Solving Portfolio Optimization Problems with Particle Filter

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Abstract. In order to improve precision of the solution to the portfolio optimization problem, a optimization scheme based on particle filter is proposed. Portfolio optimization problem is modeled by the Markowitz's portfolio theory, and it is a nonlinear optimization problem with multiple constraints. In this paper, particle filter is considered, to solve portfolio optimization problem. The nonlinear optimization problem is converted to filtering problem of particle filter. Then the nonlinear optimization problem can be solved by particle filter method. To solve portfolio optimization problem, a optimization scheme based on particle is proposed. To improve precision of the solution, crossover and mutation of genetic algorithm is considered in the proposed scheme. Lastly, results of simulation have demonstrated that the proposed optimization scheme outperforms other traditional methods in the precision of the solution of the portfolio optimization problem.

Keywords: Particle filter · Portfolio investment · Nonlinear function · Optimization problem

1 Introduction

Securities investment is a form of investment that obtains dividends, interest and capital gains by buying securities such as stocks, bonds, fund bonds and their derivatives. The goal of securities investment is to obtain returns and reduce risks, which requires the optimal combination of benefits and risks. One of the most important theories of securities investment is the portfolio theory proposed by Harry Markowitz. Portfolio refers to investors take appropriate methods to select multiple securities as investment objects based on the risks and benefits. The goal of the portfolio is to minimize the investment risk under the premise of ensuring the expected return, or to maximize the return of the investment under the premise of controlling the risk. In Markowitz's portfolio theory, the returns are supposed to obey Gaussian distribution, and the portfolio optimization problem is a nonlinear programming problem with multiple constraints.

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However, the real investment seldom follow a Gaussian distribution [1], and subject to multiple constraints. The traditional methods [2,3] use the gradient-based optimization method to solve the multiple-constraints optimization problem, but they are often not very effective. There are many researches take another tack to optimized the portfolio problem, such as apply heuristic algorithms to optimize the portfolio optimization problem. In literature [4–6], the portfolio optimization problem was optimized by genetic algorithm (GA) based methods. GA is an efficient and fast optimization method, but it is prone to trap into local extreme points, and converges prematurely. In literature [7–9], particle swarm optimization (PSO) is considered, to solve the portfolio optimization problem. So that a optimal portfolio strategy can be effective found out. The PSO algorithm is easy to program and converges quickly, and have a good balance in the global search and local search. But still, the performance of local search of PSO is not good enough and lead to a low search accuracy in the optimization. In literature [10,11], simulated annealing (SA) algorithm is considered. SA algorithm is simple but effective, and has good robustness, but has a slow convergence speed. GA, PSO and SA, they have some defects in local search and convergence speed, which will decrease the precision of the optimization. For GA, PSO and SA, they are parametric methods, which performance relies on the parameter settings. A complex parameter settings lead to a complex optimization and a unstable solution.

Particle filter (PF) is a non-parametric method, which is widely deployed in localization [12,13], target tracking [14,15] and navigation [16,17]. Particle filter is a estimation method, to estimate the state of the nonlinear and non-Gaussian systems [18–20]. Particle filter was deployed to solve filtering problem, instead of optimization problem. Therefore, conversion from optimization problem to filtering problem is needed.

In this paper, a optimization scheme is proposed, which is based on particle filter method, to solve the portfolio optimization problem. The portfolio optimization problem is converted to a filtering problem, then particle filter can be deployed in solving the portfolio problem. Crossover and mutation of genetic algorithm is considered in the proposed scheme to improve precision of the solution. To test the performance of the particle filter based optimization scheme, several simulations are formulated.

The content of this paper is organized as follows: Sect. 2 formulates the optimization problem. Section 3 introduces the theory of particle filter. Section 4 proposes the optimization scheme based on particle filter. Section 5 conducts several simulations. Section 6 draws a conclusion to this paper.

2 Problem Formulation

Markowtiz used the mean of the expected return rate to represent the level of the expected return, and used the variance of the return rate to assess the degree of the investment risk. Thereby the mean-variance model of the portfolio investment was established. Investors are normally making choice with uncertain returns and risks. In Markowtiz's portfolio model, mean and variance are used to describe returns and risks. In the mean-variance model, the mean value refers to the expected rate of return of the portfolio. The mean value is the weighted average of the expected rate of return of a single security. The weight is the corresponding investment ratio. The variance refers to the variance of the return rate of the investment portfolio.

