An Approach to the Synchronization of Dynamic Complex Network Combining Degree Distribution and Eigenvector Criteria

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Abstract. Synchronization is an important behavior of a dynamic complex network. Traditional methods, like changing network structure, enhancing coupling capability and utilizing external control strategies, etc., cannot achieve complete network synchronization. As it is easier for the small-world networks to achieve topology synchronization than regular networks and random networks, from the viewpoint of the topology of complex networks, we propose the method of Degree Distribution Connection (DDC) for balancing the degree distribution of nodes in a network. And further, we propose the method of Enhanced Synchronization Small-World (ESSW) for constructing the network combining degree distribution and eigenvector criteria, deleting connecting edge by preferentially selecting the node with a larger degree and reconnecting the edge according to the eigenvector criterion. The experimental results show that our solution is effective that can solve the network synchronization problem well by improving network topology.

Keywords: Dynamic complex network · Synchronization · Degree distribution · Eigenvector criterion · Degree Distribution Connection (DDC) · Enhanced Synchronization Small-World (ESSW)

1 Introduction

A dynamic complex network is usually a network of basic units formed by a series of nodes and their interconnecting edges. The phenomenon that the behaviors of different nodes gradually tend to be the same or similar over time is called synchronization, which is a significant characteristic of behavior of dynamic network. As synchronization behavior and its regular patterns can explain how complex systems work together, the study of network synchronization has always been an important research direction in the field of complex networks, which has a wide range of applications, including industrial robots' cooperation in factory, unmanned aerial vehicles (UAVs) formation, epidemic spread suppression, etc.

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Many traditional methods were proposed to improve the capability of synchronization, including enhancing coupling capability [1–3], changing network structure [12,18,20–23], etc. In addition, when a network itself could not reach its state of synchronization, some external control strategies [4–11] were also applied to promote network to synchronization state. Since it was not easy to change the coupling mode and the dynamic properties, also it was mostly uncontrollable to dominate the results of external control, these methods could not fundamentally achieve complete synchronization.

Watts and Strogatz [12] pioneerely proposed their significant discovery of small-world and established a WS small-world network model. Their results proved that small-world network had both large clustering coefficient and small average path length, which had great advantages for many applications related to collaboration. Based on their work, many researchers had devoted themselves to studying the synchronization problem of complex networks with the properties of small-world; and found that the small-worlds could significantly improve the synchronization capability of dynamic networks [13], which were generally believed that the topology of small-world was easier to achieve synchronization than regular network and random network.

Analyzing the internal causes for the dynamic network synchronization, we think that, from the perspective of basic components of complex networks and their interactions, subtle changes in network topology will have a greater impact on the synchronization of complex networks, and it is required for us to consider both local dynamics of nodes and global interactions among topologies. Therefore, we propose an approach to small-world network synchronization that combines degree distribution and eigenvector criteria. Nodes are no longer selected randomly for reconnection, but selected that can keep degree distribution of network as uniform as possible. At the same time, reconnection edges are handled by the eigenvector criteria. This method can improve synchronization capability of the small-world network.

The rest of the paper is organized as follows. Section 2 discusses related work. Section 3 presents the overview framework of our method. Our methods of DDC and ESSW are proposed in Sect. 4 and Sect. 5, respectively. Section 6 gives experiment results. Conclusions are finally presented in Sect. 7.

2 Related Work

Focusing on solving the network synchronization problem, many researchers have carried out a lot of work from different aspects such as node degree distribution, adding edges and network topology, etc.

Node Degree Distribution. Nishikawa et al. [14] proposed that large variance of node degree distribution would inhibit network synchronization, even if average path length of network was long, a uniform node degree distribution could still make network better synchronization. But, Hong et al. [15] pointed out that a short average path length could improve synchronization, and if network degree distribution showed an uneven state, it would also enhance synchronization capability. Lv and Li [16] once designed a PA algorithm for the construction of small-world network through a selective reconnection strategy. Their method randomly selected node connections according to a weight proportional to node degree, however, synchronization was slightly lower than WS small-world. Their results showed that the uneven degree distribution could inhibit network synchronization.

