Termination for Belief Propagation Decoding of Polar Codes in Fading Channels

Chen Zhang, Yangzhi Luo, and Liping Li(✉)

Key Laboratory of Intelligent Computing and Signal Processing of the Ministry of Education of China, Anhui University, Hefei, China
liping_li@ahu.edu.cn

Abstract. In additive white Gaussian channel (AWGN), the performance of polar codes under the successive cancellation (SC) decoding is not as good as that of the belief propagation (BP) decoding. However, in a fading channel, the performance of BP decoding is found in our study to be worse than the SC decoding. In this work, we propose a termination criterion for the BP decoding of polar codes to improve the performance and the average number of iterations at the same time. Simulation results show that BP decoding can still achieve a better performance than the SC decoding in fading channels with the proposed termination.

Keywords: Polar code · Belief propagation · Successive cancellation decoding · Fading channels

1 Introduction

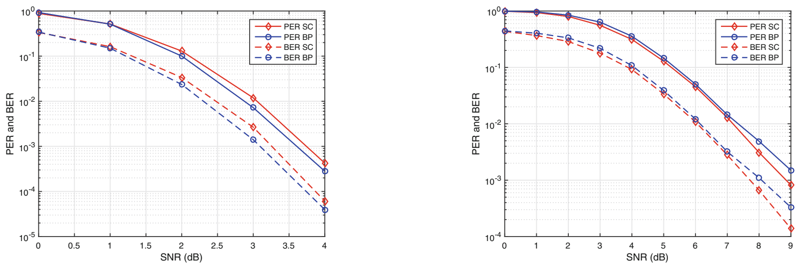
Polar codes proposed by Arıkan [3] is the first constructive and provable coding scheme which achieves the capacity of binary-input discrete memoryless symmetric channels (B-DMCs) with a low complexity successive cancellation (SC) decoding. The SC decoding is a particular instance of the belief propagation (BP) decoding shown in [6]. Overall, the performance of the SC decoding of polar codes is not as good as that of the BP decoding [2,6].

The existing decoding techniques of polar codes, including the SC decoding, the BP decoding [5,6], the successive cancellation list (SCL) decoding [12], and the CRC aided SCL decoding [9] are all studied under additive white Gaussian noise (AWGN) channels. With the adoption of polar codes as the coding scheme for the control channel of the enhanced mobile broadband (eMBB) scenario of 5G [1], it is of importance to study the performance of polar codes in fading channels. In [11], a new coding technique considering the fading characteristics is proposed to improve the performance of polar codes. In [10], the construction of polar codes is proposed for polar codes in fading channels. Until now, there is no work reported to study the performance of the BP decoding of polar codes in fading channels.

G. Gui and L. Yun (Eds.): ADHIP 2019, LNICST 301, pp. 20–30, 2019.
https://doi.org/10.1007/978-3-030-36402-1_3

In this paper, polar codes with the BP decoding are studied under fading channels. The first observation of our study shows that the BP decoding performance even deteriorates compared to the SC decoding in fading channels. For example, with the code block length $N = 256$ and the code rate $R = 0.36$, the BP performance is worse than that of the SC performance when the number of iterations is fixed to be 30 in fading channels. This is shown in Fig. 1. From Fig. 1, it can be observed that the BP decoding outperforms the SC decoding in AWGN channels. With the same fixed number of iterations, BP decoding does not hold this advantage any more over the SC decoding in fading channels.

To improve the BP decoding performance of polar codes in fading channels, this paper proposes the following procedures: (1) increase the maximum number of iterations of the BP decoding; (2) set an early termination criterion. The early termination of the BP decoding of polar codes is studied in [5,13]. Both criterions in [5,13] are studied in AWGN channels. This paper investigates the characteristics of the early termination criterions of BP decoding and proposes a new termination criterion. The results show that the BP decoding of polar codes with the proposed termination achieves the best error performance among the existing termination criterions in fading channel.



(a) AWGN Channel (b) Fading Channel

Fig. 1. Performance comparisons of polar codes with the code length $N = 256$, the code rate $R = 0.36$. For BP decoding, the number of iterations is fixed to be 30 in both AWGN and fading channels.

The rest of this paper is organized as follows. Section 2 provides the preliminaries of polar codes. In Sect. 3, the early termination of BP decoding is introduced. Based on our study, a new early terminations is proposed for fading channels. Section 4 provides the simulation results of the BP decoding with the proposed procedures. Finally, Sect. 5 concludes the paper.