Investors make a choice to select portfolio by measuring returns and risks in the investment. Investors might select portfolio by maximizing the expected return based on a certain expected risk, or minimizing the expected risk based on a certain expected return. A basic principle of Markowtiz's portfolio theory is to minimize the risk after investment for a specific expected rate of return R.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij}. \tag{1}$$

Subject to

$$\begin{cases} \sum_{i=1}^{n} \omega_i \mu_i = R, \\ \sum_{i=1}^{n} \omega_i = 1, \\ 0 \leq \omega_i \leq 1. \end{cases} \tag{2}$$

Here n represents the number of securities to be selected by investors, ω_i refer to the investment ratio coefficient of the security i, μ_i refer to the expected rate of return of security i. σ_{ij} represents the covariance of the return rate of correct i and the return rate of security j, and R refer to the expected rate of return.

3 Basis Theory of Particle Filter

Particle filter is a state estimation method, which is used to estimate the state of the nonlinear and non-Gaussian systems. Particle filter was used to solve filtering problem, to make a optimal estimation to the state. The optimal estimation to the state is approximated by updating probability density of the state.

Update function of a nonlinear and non-Gaussian system is expressed as follows

$$x_k = f_k(x_{k-1}, w_k), \tag{3}$$

its measurement function is expressed as follows

$$z_k = h_k(x_k, v_k). \tag{4}$$

Here, the update function f_k is a nonlinear function, and w_k is the process noise. The measurement function h_k a nonlinear function, and v_k is the detecting noise. k is a discrete time. The state x_k of the system is estimated according to samples of the solution domain of the measurement function, under noise scenario. Figure 1 is a flowchart of particle filter. The probability density function $p(x_{k-1}|z_{k-1})$ is a prediction of next location of the state, it is approximated

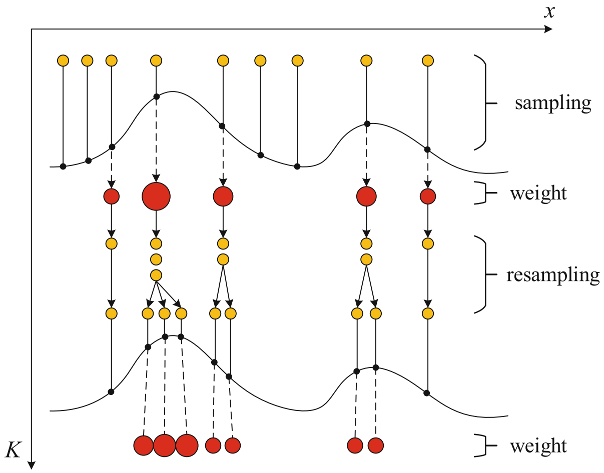

Fig. 1. Flowchart of particle filter

by the particles and their weights. By iteratively updating the probability density function of the state and the weight of particles, the minimum variance estimation of the state can be obtained.

The prediction probability density $p(x_k|x_{k-1})$ of is calculated by the update function f_k of and the update function $p(x_{k-1}|z_{k-1})$, that is

$$p(x_k|z_{k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|z_{k-1})\,\mathrm{d}x_k. \tag{5}$$

The update function $p(x_{k-1}|z_{k-1})$ is calculated by the measurement function h_k and the prediction probability density, that is

$$p(x_k|z_k) = \frac{p(z_k|x_k)p(x_k|z_{k-1})}{p(z_k|z_{k-1})}. \tag{6}$$

Assumed that the initial probability density $p(x_0)$ is given by priory knowledge, then the filtering problem can be solved. Particle is a Monte Carlo method, which use a lot of particles to approximate the probability density of the state. By the Bayesian theory, the estimated state can be expressed as

$$\hat{x} = \int x_k p(x_k|z_k)\,\mathrm{d}x_k. \tag{7}$$

Analytical solution of the Eq. (7) is limited by the integral calculation.