Adding Edges. Hagberg and Schult [17] selected those edges that had the greatest impact on synchronization based on eigenvector of Laplacian matrix of network, and achieved network synchronization by deleting the selected edges. The method could effectively maintain the number of edges in network, but the effect of synchronization was not as good as those methods of adding edges. Dai and Wang [18] presented that clustering coefficient and average path length could not determine network synchronization independently. Based on eigenvector criterion, they proposed small-world network algorithm SOSW-I and SOSW-II with the improved strategy of adding edge and reconnecting edge, respectively, which made good effects of synchronization. Wang et al. [19] improved synchronization by adding new edges to nodes with smaller degree in the nearest neighbors, but the method required to change some characteristic parameters of network, which would have a certain impact on process of synchronization evolution. Zeng et al. [20] proposed a method of residual edge-betweenness gradient (REBG) to select edge according to the betweenness of edge to improve synchronization, but the method might be hindered under some certain conditions, causing the network to not be fully synchronized.

Network Topology. Many researchers tried to solve the network synchronization problem by changing network topology [12,18,20–23]. Classical WS model suggested to modify the network topology by rewiring certain amount of the existing links based on a certain probability [12]. Dai and Wang made their experiments on SOSW-I and SOSW-II, which showed that different networks with similar topological characteristics could have different synchronization capabilities [18]. Through the comparison of small-world algorithms, Allan presented that synchronization could be achieved if it was promoted by manipulating topology appropriately [21]. Some other related methods were also proposed. Based on Zeng et al.'s work [20], Hou et al. [22] further divided candidate nodes into four different categories, improved network structure using a maximum forward matching strategy, and improved the overall network with local topological features. Wen et al. [23] studied the network synchronization problem with directed switching topology, and proposed a Lyapunov function to analyze synchronization. Reviewing these methods, it was necessary to find a solution by exploring the internal causes for the improvements of synchronization capability of network.

3 Overview Framework

3.1 Network Synchronization Problem

Suppose a continuous-time coupled dynamic network with N nodes is $G(V, E)$, where V is a set of nodes and E is a set of edges connecting nodes. The state equation of node i in G is represented by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}x_j, \quad i = 1, 2, \ldots, N \tag{1}$$

where x_i is the state variable of node i, f is the function of state transition, c is constant of coupling strength of network, $c > 0$.

The network topology is represented by a negative Laplace matrix, called coupling matrix, which is defined as $A = (a_{ij}) \in R^{N\times N}$. If there exists an edge between node i and node j, then it corresponds to the value of element a_{ij} in A. Set the weight of edge be m, $m > 0$. Since coupling matrix is undirected, which is a symmetric matrix, then $a_{ji} = a_{ij}$. If there is no edge between node i and node j, then element a_{ij} in A is 0. Similarly, $a_{ji} = 0$. The diagonal element of a negative Laplacian matrix is the inverse of the sum of elements in each row, and its value is less than 0, i.e. $a_{ii} = -\sum\limits_{j=1, j\neq i}^{N} a_{ij},\ i = 1,\ 2,\ ...,\ N$. When $t \to \infty$, if the dynamic network defined by Eq. (1) satisfies Eq. (2), then it reaches a fully synchronized state, that is,

$$x_1(t) = x_2(t) = \ldots = x_N(t) = s(t) \tag{2}$$

where $x_i(t)$ represents the state of node i in network at time t, and $s(t)$ represents the synchronization state of dynamic network (1), that is, a solution that satisfies the network conditions.

Definition 1 (Network Synchronization Problem, NSP): Let the size of nodes of dynamic network be N, $x_i(t)$ is the state variable of node i in network at time t, $s(t)$ is a solution where the isolated node satisfies the coupling condition. If there is $t > 0$, that the state of the nodes of the entire network are roughly consistent, approaching a certain same state, then the network is called to achieve identity synchronization, satisfying

$$\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 \to 0, \quad i = 1, 2, \ldots, N \tag{3}$$

where $\|x_i(t) - s(t)\|_2$ is the 2-norm of $x_i(t) - s(t)$.