2 Preliminaries

For any B-DMC W: $\mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X} = \{0, 1\}$ denotes the input alphabet and $\mathcal{Y}$ denotes the output alphabet, the channel transition probability is defined as

$W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. According to the channel combining and the splitting process, N binary input channels $W_N^{(i)}$ can be obtained with $i = 1, 2, ..., N$. For the symmetric channels [3], the K most reliable bit channels with indices in the information set $\mathcal{A}$ are used to transmit information bits. The remaining bit channels with indices in the complementary set $\mathcal{A}^{\mathcal{C}}$ transmit known bits which are called frozen bits.

Polar code encoding is performed as $x_1^N = u_1^N G_N$, where G_N denotes the generator matrix of the polar code, $u_1^N \in \{0,1\}^N$ denotes the source vector $(u_1, u_2, ..., u_N)$ and $x_1^N \in \{0,1\}^N$ denotes the codeword. The source vector u_1^N consists of information bits $u_{\mathcal{A}}$ and frozen bits $u_{\mathcal{A}^c}$. Here $u_{\mathcal{A}}$ means the sub-vector of u_1^N taking elements specified by the set $\mathcal{A}$. The generator matrix is defined as $G_N = B_N F_2^{\otimes n}$. Here B_N is the bit-reversal permutation matrix, $F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\otimes$ denotes the Kronecker product [3].

2.1 Successive Cancellation Decoding (SC)

For any given polar code (N, K, A, u_{A^c}) [3], the SC decoding estimates bit u_i (denoted as $\hat{u}_i$) based on the following:

$$\hat{u}_i \triangleq \begin{cases} u_i, & \text{if } i \in \mathcal{A}^c \\ h_i(y_1^N, \hat{u}_1^{i-1}), & \text{if } i \in \mathcal{A}, \end{cases} \tag{1}$$

where h_i is a decision function defined as:

$$h_i(y_1^N, \hat{u}_1^{i-1}) \triangleq \begin{cases} 0, & \text{if } L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) \geq 1 \\ 1, & \text{otherwise}, \end{cases} \tag{2}$$

The likelihood ratio (LR) is defined as:

$$L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) = \frac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|1)}, \tag{3}$$

which can be recursively calculated [3]:

$$\begin{aligned} &L_N^{(2i-1)}(y_1^N, \hat{u}_1^{2i-2}) \\ &\quad = \frac{L_{N/2}^{(i)} y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}) + 1}{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2})}, \end{aligned} \tag{4}$$

$$\begin{aligned} &L_N^{(2i)}(y_1^N, \hat{u}_1^{2i-1}) \\ &\quad = [L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})]^{1-2\hat{u}_{2i-1}} \cdot L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}). \end{aligned} \tag{5}$$

The initial LR is:

$$L_1^{(1)}(y_i) = \frac{W(y_i|0)}{W(y_i|1)}. \tag{6}$$

2.2 Belief Propagation (BP) Decoding

The BP decoding for polar codes [2,5] is based on the factor graph. Figure 2 shows an example when $N = 8$, where circles represent variable nodes (VNs) and squares are check nodes (CNs). Given a polar code of length $N = 2^n$, messages are passed from layer 0 to layer n, then from layer n to layer 0 iteratively. Here layer 0 is the layer with inputs from the underlying channels. The messages passed in the graph are log likelihood ratio (LLR) values.

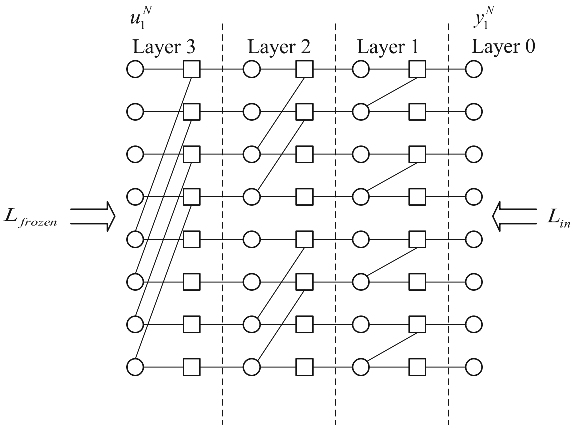


Fig. 2. BP decoding graph with the code length $N = 8$.

