, and $\mu_1,\ \mu_2,\ g,\ f$ are given functions which satisfy some certain conditions. Moreover, the authors established an approximation of weak solution by the asymptotic expansion method, satisfying an estimation with N order. Before, Long and his colleagues [7] have discussed the equation (6) in case of $\lambda = 0,\ \mu_1 = \mu_2 = 1,\ F \equiv f(x,t,u) - \left|u_t\right|^{q-2}u_t$, associated with mixed nonhomogeneous boundary conditions and initial conditions (3).

Then they proved the global existence and exponential decay of weak solution of the corresponding model, in addition an example was given to present the numerical approximating solution of the problem.

Although a numerous of works studying of solution properties to viscoelastic problems were published, however, it seems that few publications of this type coupled with numerical results are considered. For some last decades, there has been much effort to develop different numerical methods for the solution of partial differential equations with viscoelastic term, see for instance as in [7,13, 16,19,20] and their references. In [19], Quynh et al. considered a specific form of (6) with $\lambda = 0,\ \mu_1 = \mu_2 = 1,\ F \equiv f(x,t,u) - \alpha\,|u_t|^{p-2}\,u_t$ together with Robin boundary conditions

$$\begin{aligned} u_{tt} - u_{xx} + \alpha\,|u_t|^{p-2}\,u_t + \int_0^t g(t-s)u_{xx}(x,s)ds \\ = f(x,t,u), 0 < x < 1, 0 < t < T. \end{aligned} \tag{7}$$

The authors established the existence of N-order recurrent sequence associated with the Eq. (7) and proved that this sequence converges to the unique solution of the Eq. (7). Moreover, by using finite-difference formulas, they constructed an algorithm in order to find numerical solutions given by the 2-order iterative scheme (when $N = 2$). Before, in [7,13], the authors have also used finite-difference approximation to establish numerical solution for wave equations with viscoelastic term or with integral boundary conditions, respectively. Beside finite-difference method, several methods were developed to solve partial differential equations such as cubic spline approximation or finite-element method. For example, Mohanty and Gopal [8] applied the high accuracy cubic spline finite difference method for one-dimensional nonlinear wave equation, in which the application of the proposed method to telegraphic equation and wave equation in polar coordinates was considered, and their stability was also analyzed. Furthermore, the comparisons between the numerical results of proposed high accuracy cubic spline finite difference method and the corresponding second order accuracy cubic spline method were discussed. In [20], Saedpanah studied the initial-boundary value problem with memory

$$\begin{aligned} &u''(t) - Au(t) + \int_0^t K(t-s)Au(s)ds = f(t), t \in (0,T), \\ &u(0) = u^0,\ u'(0) = u^1, \end{aligned} \tag{8}$$

(u' is used for $\dfrac{du}{dt}$), associated with a homogeneous Dirichlet boundary condition or with a mixed Dirichlet-Neumann boundary condition, where A is a self-adjoint, positive definite, second order elliptic linear operator on a certain separable Hilbert space. The kernel K is considered to be either smooth (exponential), or no worse than weakly singular. By using Picard's iteration, the existence and uniqueness of the spatial local and global Galerkin approximation of the problem (8) were proved. Then, spatial finite element scheme of the problem was constructed and optimal order a priori estimates were proved by the energy method. Finally, the author gave a numerical example to illustrate the order of convergence of the spatial finite element discretization to a concrete problem with smooth convolution kernel. In our paper, in Sect. 2, we first introduce some notations and inequalities of compact imbeddings in Hilbert spaces. Next, in Sect. 3, we present the theorem of existence and uniqueness of the problem (8) of which their proofs are the same as in [14,15,19]. In Sect. 4, we construct an algorithm in order to find the approximate solution of the problem (1)–(3) by using the finite-difference formulas, and an example is also given to illustrate the exact solution and the finite-difference approximate solution. Finally, in Sect. 5, we summarize the obtained results in our paper.

2 Preliminaries

In this section, we present some notations and materials in order to present main results.