A is a symmetric irreducible matrix, and A of dynamic network (1) is a negative Laplace matrix. Since the minimum eigenvalue of Laplacian matrix is 0, the sum of each row of A is equal to 0, and the maximum eigenvalue is 0, other eigenvalues are negative. So, the N eigenvalues of A satisfy

$$0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_N \tag{4}$$

It can be seen that the second eigenvalue λ_2 is a negative number. The smaller the value of λ_2, the larger its absolute value $|\lambda_2|$.

When dynamic network (1) satisfies Eq. (5), then its state reaches asymptotic stability [24].

$$c \geqslant |\frac{r}{\lambda_2}| \tag{5}$$

where r is a constant determined by dynamic equation of node. Therefore, c is actually determined by λ_2. For dynamic network (1), if A is constant, then network can achieve synchronization with a small coupling strength c, and synchronization capability of network is strong. Inequality (4) shows that synchronization capability of dynamic network can be measured by λ_2 of its coupling matrix. The smaller the value of λ_2 is, the stronger the synchronization capability is.

3.2 Our Method

The goal of the paper is to propose an improved small-world network based on the combination of degree distribution and eigenvector criterion to improve the capability of network synchronization.

After analyzing the influence of node degree distribution on network synchronization, we can see, uneven degree distribution of nodes will inhibit network synchronization to a certain extent. So, we propose a method of degree distribution based on connection (DDC). Define and calculate the variance of network degree distribution. According to the value of variance, the node with smaller degree is preferentially selected, and degree distribution of nodes in network is kept as uniform as possible. On this basis, combined with the eigenvector criterion, we further propose the method of modeling Enhanced Synchronization Small-World (ESSW) network. Select nodes to delete edges according to the probability proportional to the value of weight of node. When reconnecting edges, select two nodes according to the eigenvector criterion to connect edges selectively. We can further analyze how to improve network synchronization capability by changing small-world network topology, so as to find a way to solve the network synchronization problem.

4 Network Connection Based on Degree Distribution

4.1 Degree Distribution

Degree distribution of node in network is an important characteristic parameter used to describe network topology, and its distribution can be judged by the degree distribution variance.

Definition 2 (Degree distribution variance): Suppose the total number of network nodes is N, and d_i is degree of node i. The variance of degree distribution σ satisfies

$$\sigma^2 = \frac{\sum_{1}^{N} d_i^2 - \frac{(\sum_{1}^{N} d_i)^2}{N}}{N} \tag{6}$$

Hong et al. [15] presented that making node degree distribution of network uneven could improve network synchronization capability. Lv and Li [16] presented a different opinion. They proposed a PA algorithm for the construction of small-world. And their experiments showed that the gradual increase in the variance of degree distribution was not the main reason for enhancing synchronization capability of small-world network.

When the reconnection probability P is small, with reconnection of edges, the average path length of small-world network will decrease rapidly. So, synchronization capability of network is mainly determined by the change of average path length. However, when the change rate of average path length of small-world network is almost 0, if the variance of degree distribution of the current network is greater than a certain threshold, synchronization capability of network will be then inhibited. Therefore, we think, the uneven node degree distribution of small-world network will inhibit the improvement of synchronization capability of network to some extent.

4.2 The Method of DDC

The reconnection edge of WS small-world was not prioritized, but connected completely randomly. In the paper, we design a network connection method based on degree distribution. In the reconnection, the node with a smaller degree is preferentially selected for connection, and the node degree distribution of network is kept as uniform as possible.

Give preference to those nodes that can make the node degree distribution more uniform, which can reduce the uneven distribution of degrees during the reconnection process. The weight is set for node by the probability inversely proportional to the network node degree, and then the reconnection strategy is set according to the probability proportional to the node weight. Nodes with a larger degree are not easy to participate in edge connection, so the degree of node generally does not increase; Nodes with a smaller degree have a greater probability of connecting new edges, and the degree will increase. Thus, the node degree distribution will tend to be average.