The initialization of the LLR is done:

$$L_{in}(i) = \ln \frac{P(y_i|x_i = 0)}{P(y_i|x_i = 1)}, \tag{7}$$

$$L_{frozen} = \begin{cases} \infty, & j \in A^c \\ 0, & j \in A, \end{cases} \tag{8}$$

where L_{in} are the LLR calculated from the channel. The initial LLRs are fed to the factor graph from the right to the left. The initial LLR L_{frozen} are the input to the factor graph from the left to the right, as shown in Fig. 2. Let $L_{v\rightarrow c}(\lambda, i)$ denote the message from the VN on the i-th row of the λ-th VN layer to the CN on the i-th row of the $(\lambda + 1)$-th CN layer ($1 \le i \le N$, $0 \le \lambda \le n - 1$). Similarly, let $L_{c\rightarrow v}(\lambda, i)$ denote the message from the CN on the i-th row of the $(\lambda + 1)$-th CN layer to the VN on the i-th row of the λ-th VN layer. Finally, let $L_{v\rightarrow c}(n, i)$ be the messages flooded to the n-th VN layer, which is used for estimation of the information bits. For bit u_i, $L_{v\rightarrow c}(n, i) + L_{frozen}(i)$ is taken as the decision metric for $\hat{u}_i$ (the estimation of u_i), and $L_{c\rightarrow v}(0, i) + L_{in}(i)$ is taken as the decision metric for the estimated coded symbol $\hat{x}_i$:

$$\hat{u}_i = \begin{cases} 0, & L_{v\to c}(n,i) + L_{frozen}(i) > 0 \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

$$\hat{x}_i = \begin{cases} 0, & L_{c\to v}(0,i) + L_{in}(i) > 0 \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

At the beginning of the BP decoding, $L_{in}(i)$ is put into $L_{v\to c}(0,i)$. Then update all $L_{v\to c}(\lambda,i)$ messages from the right to the left, then update all $L_{c\to v}(\lambda,i)$ messages from the left to the right. Repeat the steps until an early termination condition is met, or the maximum number of iterations is reached.

3 Termination for BP Decoding

In order to decode more efficiently, it is desirable to terminate the BP iterations early when the decoding converges. In this section, we first show the two existing early termination criterions [5,13] for the BP decoding of polar codes. A new termination criterion for polar codes with BP decoding is then proposed.

3.1 Termination Based on Frozen Bits

In [5], the authors proposed that if the estimated codeword $\hat{x}_1^N$ can produce all zero frozen bits, then the BP iteration can be terminated. Note that in [5], the frozen bits are set to be all zeros in the encoding process. Here we formally state this termination criterion as Termination I.

Termination I: Let G_N^{-1} be the inverse matrix of the generator matrix G_N. Let $\tilde{u}_1^N$ be the vector calculated from the estimated codeword $\hat{x}_1^N$: $\tilde{u}_1^N = \hat{x}_1^N G_N^{-1}$. Then the estimated codeword $\hat{x}_1^N \in \{0,1\}^N$ is a real codeword, if and only if: $\tilde{u}_i = 0, \forall i \in \mathcal{A}^c$.

Note that Termination I requires a matrix inversion, which can be avoided by employing the special features of the generator matrix G_N. In [7], it is proven that the inverse of $F^{\otimes n}$ is itself. In [3], it is also shown that $B_N^{-1} = B_N$. Therefore

$$G_N^{-1} = (B_N F^{\otimes n})^{-1} = (F^{\otimes n})^{-1} B_N^{-1} = F^{\otimes n} B \tag{11}$$

Figure 3 shows the number of iterations of a polar system employing Termination I. Here the maximum number of iterations is set to be 70. The plot shows the number of iterations for each polar block. The red cross shows that a block is in error. It can be seen that there are quite a large number of blocks in error with a small iteration number when employing Termination I. There are also blocks that are in error while the maximum number of iterations is reached. These are the cases where BP decoding does not converge, and therefore can not be corrected by the BP decoding.

3.2 Termination Based on Estimated Codewords

Before introducing Termination II, recall that the encoding process is $x_1^N = u_1^N G_N$ for a given source vector u_1^N. Hence if $\hat{u}_1^N$ and $\hat{x}_1^N$ are the valid estimates, $\hat{x}_1^N = \hat{u}_1^N G_N$ must also hold. Therefore, $\hat{x}_1^N = \hat{u}_1^N \cdot G_N$ can be used to detect valid $\hat{u}_1^N$ and $\hat{x}_1^N$ [13].

Termination II: Let the calculated codeword be $\tilde{x}_1^N = \hat{u}_1^N G_N$. Then the condition for the termination of the BP iterations is when $\hat{x}_1^N = \tilde{x}_1^N$.