For particle filter, the update function $p(x_k|z_k)$ is approximated by a lot of samples of the measurement function and their corresponding weight. By the law of large numbers, the approximated update function is denoted as follows

$$\hat{p}(x_k|z_k) = \frac{1}{N}\sum_{m=1}^{M}\delta(x_k - x_k^m) = \sum_{m=1}^{M} w_k^m \delta(x_k - x_k^m), \tag{8}$$

where $\delta(x)$ is Dirichlet function. w_k^m is weight of particle x_k^m, which subject to

$$w_0^m = \frac{1}{M}, \tag{9}$$

$$w_k^m = w_{k-1}^m \frac{p(z_k|x_k^m)p(x_k^m|x_{k-1}^m)}{p(x_k^m|x_{k-1}^m, z_k)}. \tag{10}$$

Then, the state is approximated as follows

$$\hat{x} = \int x_k p(x_k|z_k)\, \mathrm{d}x_k = \sum_{m=1}^{M} w_k^m x_k^m. \tag{11}$$

The state of system is estimated by the update function and the samples of the measurement function, particle filter used a lot of samples particles to update the state, and obtained an optimal estimation.

4 PF Based Optimization Scheme

Particle filter is deployed to estimate state of nonlinear and non-Gaussian systems. Particle filter was designed for solving filtering problem, instead of optimization problem. To solve a optimization problem, conversion from optimization problem to filtering problem is needed.

4.1 Basis Principle of PF Based Optimization Scheme

The portfolio optimization problem is a nonlinear optimization problem with multiple constraints. The nonlinear optimization problem is converted to a estimation problem of the state. Then particle filter can be deployed in solving the portfolio optimization problem.

Consider an objective function as follows

$$\min fitness(x), \tag{12}$$

which subject to

$$a_i \le x^i \le b_i, i \in n. \tag{13}$$

here $fitness(.)$ is the fitness function, x^i is the variable, $[a_i, b_i]$ is the domain of the variable x^i.

To converted optimization problem to filtering problem, the optimization is considered as a nonlinear dynamic time-varying system, which the change of the global optimal solution is a nonlinear process. For the dynamic time-varying system, its discrete time is the iterations of the optimization. The state to the system is a solution of the optimization, and the update function is the changes of the solution during the iterations. The measurement function is the fitness function of the optimization. Consequently, the optimization problem is converted to a estimation problem of the state, and the global optimal solution can be obtained when the iteration is accomplished.

Consider an optimization problem, which its fitness function is $fitness(x)$. For particle filter, the state update function is denoted as follows

$$x_k = f_k(x_{k-1}, w_k). \tag{14}$$

The measurement function is expressed as

$$y_k = fitness(x), \tag{15}$$

where x_k is the state, w_k is process noise. The update function $f_k(.)$ is the change of the global optimal solution during the iteration, which is a nonlinear function. The measurement function is $fitness(x)$, which is the same as fitness function of the optimization problem. The solution of the optimization problem continues to approximate the optimal solution during the iteration, and lastly converge on the global optimal solution.

4.2 PF for Optimization

In this paper, a optimization scheme based on particle is proposed for solving portfolio optimization problem, which can improve precision of the solution. The objective function of the portfolio optimization problem is given in Eq. (1) and Eq. (2), which is a nonlinear function with multiple constraints. Flow chart of the optimization scheme based on particle filter is represented in Fig. 2. The following is the specific steps of the process.

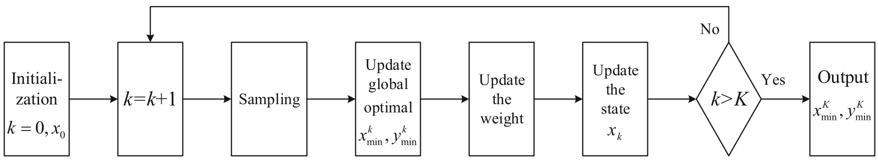

Fig. 2. Schematic of the proposed scheme

step 1: Initialization: The state is initialized as a random sample of the solution domain the measurement function (1), which subject to (2). Amount of particle is M. Maximum iterations is K, and initial iteration is $k = 0$.

step 2: Iteration: The iteration is $k = k + 1$. The solution of portfolio optimization problem is iteratively updated, and converge on global optimal solution when the iteration is accomplished.