Let $\Omega = (0,1),\ Q_T = (0,1) \times (0,T)$ and we define the scalar product in L^2 by

$$\langle u,\ v\rangle = \int_0^1 u(x)v(x)dx,$$

and the corresponding norm $\|\cdot\|$, i.e., $\|u\|^2 = \langle u,\ u\rangle$. Let us denote the standard function spaces by $C^m(\overline{\Omega})$, $L^P = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \le p \le \infty$ and $m \in \mathbb{N}$. Also, we denote that $\|\cdot\|_X$ is a norm in a certain Banach space X, and $L^p(0,T;X)$, $1 \le p \le \infty$, is the Banach space of real functions $u : (0,T) \to X$ measurable with the corresponding norm $\|\cdot\|_{L^p(0,T;X)}$ defined by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < \infty\ for 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup}\, \|u(t)\|_X \ \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

For $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$, $f = f(x,t,y_1,y_2)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{2+i} f = \frac{\partial f}{\partial y_i}$, $i = 1,2$ and $D^\alpha f = D_1^{\alpha_1} ... D_4^{\alpha_4} f$; $\alpha = (\alpha_1, ..., \alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| = \alpha_1 + ... + \alpha_4 = k$, $D^{(0,\ldots,0)} f = D^{(0)} f = f$.

Put

$$V = \{v \in H^1 :\ v(1) = 0\}, \tag{9}$$

which is a closed subspace of H^1.

We consider a symmetric bilinear form $a(\cdot,\cdot)$ defined by

$$a(u,v)=\int_0^1 u_x(x)v_x(x)dx+u(0)v(0),\ \text{with } u,v\in V, \tag{10}$$

and corresponding norm $\|v\|_V=\sqrt{a(v,v)}$.

Then, it is easy to prove the following inequalities (see [15])

(i) $\|v\|_{C^0(\bar{\Omega})}\leq\|v_x\|\leq\|v\|_V$, for all $v\in V$,
(ii) $|a(u,v)|\leq 2\|u\|_V\|v\|_V$, for all $u,\ v\in V$,
(iii) $a(v,v)\geq\|v_x\|^2$, for all $v\in V$.

3 Existence and Uniqueness

In this section, we consider the local existence and uniqueness of the problem (1)–(3). By using the linear approximate method and the Faedo-Galerkin method, we shall prove that there exists a recurrent sequence converging to the weak solution of (1)–(3) defined as below.

Definition 1. *A function u is called a weak solution of initial-boundary value problem (1)–(3) if* $u\in W_T=\left\{u\in L^\infty(0,T;H^2\cap V):u'\in L^\infty(0,T;V),u''\in L^\infty(0,T;L^2)\right\}$ *and satisfies the variational equation*

$$\langle u''(t),v\rangle+A(t;u(t),v)-\int_0^t g(t-s)\bar{A}(s;u(s),v)ds$$
$$=\langle f(t,u(t),u'(t)),v\rangle,\forall v\in V, \tag{11}$$

together with initial conditions

$$u(0)=\tilde{u}_0,u'(0)=\tilde{u}_1, \tag{12}$$

where

$$A(t;u,v)=\langle\mu(t)u_x,v_x\rangle+\mu(0,t)u(0)v(0), \tag{13}$$
$$\bar{A}(t;u,v)=\langle\bar{\mu}(t)u_x,v_x\rangle+\bar{\mu}(0,t)u(0)v(0), \tag{14}$$

Consider a fixed constant $T^*>0$, we make the following assumptions:

(H_1) $(\tilde{u}_0,\tilde{u}_1)\in\left(V\cap H^2\right)\times V,\ \ \tilde{u}_{0x}(0)-\tilde{u}_0(0)=0$;
(H_2) $g\in H^1(0,T^*)$;
(H_3) $f\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^2)$;
(H_4) $\mu\in C^2([0,1]\times[0,T^*])$ and there exists a constant $\mu_0>0$ such that $\mu(x,t)\geq\mu_0$ for all $(x,t)\in[0,1]\times[0,T^*]$;
(H_5) $\bar{\mu}\in C^1([0,1]\times[0,T^*])$.

For every $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T; V \cap H^2) : v_t \in L^\infty(0,T;V),\ v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } \max\{\|v\|_{L^\infty(0,T;V\cap H^2)}, \|v_t\|_{L^\infty(0,T;V)}, \|v_{tt}\|_{L^2(Q_T)}\} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \end{cases}$$

in which $Q_T = \Omega \times (0,T)$.