The weight W_i of a node i $(i = 1, 2, \ldots, N)$ in network is represented by

$$W_i = \begin{cases} \left(\dfrac{1}{d_i}\right)^x & , d_i \neq 0 \\ \quad 1 & , \text{othervise} \end{cases} \tag{7}$$

where x is weight coefficient of the adjusted node. The larger x is, the easier it is for those nodes with smaller degrees to be selected. d_i is degree of node i. When degree is 0, then weight $W_i = 1$.

4.3 The Algorithm of DDC

Let the total number of network nodes be N, the initial number of neighbors is k (must be an even number), reconnection probability is P, and weight coefficient is x. The steps of the implementation of DDC algorithm are described as follows.

Step 1: According to k, each node in network is connected to the $k/2$ nearest neighbor nodes, and the initial network G is obtained.

Step 2: Traverse the edge set E of G, and generate a random number p for each edge in E. If $p > P$, then skip this edge and continue to traverse the next edge in the set; otherwise, perform a selective reconnection operation.

Step 3: Calculate the degree distribution of node of network. Calculate the inverse of the degree of each node, i.e.,$1/d_i$. If the degree is 0, then set its reciprocal to 1.

Step 4: Calculate the x-th power of the reciprocal of the degree of each node, that is $(1/d_i)^x$, to obtain the weight value of the node.

Step 5: Select the node to connect an edge according to the probability proportional to the value of node weight.

Step 6: Update the current network and check whether all edges in E have been traversed. If not, then continue to traverse the next edge and return to **Step 2**; otherwise, the algorithm ends.

5 Enhanced Synchronization Small-World

On the basis of Sect. 4, further combining the eigenvector criterion, we propose our modeling ESSW method, which has stronger capability of synchronization, but not losing the small-world characteristics.

5.1 Eigenvector Criterion

It can be known from inequality (4) that, in order to improve the capability of network synchronization, the second eigenvalue λ_2 of the coupling matrix A of the network should be made as small as possible. Suppose that n edges with weight m are added to the current network. If the newly added edges can minimize λ_2 of A, then the added edges are the optimal solution for network synchronization.

The coupling matrix of the network after adding an edge is defined as $A(m) = A + m\Delta A$, where $\Delta A = (\Delta a_{ij})_{N\times N}$, representing the change of A after adding edges to the network. If an edge is added between node i and node j, and the weight is m, then $\Delta a_{ij} = \Delta a_{ji} = m$; otherwise $\Delta a_{ij} = \Delta a_{ji} = 0$. Diagonal element is $\Delta a_{ii} = -\sum_{j=1,j\neq i}^{N} \Delta a_{ij},\ i = 1,\ 2,\ ...,\ N$. After adding edges, the eigenvalues of $A(m)$ are denoted as $0 = \lambda_1(m) > \lambda_2(m) \geq \lambda_3(m) \geq ... \geq \lambda_N(m)$. Since a new edge is added to the original network, the value of λ_2 always decreases, i.e., $\lambda_2(m) < \lambda_2 < 0$. In order to improve the capability of network synchronization, it is necessary to change the network topology by adding edges to satisfy $min(\lambda_2(m))$.

Give the corresponding unit eigenvector $\xi(m)$ for $\lambda_2(m)$ of $A(m)$. It can be obtained $A(m)\xi(m) = \lambda_2(m)\xi(m)$ from the matrix definition, and after transformation, it can be obtained as

$$\lambda_2(m) = \xi(m)^T(A + m\Delta A)\xi(m) \tag{8}$$

Taking the partial derivative with respect to m on both sides of Eqs. (8), (9) is obtained.

$$\frac{\partial \lambda_2(m)}{\partial m} = \xi(m)^T \varDelta A \xi(m) + 2\lambda_2(m)\xi(m)^T \frac{\partial \xi(m)}{\partial m} \tag{9}$$

Due to $\xi(m)^T\xi(m) = 1$, Eq. (9) can be simplified as

$$\frac{\partial \lambda_2(m)}{\partial m} = \xi(m)^T \varDelta A \xi(m) \tag{10}$$

In the formula, if m is small enough, then the size of λ_2 has nothing to do with the weight m of the edge. Therefore, the problem can turn into how to add a new edge to the original network to get a new coupling matrix $A(m)$, which minimizes $\lambda_2(m)$.