step 3: Sampling: Particle $m \in \{1, M\}$ is sampled according to $x_k^m \sim p(x_k^m|z_k)$. Here, a uniform distribution $U(x_k^m - c_k, x_k^m + c_k)$ is used instead of $p(x_k^m|z_k)$. It is a random sampling, a new particle is generated as follows

$$x_k^m = \frac{\Lambda \cdot rand}{a^k} + x_{k-1} - \frac{1}{2} \cdot \frac{\Lambda}{a^k}. \tag{16}$$

here, a is a number slightly larger than 1. Λ is argument domain to the measurement function (1). The sampling region Λ is narrowed with increasing the iteration k.

step 4: Crossover and Mutation: Crossover operation and mutation operation of GA is considered in the proposed scheme to improve precision of the solution,. By introducing crossover operation and mutation operation, variety of particles is enriched, which contribute to avoid local optimal solutions. Details of crossover operation and mutation operation is given as follows

– *Crossover*: The crossover operation is a single-point crossover. Any two particles are choosed to pair. Elements in the same position of the variable of the particles are exchanged, which can generate two new particles.
– *Mutation*: Select a variable from the set of variables and replace it with a new variable within the range of its values.

step 5: Optimal solution : According to the measurement function (1), measurement value y_k^m of particle x_k^m is calculated. Then, the minimum fitness value y_{min}^k among y_k^m can be found, so the corresponding solution x_{min}^k.

step 6: Weights updated: Measurement value y_{k-1} of x_{k-1} is calculated according to the measurement function (1), which will be the observed value. Measurement values y_k^m of the particles x_k^m are compared with the observed value y_{k-1} to update the weights as follows

$$w_k^m = \begin{cases} 0, & y_k^m \leq y_{k-1} \ or \text{ fail to meet the constraints (2)} \\ 1/q_k^m, & \text{otherwise}, \end{cases} \tag{17}$$

where

$$q_k^m = \frac{1}{s\sqrt{2\pi}} exp(-\frac{(y_k^m - y_{k-1})^2}{2\sigma^2}). \tag{18}$$

Measurement values $\{y_k^m\}$ of $x_k^m, m \in \{1, M\}$ are considered to follow normal distribution $N(y_{k-1}, s^2)$, and σ is variance of measurement values $\{y_k^m\}$. Then, the weights of the particles are normalized as follows

$$w_k^m = \frac{w_k^m}{\sum_{m=1}^{M} w_k^m}. \tag{19}$$

step 7: Resampling: As iteration goes on, effective particles might decrease significantly, which lead to the optimization converged prematurely. Number of effective particles is calculated as follows

$$N_{eff} = \frac{1}{\sum_{m=1}^{M} (w_k^m)^2} \tag{20}$$

where threshold $N_{th} = 2M/3$. The resampling is started when $N_{eff} < N_{th}$, and independent resampling [21] is considered.

step 8: State updated: The state x_k is approximated by the particles and their corresponding weights as follows

$$\hat{x} = \sum_{m=1}^{M} w_k^m x_k^m. \tag{21}$$

step 9: If iteration $k \leq K$, go back to step 2 and continue the iteration. When $k > K$, the optimization is accomplished, and return a global optimal solution x_K^{min} and its corresponding fitness value y_K^{min}.

The specific steps of the optimization scheme based on particle filter is organized in Algorithm 1. For the portfolio optimization, the nonlinear optimization problem is converted to the filtering problem of particle filter, and solve by particle filter method.

Algorithm 1

Set: K, M
Initialize state particle x_0
for $(k = 1 : K)$ **do**
  Random sampling $\{x_k^m\}$
  Crossover and Mutation
  Update optimal solution: $[y_{min}^k, index] = min(y_k^m)$, $x_{min}^k = \{x_k^m\}(index)$
  Update the weight of the particle according to the Eqs. (18, 17, 19)
  Calculate N_{eff} **according to the Eqs. (20)**
  if $(N_{eff} < 2M/3)$ **then**
    Independent resampling
  end if
  Update the state particle according to Eqs. (21)
end for
Output: The optimal solution: x_K^{min} and y_K^{min}.

5 Simulation Results

To test the performace of the optimization scheme based on particle filter, several simulations are conducted. Outcomes of the proposed optimization scheme are compared with other heuristic algorithm based methods, such as GA based method in [4], PSO based method in [7] and SA based method in [11]. In the mean-variance model, the mean represents the expected return rate of the portfolio, and the variance represents the variance of the expected return rate.