We consider the recurrent sequence $\{u_m\}$ satisfying the first term $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T), \tag{15}$$

then we find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + A(t; u_m(t), w) + \int_0^t g(t-s)\bar{A}(s; u_m(s), w)ds \\ \qquad\qquad\qquad = \langle F_m(t), w\rangle, \forall w \in V, \\ u_m(0) = \tilde{u}_0, u_m'(0) = \tilde{u}_1, \end{cases} \tag{16}$$

where

$$F_m(x,t) = f[u_{m-1}](x,t) = f\left(x,t,u_{m-1},u_{m-1}'\right). \tag{17}$$

In the next part, we present two theorems that confirm the existence and uniqueness of solutions of the problem (1)–(3). Actually, the first theorem (Theorem 1) claims the existence of the recurrent sequence defined by (15)–(17), after that the second theorem (Theorem 2) shows that the recurrent sequence converges to the unique weak solution of the problem (1)–(3). The proofs of these theorems can be proved similarly to the ones given in [14,15,19].

Theorem 1. *Suppose that $(H_1)-(H_5)$ hold. Then there exist positive constants M and T such that, with $u_0 \equiv \tilde{u}_0$, there exists the recurrent sequence $\{u_m\}$ defined by (15)–(17).*

Using the result of Theorem 1 and the compact imbedding theorems, we can establish the existence and uniqueness of weak solution of the problem (1)–(3) which is given the following theorem.

Theorem 2. *Suppose that $(H_1)-(H_5)$ hold. Then, the problem (1)–(3) has a unique weak solution $u \in W_1(M,T)$, where the constants $M > 0$ and $T > 0$ are suitably chosen.*

Moreover, the recurrent sequence $\{u_m\}$ defined by (15)–(17) strongly converges to u in the Banach space

$$W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\},$$

and the following estimate is confirmed

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m,\ for\ all\ m \in \mathbb{N}, \tag{18}$$

where $k_T \in [0,1)$ and C_T is a positive constant which depends on T, f, g, $\tilde{u}_0$, $\tilde{u}_1$ and k_T.

4 Numerical Results

Consider the following problem:

$$\begin{cases} u_{tt} - \dfrac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + \displaystyle\int_0^t g(t-s)\frac{\partial}{\partial x}\left(\bar{\mu}(x,s)u_x(x,s)\right)ds \\ \qquad\qquad\qquad\qquad = f(x,t,u,u_t), 0 < x < 1, 0 < t < T, \\ u_x(0,t) - u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{19}$$

where the functions $\mu(x,t), \bar{\mu}(x,t), g(t), f, \tilde{u}_0$ and $\tilde{u}_1$ are defined by

$$\begin{cases} \mu(x,t) = 1 + (1+x)e^{-\mu_0 t}, \bar{\mu}(x,t) = 1 + (1+x)e^{-\bar{\mu}_0 t}, g(t) = e^{-g_0 t}, \\ f(x,t,u,u_t) = -u_t^3 + |u|^3 u + F(x,t), \\ \tilde{u}_0(x) = (1-x)e^{2x}, \tilde{u}_1(x) = -(1-x)e^{2x}, \\ F(x,t) = (1-x)e^{2x}\left[1 - (1-x)^2 e^{4x} - (1-x)^3 e^{6x}\right] \\ \qquad + e^{2x-t}\left[4 - \left(1 - 6x - 4x^2\right)e^{-\mu_0 t}\right] - \dfrac{4}{g_0 - 1}e^{2x}\left(e^{-t} - e^{-g_0 t}\right) \\ \qquad + \dfrac{1}{g_0 - \bar{\mu}_0 - 1}\left(1 - 6x - 4x^2\right)e^{2x}\left(e^{-(\bar{\mu}_0+1)t} - e^{-g_0 t}\right), \\ g_0 = 2, \mu_0 = \bar{\mu}_0 = 1. \end{cases} \tag{20}$$