It is known from Eq. (10) that making the value of the left side of the equation smaller $\lambda_2(m)$ is equivalent to how to construct $\varDelta A$, so that the value of the right side of the equation $\xi^T \varDelta A \xi$ is the smallest. Since the result is independent of the size of m, here $\xi \triangleq [\xi_1,\ \xi_2,\ ...,\ \xi_N]$ is the unit feature vector corresponding to λ_2 of A of the initial network.

For the convenience of calculation, we only add one edge to the original network each time. After the new coupling matrix is obtained, it is used as the new network coupling matrix, and then the calculation is performed, and the subsequent edge is added. The eigenvector criterion for adding edge is defined as shown in Eq. (11).

$$\min \xi^T \varDelta A \xi = \min_{i,j \in E} \{-(\xi_i - \xi_j)^2\} \tag{11}$$

where E is the set of edges of network. In order to minimize the value of the right side of the equation, two nodes i and j should be selected so as to minimize $-(\xi_i - \xi_j)^2$, that is, to maximize $|\xi_i - \xi_j|$. Therefore, in the process of network construction, we can refer to the nodes obtained by the eigenvector criterion and connect them, the value of $\lambda_2(m)$ can be then minimized, and the capability of network synchronization can be improved to the greatest extent.

5.2 The Method of ESSW

For the classical WS small-world network, the reconnection edges are random, that is, the connection nodes are not selected, but reconnected randomly, which depends on the reconnection probability P. Although random reconnection can improve the capability of network synchronization, it is not the optimal. In order to solve the problem of complex network synchronization, we propose an ESSW small-world network modeling method that combines degree distribution and eigenvector criteria. When constructing a small-world network, reconnect edges selectively, so that network can be more synchronized than those networks by random reconnection edges.

Give scale of the given network N, number of neighbors k, and reconnection probability P. Calculate the maximum number of iterations T of the algorithm according to Formula (12), where T is also the number of reconnected edges in ESSW.

$$T = \frac{P \times N \times k}{2} \tag{12}$$

When reconnecting edges, the selection is made according to the eigenvector criterion. For the network G_t of the t-th iteration, find λ_2 corresponding to A and the corresponding unit eigenvector ξ, where $\xi \triangleq [\xi_1,\ \xi_2,\ ...,\ \xi_N]^T$. If the corresponding element $a_{ij} = 0$ in A, it means that there is no connection between nodes i and j. Then, the distance gap between the components ξ_i and ξ_j of ξ corresponding to i and j is calculated.

Traverse the nodes in network, and add an edge between i and j for the node pair (i, j) with the largest distance gap, satisfying

$$\max gap = \max_{i,j \in E} |\xi_i - \xi_j| \tag{13}$$

where E is the set of edges of network.

The network obtained by our method can make it converge the fastest, get the optimal solution, and have stronger synchronization capability.

5.3 The Algorithm of ESSW

Let the total number of nodes in network be N, the number of neighbors be k (must be an even number), the maximum number of iterations be T, and reconnection probability is P. The steps of the implementation of ESSW are described as follows.

Step 1: According to k, each node in network is connected to the $k/2$ nearest neighbor nodes, and the initial network G is obtained.

Step 2: According to P and the number of edges e of the initial network, the number T of iterations of the algorithm is calculated.

Step 3: In the $(t+1)$-th iteration, the network obtained in the t-th iteration is taken as the network G_t of this iteration.

Step 4: Find the node degree distribution of the current network G_t, and select a node according to the probability proportional to the node degree, and delete the edge connected to it.

Step 5: Obtain A of network $G_t^{'}$ after deleting edge, calculate the λ_2 of A, and obtain the unit feature vector ξ corresponding to λ_2.

Step 6: According to the reconnection policy of ESSW, add a new edge between node i and node j.

Step 7: Update the network and check whether the algorithm has reached the maximum number of iterations T. If it is not reached, then return to **Step 3**, and take the network G_{t+1} obtained by the $(t + 1)$-th iteration as a new network, and continue to perform the reconnection operation; otherwise, the algorithm ends.