Table 1. The expected return rate and its corresponding covariance of the portfolio which is composed of 3 securities.

Securities	Return rate (%)	Covariance		
		1	2	3
1	5.8	0.004	0.018	0.022
2	8.8	0.018	0.118	0.074
3	7.9	0.022	0.074	0.081

Simulation 1: A portfolio is assumed composed of 3 securities, expected return rate of investment is 6.8%. The expected return rate and its corresponding covariance are shown in Table 1. In the mean-variance model, the expected return rate refers to mean value, and the covariance of the expected return rate refers to the covariance. The goal of the optimization problem in this simulation is to minimize the investment risk under the premise of ensuring the expected return, that is minimize the covariance. We used the proposed PF based scheme to optimize the portfolio optimization problem, and compare with other methods. The minimum investment risk ratio coefficient vector of the portfolio obtained by the proposed PF based scheme is $\omega = [0.60916817456292, 0.19805907398015, 0.19277275145693]$. The global optimal solutions are shown in Table 2. It can be observed from Table 2 that the optimal solution obtained by our proposed PF based scheme is smaller, which means that investment risk of the portfolio of our scheme is smaller.

Table 2. Results of *Simulation 1* obtained by the PF based scheme and other methods.

Methods	The global optimal solution
PF based scheme	0.03764344490657
GA based method in [4]	0.03764370000000
PSO based method in [7]	0.03764370000000
SA based method in [11]	0.037654411009027

Simulation 2: In this simulation, a portfolio is assumed composed of 6 securities, expected return rate of investment is 20.5%. The expected return rate and its corresponding covariance are shown in Table 3. The portfolio in here has more securities, that means the dimensionality of the variable to be optimized increases, which will lead to a more complicated optimization. The goal of the portfolio optimization problem in this simulation is to minimize the investment risk under the premise of ensuring the expected return, that is minimize the covariance. We used the proposed PF based scheme to optimize the portfolio optimization problem, and the minimum investment risk ratio coefficient

Table 3. The expected return rate and its corresponding covariance of the portfolio which is composed of 6 securities.

Securities	Return rate (%)	Covariance					
		1	2	3	4	5	6
1	18.5	0.210	0.210	0.221	−0.216	0.162	−0.215
2	20.3	0.210	0.225	0.239	−0.216	0.168	−0.219
3	22.9	0.210	0.239	0.275	−0.246	0.189	−0.247
4	21.8	−0.216	−0.216	−0.246	0.256	−0.185	0.254
5	16.7	0.162	0.168	0.189	−0.185	0.142	−0.188
6	23.1	−0.215	−0.219	−0.247	0.254	−0.188	0.266

vector of the portfolio obtained by the proposed PF based scheme is $\omega = [0.07459492705459, 0.00000000001454, 0.16811832788656, 0.24520471325523, 0.29758477229673, 0.21449725949235]$. The global optimal solutions obtained by our proposed PF based scheme and other methods are shown in Table 4. According to Table 4, the investment risk of the portfolio optimized by our proposed PF based scheme is smaller than other methods. Although the optimization problem becomes more complicated and the dimension of the variable to be optimized increases, but still our proposed PF based scheme can obtain a higher precision global optimal solution compare with other methods.

Table 4. Results of *Simulation 2* obtained by the PF based scheme and other methods.

Methods	The global optimal solution
PF based scheme	0.00363465183449
GA based method in [4]	0.00363475500000
PSO based method in [7]	0.00363475500000
SA based method in [11]	0.003634848401176

6 Conclusion

In order to improve precision of solution of portfolio optimization problem, and avoid the optimization trap into local optimal solution, a optimization scheme based on particle filter is proposed. Portfolio optimization problem is a nonlinear optimization problem with multiple constraints, and it is converted into the state estimation problem of particle filter. To solve the portfolio optimization problem, a particle filter based optimization scheme is proposed. Crossover operation and mutation operation of genetic algorithm is introduced to improve precision of the solution. Simulation results have demonstrated that the proposed PF based scheme outperforms the GA, PSO and SA based methods. The investment risk optimized by our proposed PF based scheme is smaller than other methods. The PF based optimization scheme provide a more effective and high precision way to optimize the portfolio problem.

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