The exact solution of the problem (19), with $\mu(x,t)$, $\bar{\mu}(x,t)$, $g(t)$, f, $\tilde{u}_0$ and $\tilde{u}_1$ defined in (20) respectively, is the function u_{ex} given by

$$u_{ex}(x,t) = (1-x)e^{2x-t}. \tag{21}$$

In order to find the numerical solution of the problem (19), we use the difference given by [18] (pages 36 and 43) to approximate the second-order spatial derivatives and then transfer (19) to the following system of first-order ordinary differential equations with the unknowns $u_i(t) \equiv u(x_i,t)$, $v_i(t) = u_i'(t)$

$$\begin{cases} u_i'(t) = v_i(t), i = \overline{1,N}, \\ v_1'(t) = \dfrac{1}{h^2}\left[\alpha_1^*(t)u_1(t) + \alpha_2(t)u_2(t)\right] \\ \qquad -\dfrac{1}{h^2}\displaystyle\int_0^t g(t-s)\left[\bar{\alpha}_1^*(s)u_1(s) + \bar{\alpha}_2(t)u_2(s)\right]ds + f_1\left(t, u_1(t), v_1(t)\right), \\ v_i'(t) = \dfrac{1}{h^2}\left[\alpha_i(t)u_{i-1}(t) + \beta_i(t)u_i(t) + \alpha_{i+1}(t)u_{i+1}(t)\right] \\ \qquad -\dfrac{1}{h^2}\displaystyle\int_0^t g(t-s)\left[\bar{\alpha}_i(s)u_{i-1}(s) + \bar{\beta}_i(s)u_i(s) + \bar{\alpha}_{i+1}(s)u_{i+1}(s)\right]ds \\ \qquad + f_i\left(t, u_i(t), v_i(t)\right), i = \overline{2, N-1}, \\ v_N'(t) = \dfrac{1}{h^2}\left[\alpha_N(t)u_{N-1}(t) + \beta_N(t)u_N(t)\right] \\ \qquad -\dfrac{1}{h^2}\displaystyle\int_0^t g(t-s)\left[\bar{\alpha}_N(s)u_{N-1}(s) + \bar{\beta}_N(s)u_N(s)\right]ds + f_N\left(t, u_N(t), v_N(t)\right), \\ u_i(0) = \tilde{u}_0(x_i), v_i(0) = \tilde{u}_1(x_i), \ i = \overline{1,N}, \end{cases} \tag{22}$$

where $x_i = ih$, $h = \dfrac{1}{N+1}$, $i = \overline{0, N+1}$ and

$$\begin{aligned}
&\alpha_i(t) = \mu_i(t) = \mu(x_i, t),\\
&\beta_i(t) = -\alpha_i(t) - \alpha_{i+1}(t),\\
&\alpha_1^*(t) = -\frac{h}{1+h}\alpha_1(t) - \alpha_2(t),\\
&\bar{\alpha}_i(t) = \bar{\mu}_i(t) = \bar{\mu}(x_i, t),\\
&\bar{\beta}_i(t) = -\bar{\alpha}_i(t) - \bar{\alpha}_{i+1}(t),\\
&\bar{\alpha}_1^*(t) = -\frac{h}{1+h}\bar{\alpha}_1(t) - \bar{\alpha}_2(t),\\
&f_i\left(t, u_i(t), v_i(t)\right) = f\left(x_i, t, u_i(t), v_i(t)\right) = -v_i^3(t) + \left|u_i(t)\right|^3 u_i(t) + F(x_i, t), i = \overline{1, N}.
\end{aligned} \tag{23}$$

Then, the system (22) is equivalent to

$$\begin{cases} X'(t) = \hat{B}(t)X(t) - \displaystyle\int_0^t g(t-s)\hat{A}(s)X(s)ds + \mathcal{F}(t, X(t)),\\ X(0) = X_0, \end{cases} \tag{24}$$

where

$$\begin{cases} X(t) = \left(u_0(t), \cdots, u_N(t), v_0(t), \cdots, v_N(t)\right)^T \in \mathfrak{M}_{2N\times 1},\\ \mathcal{F}(t, X(t)) = \left(0, \cdots, 0, \mathcal{F}_1(t, X(t)), \cdots, \mathcal{F}_N(t, X(t))\right)^T \in \mathfrak{M}_{2N\times 1},\\ \mathcal{F}_i(t, X(t)) = f_i\left(t, u_i(t), v_i(t)\right) = -v_i^3(t) + \left|u_i(t)\right|^3 u_i(t) + F(x_i, t), i = \overline{1, N},\\ X_0 = \left(\tilde{u}_0(x_1), \cdots, \tilde{u}_0(x_N), \tilde{u}_1(x_1), \cdots, \tilde{u}_1(x_N)\right)^T \in \mathfrak{M}_{2N\times 1}, \end{cases} \tag{25}$$

$$\hat{A}(t) = \begin{bmatrix} O & O \\ \frac{1}{h^2}\bar{A}(t) & O \end{bmatrix} \in \mathfrak{M}_{2N}, \hat{B}(t) = \begin{bmatrix} O & E \\ \frac{1}{h^2}A(t) & O \end{bmatrix} \in \mathfrak{M}_{2N}, \tag{26}$$