6 Experiments

6.1 Evaluation Indexes

Parameters, such as average path length of network, clustering coefficient, network betweenness, maximum node degree, and network diameter, determine the basic characteristics of topology of complex network. They are reflected in the network topology, which makes network have topological properties different from other networks, and promotes or inhibits certain dynamic behaviors of network.

Definition 3 (Average Path Length): It is defined as the minimum number of edges through which two pairs of nodes are related to each other in network, denoted by L. It can be calculated by Formula (14).

$$L = \frac{\sum\limits_{i,j} d(v_i, v_j)}{N(N-1)} \tag{14}$$

where N is the total number of nodes in network. When $v_i = v_j$ or no path between v_i and v_j, then $d(v_i, v_j) = 0$.

Definition 4 (Clustering Coefficient): It is defined as the ratio of the actual number of connections between nodes to the total number of connections, denoted as C_i, which is expressed as the degree of association between node i and its neighbor nodes. It can be calculated by Formula (15).

$$C_i = \frac{2n_i}{k_i(k_i - 1)} \tag{15}$$

where k_i represents the degree of node i, and n_i represents the actual number of connections between node $i$and all adjacent nodes.

Definition 5 (Network Betweenness): The betweenness of node k in network satisfies Formula (16).

$$BC_k = \frac{\sum\limits_{i \neq j} \frac{n_k(i,j)}{n(i,j)}}{(N-1)(N-2)} \tag{16}$$

where $n(i, j)$ is the number of shortest paths between node i and node j. $n_k(i, j)$ is the number of shortest paths between node i and node j through node k. The larger the node betweenness is, the more important the node plays in interacting with other nodes.

Definition 6 (maximum node degree): It is defined as the maximum value of degree distribution of node in network, denoted as d_{max}, which can be calculated by Formula (17). To a certain extent, it reflects the unevenness of degree distribution of network.

$$d_{\max} = \max_{v_i \in Nodes} d_i \tag{17}$$

where $Nodes$ is a collection of nodes in network. d_i is degree of node i as defined in the **Definition** 2.

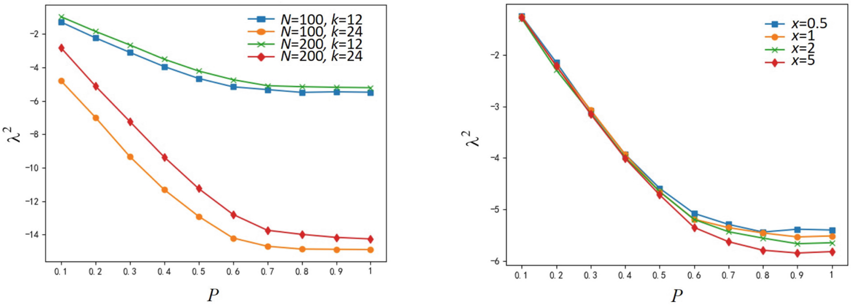
Definition 7 (Network diameter): It is defined as the maximum distance between any two node pairs in network, denoted as D, which has an important impact on the stability of the entire network. It can be calculated by Formula (18).

$$D = \max_{v_i, v_j \in Nodes} l(v_i, v_j) \tag{18}$$

where $l(v_i, v_j)$ represents the distance between node i and node j.

6.2 The Experiments of DDC

Figure 1 shows the results of the effects of number of network nodes N, number of neighbors k and weight coefficient x on the synchronization of small-world network based on degree distribution.



(a) Numbers of nodes and neighbors (b) Weight coefficient (N = 100, k = 12)

Fig. 1. The influence of various parameters on the synchronization of small-world network based on degree distribution

Figure 1(a) represents the effects of N and k on the capability of network synchronization. From the figure, each curve presents a trend of monotonically decreasing, indicating that, under different parameter combinations, as P increases, λ_2 becomes smaller, and synchronization capability of small-world network is gradually improved. For the same N, the larger the k is, the stronger the network synchronization is. For the same k, the smaller N is, the stronger the network synchronization is.