Note that from [4], it is observed that systematic polar codes achieve better BER performance than the counterpart of the normal non-systematic polar codes. Observe that in Termination II, the operation $\tilde{x}_1^N = \hat{u}_1^N G_N$ is exactly what systematic polar codes perform after obtaining estimates of $\hat{u}_1^N$. This operation produces less errors in $\tilde{x}_1^N$ than those in $\hat{u}_1^N$. Therefore, it can be predicted that Termination II results in less errors than Termination I.

With Termination II, the iteration number is shown in Fig. 4. It can see observed that the error cases with small numbers of iterations are far less than that employing Termination I. However there are still a few blocks in error with small numbers of iterations. These are the cases where the early termination criterion fails.

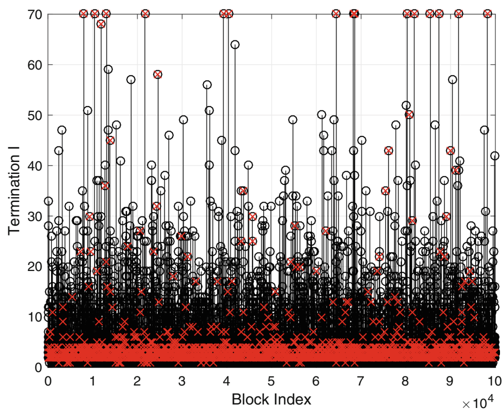


Fig. 3. The number of iterations of the BP decoding of the polar code with the code length $N = 256$ and the code rate $R = 0.36$. The maximum number of iterations is 70 in the fading channel. Termination I is employed to determine whether BP should stop.

3.3 Proposed Termination Criterion

The termination criterion is to detect a valid codeword. It is equivalent to determine whether the estimated information bits are correct. In this sense, the cyclic redundancy check (CRC) [8] can perform the same role: determine whether the estimated codeword is correct.

In this paper, the CRC based early termination is proposed to be used in the BP decoding of polar codes. This requires that before the polar code encoding, CRC check bits are added to the information bits. Let K_{crc} be the number of CRC check bits. If the original information set $\mathcal{A}$ has a size of K, then $K - K_{crc}$ information bits plus the K_{crc} check bits go to the polar encoder. In the decoding side, when the estimates of the K bits are obtained in every BP decoding iteration, CRC check can be done: if the estimated information bits pass the CRC check, then the BP decoding can stop. Otherwise, the BP decoding continues until the CRC check succeeds or the maximum number of iterations is reached. In this paper, we call this termination the CRC-based Termination (written as Termination-CRC).

Figure 5 shows the number of iterations and the error cases. It can be seen that except the uncorrectable blocks, the number of blocks in error with small iteration numbers are greatly reduced compared with Termination I and Termination II, showing the effectiveness of Termination-CRC.

When the errors in the CRC check bits and the errors in the information bits align, it can happen that the CRC check succeeds while the estimated information bits are incorrect. This can be confirmed from Fig. 5: there are six blocks in error (among 10^5 polar blocks) with small numbers of iterations. To remove these error cases, we further propose to combine Termination II with CRC checks. The scheme works as the following:

– Add K_{crc} check bits to $K - K_{crc}$ information bits;
– Encoding the K bits into N coded symbols;
– Perform BP decoding to the received N samples;
– Employ Termination II and CRC check of the estimated bits. If both succeed, terminate the BP decoding. Otherwise, continue the BP iterations until both checks are successful or the maximum number of iterations is reached.

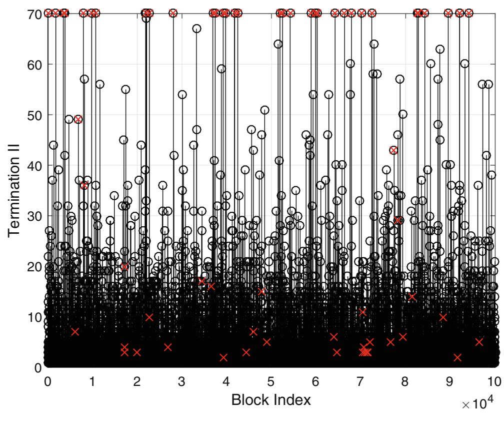


Fig. 4. The number of iterations of the BP decoding of the polar code with the code length $N = 256$ and the code rate $R = 0.36$. The maximum number of iterations is 70 in the fading channel. Termination II is employed to determine whether BP should stop.