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathfrak{M}_N, O = O_{N\times N} \in \mathfrak{M}_N,$$

$$A(t) = \begin{bmatrix} \alpha_1^*(t) & \alpha_2(t) & 0 & \cdots & \cdots & 0 \\ \alpha_2(t) & \beta_2(t) & \alpha_3(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \alpha_{N-1}(t) & \beta_{N-1}(t) & \alpha_N(t) \\ 0 & 0 & \cdots & 0 & \alpha_N(t) & \beta_N(t) \end{bmatrix} \in \mathfrak{M}_N, \tag{27}$$

$$\bar{A}(t) = \begin{bmatrix} \bar{\alpha}_1^*(t) & \bar{\alpha}_2(t) & 0 & \cdots & \cdots & 0 \\ \bar{\alpha}_2(t) & \bar{\beta}_2(t) & \bar{\alpha}_3(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \bar{\alpha}_{N-1}(t) & \bar{\beta}_{N-1}(t) & \bar{\alpha}_N(t) \\ 0 & 0 & \cdots & 0 & \bar{\alpha}_N(t) & \bar{\beta}_N(t) \end{bmatrix} \in \mathfrak{M}_N. \tag{28}$$

The nonlinear differential system (24) is solved by using the following linear recursive scheme generated by the nonlinear term $\mathcal{F}_i(t, X(t)) = f_i\left(t, u_i(t), v_i(t)\right) = -v_i^3(t) + \left|u_i(t)\right|^3 u_i(t) + F(x_i, t)$:

$$\begin{cases} \dfrac{dX^{(m)}}{dt}(t) = \hat{B}(t)X^{(m)}(t) - \displaystyle\int_0^t g(t-s)\hat{A}(s)X^{(m)}(s)ds + \mathcal{F}^{(m)}(t), \\ X^{(m)}(0) = X_0, \end{cases} \tag{29}$$

where

$$\begin{cases} X^{(m)}(t) = \left(u_1^{(m)}(t), \cdots, u_N^{(m)}(t), v_1^{(m)}(t), \cdots, v_N^{(m)}(t)\right)^T \in \mathfrak{M}_{2N\times 1}, \\ \mathcal{F}^{(m)}(t) \equiv \mathcal{F}\left(t, X^{(m-1)}(t)\right) \\ \qquad = \left(0, \cdots, 0, \mathcal{F}_1(t, X^{(m-1)}(t)), \cdots, \mathcal{F}_N(t, X^{(m-1)}(t))\right)^T \in \mathfrak{M}_{2N\times 1}, \\ \mathcal{F}_i(t, X^{(m-1)}(t)) = f_i\left(t, u_i^{(m-1)}(t), v_i^{(m-1)}(t)\right) \\ \qquad = -\left(v_i^{(m-1)}(t)\right)^3 + \left|u_i^{(m-1)}(t)\right|^3 u_i^{(m-1)}(t) + F(x_i, t), i = \overline{1, N}, \\ X_0 = (\tilde{u}_0(x_1), \cdots, \tilde{u}_0(x_N), \tilde{u}_1(x_1), \cdots, \tilde{u}_1(x_N))^T \in \mathfrak{M}_{2N\times 1}. \end{cases} \tag{30}$$

In order to find the numerical solution of the problem (29), we will approximate $\dfrac{dX^{(m)}}{dt}(t_j)$ as follows

$$\frac{dX^{(m)}}{dt}(t_j) \approx \frac{X_{j+1}^{(m)} - X_j^{(m)}}{\Delta t}, \tag{31}$$
$$X_j^{(m)} = X^{(m)}(t_j), t_j = j\Delta t, \Delta t = \frac{T}{M}, j = \overline{0, M}.$$