Figure 1(b) represents the effect of x on the capability of network synchronization. As can be seen from the figure, with the increase of P, the synchronization capability gradually improves. When $P > 0.4$, different x has different effects on the synchronization. The larger x is, the faster the value of λ_2 decreases, indicating that the synchronization is stronger.

Table 1. Comparison of the λ_2 of different weight coefficients in small-world network based on degree distribution

P	x			
	0.5	1	2	5
0.1	−1.242	−1.237	−1.259	−1.319
0.2	−2.223	−2.183	−2.205	−2.165
0.3	−3.122	−3.090	−3.166	−3.154
0.4	−3.884	−3.958	−3.946	−4.030
0.5	−4.590	−4.604	−4.719	−4.770
0.6	−5.042	−5.201	−5.305	−5.355
0.7	−5.297	−5.364	−5.480	−5.628
0.8	−5.394	−5.469	−5.570	−5.815
0.9	−5.377	−5.522	−5.644	−5.823
1	−5.360	−5.571	−5.662	−5.807

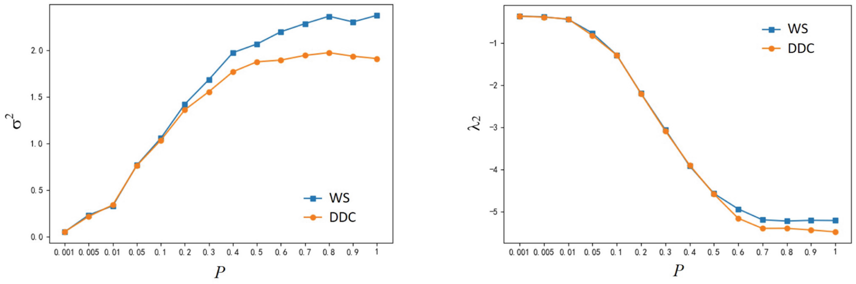
The values corresponding to Fig. 1(b) are shown in Table 1.

As can be seen from Table 1, when $P = 0.2$, the value of λ_2 for $x = 5$ is larger than that for $x = 0.5$, $x = 1$, and $x = 2$, representing that although the increase of x can effectively improve the synchronization capability of small-world network, when P is small, too much pursuit of the uniformity of degree distribution may reduce the chance of adding edges that can greatly improve synchronization in random reconnection. Therefore, in practical application, it is necessary to set $x$reasonably according to the requirements.

Set $N = 100$ and $k = 12$, Fig. 2 compares the variance of degree distribution and network synchronization as the reconnection probability increases between WS [12] and our DDC method.

As can be seen from Fig. 2(a), when $P > 0.5$, the difference between the variances of degree distributions of node of the two networks gradually widens. This is because the larger P is, the larger the proportion of reconnected edges in network is, and DDC preferentially selects nodes with smaller degree for reconnection each time. Also, as can be seen from Fig. 2(b), when $P < 0.5$, the values of λ_2 of the two networks is not significantly different. It is because there is little difference in the variance of the degree distribution of node, so the impact on the capability of network synchronization is small. When $P > 0.5$, the capability of synchronization of DDC is stronger than that of WS small-world network.

It can be seen from Fig. 3, when N and k are constant, the larger P is, the stronger the synchronization capability of small-world network is. When N and P are constant, the larger the k is, the stronger the synchronization capability of small-world network is; At the same time, when k and P are constant, the smaller N is, the stronger the synchronization capability of small-world network is. Therefore, N and k not only determine the strength of network synchronization, but also determine the strength of the network synchronization obtained at the end of the algorithm iteration. It shows that topology plays an important role in determining the synchronization of ESSW.



(a) Degree distribution variance (b) Network synchronization capability

Fig. 2. Parameters comparison of WS and DDC based on degree distribution (N = 100, k = 12)

6.3 The Experiments of ESSW

Figure 3 shows the results of the influence of different parameter combinations of number of network nodes N and number of neighbors k on the ESSW algorithm.