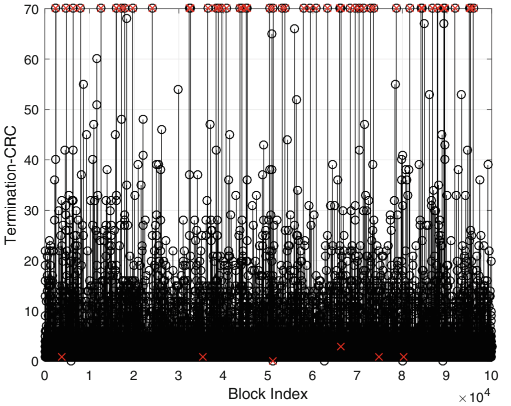


Fig. 5. The number of iterations of the BP decoding of the polar code with the code length $N = 256$ and the code rate $R = 0.36$. The maximum number of iterations is 70 in the fading channel. Termination-CRC is employed to determine whether BP should stop.

This combination of Termination II with CRC checks are termed Termination II-CRC in the paper. The iteration numbers and the error cases are reported in Fig. 6. It can be observed that except for the uncorrectable blocks, the error due to the incorrect early termination does not occur with the proposed Termination II-CRC. In the next section, the error performance of different termination criterions is presented to show the effectiveness of every termination criterion.

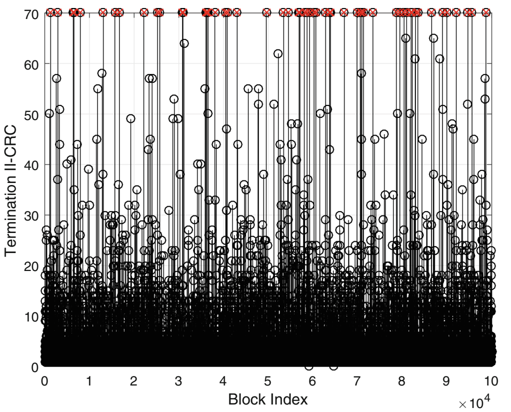


Fig. 6. The number of iterations of the BP decoding of the polar code with the code length $N = 256$ and the code rate $R = 0.36$. The maximum number of iterations is 70 in the fading channel. Termination II-CRC is employed to determine whether BP should stop.

4 Simulation Results

Binary phase shift keying (BPSK) modulation is employed to the coded symbols of polar codes. The polar code simulated has the block length $N = 256$ and the code rate R is 0.36. The maximum number of iterations for the BP decoding is 70. The modulated coded symbol x is transmitted through a fading channel: $y = hx + n$, where h follows a standard normal distribution, and n is the additive white Gaussian noise (AWGN) with mean zero and variance σ^2. With Termination-CRC and Termination II-CRC, 16 CRC check bits (with the polynomial 0×8810) are added to the information bits.

Figure 7 shows the bit error rate (BER) and packet error rate (PER) of the polar code in the fading channel. In the receiver side, assume the sign of the fading channel parameter h is estimated already from the channel estimation function, denoted as s_h. The receiver then performs the operation $\tilde{y} = s_h y = |h|x + s_h h$. After this operation, the SC or the BP decoding can start.

In Fig. 7, the line with the legend "BP I = 70" refers to the BP decoding with a fixed number of iterations of 70. It can be observed that with this fixed number of runs, the performance of the BP decoding is slightly better than the SC decoding, unlike the results in Fig. 1-b where the BP decoding performance (a fixed iteration number of 30) is worse than that of the SC decoding. This shows that in the fading channel, the BP decoding requires a larger number of iterations to converge than in the AWGN channels.

However, as shown in Figs. 3, 4, 5 and 6, the majority of blocks converge earlier than the maximum number. Given that the maximum number is already larger than that of the AWGN channels, it is desirable to stop the BP decoding when it converges.

Seen from Fig. 7, Termination I (the black solid line and the black dashed line) is not effective: it exhibits an error floor. This is due to a large number of incorrect early terminations. Termination II (the green solid and dashed lines with triangles) is much more effective than Termination I. Termination-CRC is slightly better than Termination II, also can be predicted from Figs. 4 and 5. The best criterion is Termination II-CRC (the solid and dashed lines with squares). This best performance can also be predicted from Fig. 6: there are no early termination errors for Termination II-CRC. Only those uncorrectable blocks are left with the proposed Termination II-CRC.