Therefore

$$\begin{cases} \dfrac{X_{j+1}^{(m)} - X_j^{(m)}}{\Delta t} = \hat{B}_j X_j^{(m)} - \displaystyle\int_0^{t_j} g(t_j - s)\hat{A}(s)X^{(m)}(s)ds \\ \qquad\qquad + \mathcal{F}^{(m)}(t_j), j = \overline{0, M-1}, \\ X_0^{(m)} = X_0. \end{cases} \tag{32}$$

On the other hand, we use the trapezoidal formula to approximate $\int_0^{t_j} g(t_j - s)\hat{A}(s)X^{(m)}(s)ds$ as follows

$$\int_0^{t_j} g(t_j - s)\hat{A}(s)X^{(m)}(s)ds \approx \Delta t\left[\frac{g_j\hat{A}_0X_0^{(m)} + g_0\hat{A}_jX_j^{(m)}}{2} + \sum_{\nu=1}^{j-1} g_{j-\nu}\hat{A}_\nu X_\nu^{(m)}\right]$$
$$= \Delta t\left[\frac{g_j\hat{A}_0X_0 + g_0\hat{A}_jX_j^{(m)}}{2} + \sum_{\nu=1}^{j-1} g_{j-\nu}\hat{A}_\nu X_\nu^{(m)}\right], \tag{33}$$

where $g_j = g(t_j)$.

Hence

$$\begin{cases} X_{j+1}^{(m)} = \left(\mathbb{I} + \Delta t\hat{B}_j - \frac{1}{2}(\triangle t)^2 g_0\hat{A}_j\right)X_j^{(m)} \\ \qquad - (\triangle t)^2\left[\frac{g_j\hat{A}_0X_0}{2} + \sum_{\nu=1}^{j-1} g_{j-\nu}\hat{A}_\nu X_\nu^{(m)}\right] + \Delta t\mathcal{F}^{(m)}(t_j), j = \overline{1, M-1}, \\ X_0^{(m)} = X_0, \end{cases} \tag{34}$$

where $\mathbb{I}$ the identity matrix of size $2N$.

We rewrite (34) in the form

$$\begin{cases} X_1^{(m)} = \left(\mathbb{I} + \Delta t\hat{B}_0\right)X_0 + \Delta t\mathcal{F}^{(m)}(0) \equiv \mathcal{F}_0\left[X_0\right], \\ X_2^{(m)} = \left(\mathbb{I} + \Delta t\hat{B}_1 - \frac{1}{2}(\triangle t)^2 g_0\hat{A}_1\right)X_1^{(m)} - \frac{1}{2}(\triangle t)^2 g_1\hat{A}_0X_0 + \Delta t\mathcal{F}^{(m)}(t_1) \\ \qquad \equiv \mathcal{F}_1\left[X_0, X_1^{(m)}\right], \\ X_{j+1}^{(m)} = \left(\mathbb{I} + \Delta t\hat{B}_j - \frac{1}{2}(\triangle t)^2 g_0\hat{A}_j\right)X_j^{(m)} \\ \qquad - (\triangle t)^2\left[\frac{g_j\hat{A}_0X_0}{2} + \sum_{\nu=1}^{j-1} g_{j-\nu}\hat{A}_\nu X_\nu^{(m)}\right] + \Delta t\mathcal{F}^{(m)}(t_j) \\ \qquad \equiv \mathcal{F}_j\left[X_0, X_1^{(m)}, X_2^{(m)}, \cdots, X_j^{(m)}\right], j = \overline{2, M-1}, \end{cases} \tag{35}$$

in which

$$\begin{cases} \mathcal{F}_0\left[X_0\right]=\left(\mathbb{I}+\Delta t\hat{B}_0\right)X_0+\Delta t\mathcal{F}^{(m)}(0), \\ \mathcal{F}_1\left[X_0,X_1^{(m)}\right]=\left(\mathbb{I}+\Delta t\hat{B}_1-\frac{1}{2}\left(\triangle t\right)^2g_0\hat{A}_1\right)X_1^{(m)}-\frac{1}{2}\left(\triangle t\right)^2g_1\hat{A}_0X_0+\Delta t\mathcal{F}^{(m)}(t_1), \\ \mathcal{F}_j\left[X_0,X_1^{(m)},X_2^{(m)},\cdots,X_j^{(m)}\right]=\left(\mathbb{I}+\Delta t\hat{B}_j-\frac{1}{2}\left(\triangle t\right)^2g_0\hat{A}_j\right)X_j^{(m)} \\ \qquad\qquad -\left(\triangle t\right)^2\left[\frac{g_j\hat{A}_0X_0}{2}+\sum_{\nu=1}^{j-1}g_{j-\nu}\hat{A}_\nu X_\nu^{(m)}\right] \\ \qquad\qquad +\Delta t\mathcal{F}^{(m)}(t_j),\, j=\overline{2,M-1}, \\ \mathcal{F}^{(m)}(0)\equiv\mathcal{F}\left(0,X_0\right)=\left(0,\cdots,0,\mathcal{F}_1(0,X_0),\cdots,\mathcal{F}_N(0,X_0)\right)^T\in\mathfrak{M}_{2N\times1}, \\ \mathcal{F}_i(0,X_0)=f_i\left(0,\tilde{u}_0(x_i),\tilde{u}_1(x_i)\right)=-\tilde{u}_1^3(x_i)+\left|\tilde{u}_0(x_i)\right|^3\tilde{u}_0(x_i)+F(x_i,0),\, i=\overline{1,N}. \end{cases} \tag{36}$$

With the positive integer N, M fixed, we find $\left(X_1^{(m)},X_2^{(m)},\cdots,X_M^{(m)}\right)$ by (35)–(36) such that

$$\max_{1\le j\le M}\left|X_j^{(m)}-X_j^{(m-1)}\right|_1<\varepsilon=10^{-3}, \tag{37}$$

where $|\cdot|_1$ is the norm in the space $\mathbb{R}^{2N}$ given as below

$$\left|X_j^{(m)}-X_j^{(m-1)}\right|_1=\sum_{i=1}^N\left(\left|u_i^{(m)}(t_j)-u_i^{(m-1)}(t_j)\right|+\left|v_i^{(m)}(t_j)-v_i^{(m-1)}(t_j)\right|\right), \tag{38}$$

with

$$X_j^{(m)}=X^{(m)}(t_j)=\left(u_1^{(m)}(t_j),\cdots,u_N^{(m)}(t_j),v_1^{(m)}(t_j),\cdots,v_N^{(m)}(t_j)\right)^T\in\mathfrak{M}_{2N\times1}. \tag{39}$$

If (37) holds then $X_j^{(m)}$ is chosen as follows $X_j^{(m)}\equiv X_j=(u_1(t_j),\cdots,u_N(t_j),v_1(t_j),\cdots,v_N(t_j))^T$, and then the following error is obtained

$$E_{N,M}\left(u\right)=\max_{1\le j\le M}\max_{1\le i\le N}\left|u_{ex}\left(x_i,t_j\right)-U_i(t_j)\right|. \tag{40}$$

For details, Table 1 presents the errors $E_{N,M}\left(u\right)$ of the approximate solution $u^{(m)}(x,t)$ of (19) defined by (35)–(36) and the exact solution $u_{ex}(x,t)$ defined by (21) with respect to the values of N and M as below

Table 1. Errors of the approximate solution $u^{(m)}(x,t)$ and the exact solution $u_{ex}(x,t)$

N	M	$E_{N,M}(u)$
5	25	0.105613828788385
10	100	0.036310154878634
15	225	0.020252536096917
20	400	0.013818817721289
25	625	0.010475400779207
30	900	0.008349131471897

Obviously, the errors are decreasing when the values of M and N are increasing.

With the functions given as in (20) and $0 \leq x \leq 1$, $0 \leq t \leq 1.5$ and mesh of $N = 30$ and $M = 900$, Fig. 1 illustrates the surface of the exact solution $u_{ex}(x,t)$ of the problem (19) below.