Figure 8 shows the average number of iterations of the BP decoding employing different termination criterions. The flat line with the legend "BP I = 70" refers to BP decoding with a fixed number of runs of 70. It can be seen that, on average Termination-CRC and Termination II-CRC requires almost the same number of iterations. Given that Termination II-CRC produces the best performance, it is desirable to implement this termination criterion in the BP decoding of polar codes in fading channels.

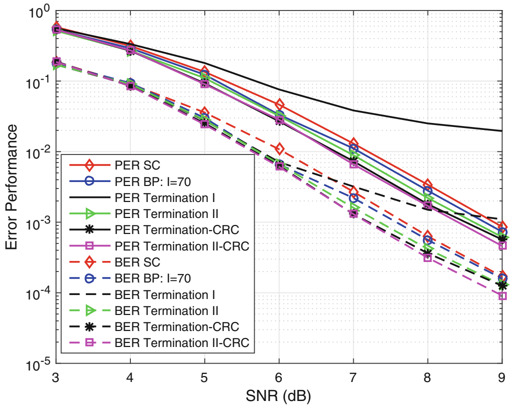


Fig. 7. BER and PER performance comparison of polar codes with the code length $N = 256$, code rate $R = 0.36$. The maximum number of iterations is 70 in the fading channel.

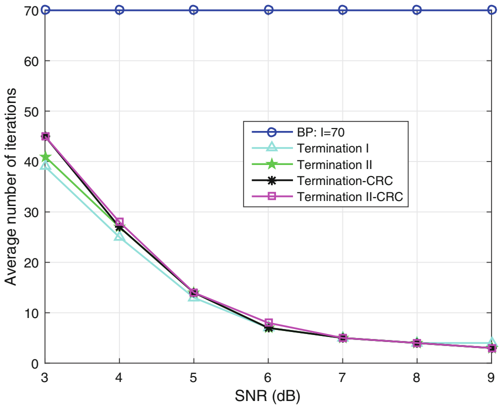


Fig. 8. Average number of iterations for BP decodings with different termination criterions. The label with "BP I = 70" is the BP decoding with a fixed number of iterations of 70.

5 Conclusion

This paper studies the performance of BP decoding in fading channels. Different termination criterions are investigated and a new termination criterion is proposed. The proposed termination can achieve the best BP decoding performance in fading channels among the existing termination criterions.

References

1. 3GPP: 3GPP TSG RAN WG1 meeting #87, chairmans notes of agenda item 7.1.5 channel coding and modulation (2016)
2. Arikan, E.: A performance comparison of polar codes and reed-muller codes. IEEE Commun. Lett. **12**(6), 447–449 (2008)
3. Arikan, E.: Channel polarization: a method for constructing capacity-achieving codes for symmetric binary-input memoryless channels. IEEE Trans. Inf. Theory **55**(7), 3051–3073 (2009)
4. Arikan, E.: Systematic polar coding. IEEE Commun. Lett. **15**(8), 860–862 (2011)
5. Guo, J., Qin, M., i Fabregas, A.G., Siegel, P.H.: Enhanced belief propagation decoding of polar codes through concatenation. In: 2014 IEEE International Symposium on Information Theory Proceedings (ISIT), pp. 2987–2991 (2014)
6. Hussami, N., Korada, S., Urbanke, R.: Performance of polar codes for channel and source coding. In: IEEE International Symposium on Information Theory (ISIT), pp. 1488–1492, June 2009
7. Li, L., Zhang, W.: On the encoding complexity of systematic polar codes. In: Proceedings of IEEE International System-on-Chip Conference (SOCC), pp. 508–513, September 2015
8. Lin, S., Costello, D.J.: Error Control Coding, 2nd edn. Pearson Prentice Hall, New Jersey (2004)
9. Niu, K., Chen, K.: CRC-aided decoding of polar codes. IEEE Commun. Lett. **16**(10), 1668–1671 (2012)
10. Deng, R., Li, L., Hu, Y..: On the polar code encoding in fading channels. Veh. Technol. Conf. **62**(24), 1–5 (2017)
11. Si, H., Koyluoglu, O.O., Vishwanath, S.: Polar coding for fading channels: binary and exponential channel cases. IEEE Trans. Commun. **62**(8), 2638–2649 (2014)
12. Tal, I., Vardy, A.: List decoding of polar codes. IEEE Trans. Inf. Theory **61**(5), 2213–2226 (2015)
13. Yuan, B., Parhi, K.K.: Early stopping criteria for energy-efficient low-latency belief-propagation polar code decoders. IEEE Trans. Sig. Process. **62**(24), 6496–6506 (2014)