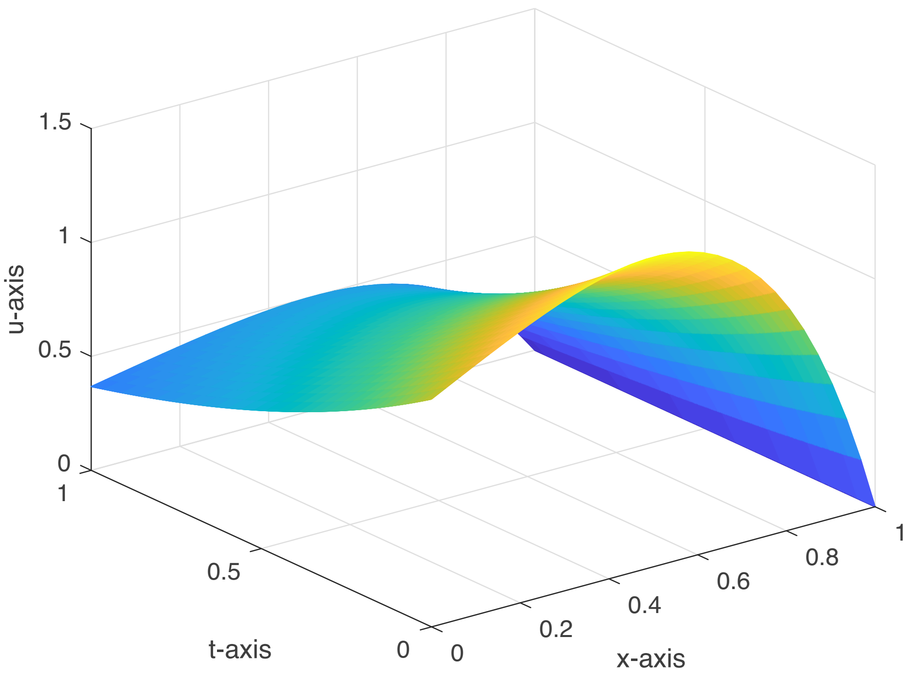

Fig. 1. The picture of the exact solution u_{ex} of (19).

With the functions given as in (20) and mesh of $N = 30$ and $M = 900$, Fig. 2 illustrates the surface of the approximate solution of $u^{(m)}(x, t)$ to the problem (19) with defined by the algorithm (35) and (36).

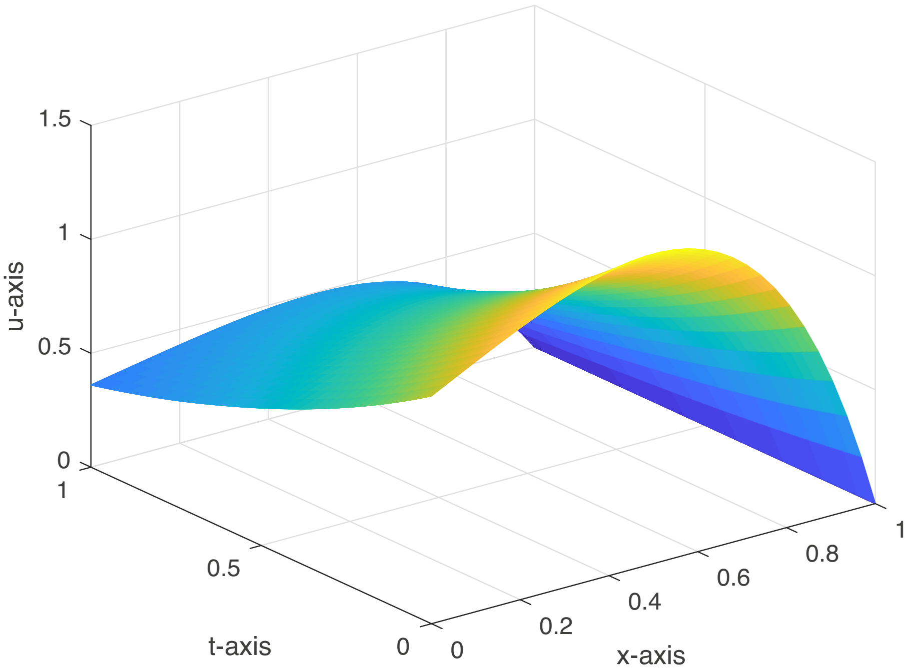

Fig. 2. The picture of the approximate solution of $u^{(m)}(x, t)$ to the problem (19) defined by the algorithm (35) and (36).

5 Conclusion

This paper is concerned with an initial-boundary value problem for nolinear wave equation with memory term. The results of existence and uniqueness of solutions to the problem are established by the linear approximate technique and the Faedo-Galerkin method, in which their proofs are the same as in [14,15,19]. An algorithm in order to find the approximate solution of the problem is constucted by the finite-difference formulas, and an example is also given to illustrate the exact solution and the finite-difference approximate solution.

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