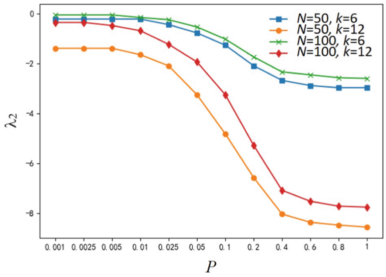


Fig. 3. Parameter analysis of ESSW

In order to demonstrate the improvement of synchronization capability of ESSW, in the following, we compare ESSW with WS [12] as well as some improved small-world networks related to synchronization, such as SOSW-II [18] and PA [16]. Given $N = 100$ and $k = 12$. All connections are assumed to be symmetrical with the same coupling strength. The comparison results of synchronization capabilities of the four algorithms are shown in Fig. 4, and the values are shown in Table 2.

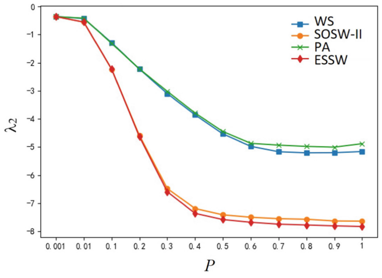


Fig. 4. Comparison of capability of network synchronization of algorithms

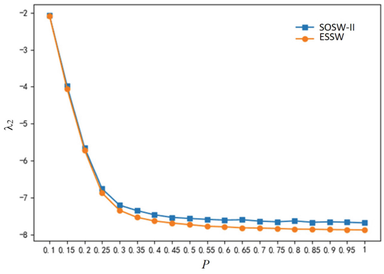


Fig. 5. Comparison of synchronization capabilities of SOSW-II and ESSW

As can be seen from Fig. 4 and Table 2, when $P = 0.8$, the difference of λ_2 between ESSW and SOSW-II is the largest, the former is 0.205 smaller than the latter, and ESSW has stronger synchronization capability. Compared with WS and PA, the λ_2 of ESSW has the largest difference at $P = 0.4$ and $P = 0.3$, the difference is 3.505 and 3.572, respectively, representing that the capability of synchronization of ESSW is stronger than WS and PA.

Table 2. Comparison of λ_2 of different algorithms

P	WS	SOSW-II	PA	ESSW
0.001	−0.356	−0.356	−0.356	−0.356
0.01	−0.426	−0.555	−0.413	−0.554
0.1	−1.284	−2.228	−1.309	−2.229
0.2	−2.225	−4.583	−2.219	−4.629
0.3	−3.097	−6.475	−3.017	−6.589
0.4	−3.854	−7.187	−3.789	−7.359
0.5	−4.532	−7.41	−4.448	−7.581
0.6	−4.973	−7.497	−4.866	−7.677
0.7	−5.166	−7.549	−4.930	−7.745
0.8	−5.206	−7.569	−4.976	−7.774
0.9	−5.200	−7.630	−5.002	−7.805
1	−5.159	−7.638	−4.879	−7.828

For SOSW-II algorithm, it also adopted the eigenvector criterion as the reconnection strategy. We compare it with our ESSW, and the result is shown in Fig. 5. When ESSW deletes edge, it selects nodes based on the node degree distribution, because it adjusts the subtle changes of topology, we can see, ESSW can always have better synchronization capability than that of SOSW-II with the increase of P.

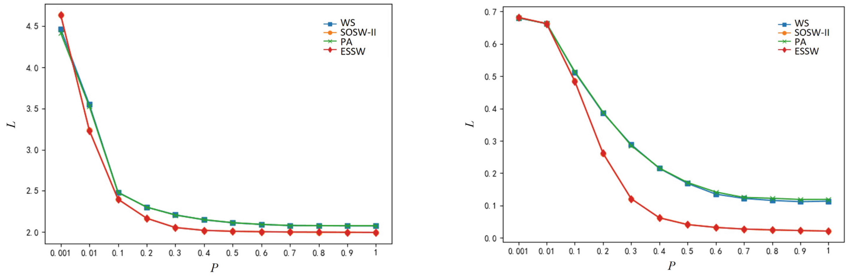
6.4 Analysis of Influence of Network Topology on Synchronization Capability

Set number of nodes $N = 100$, number of neighbors $k = 12$, reconnection probability $P \in (0.001, 1)$. Using four different small-world network construction algorithms, include WS, SOSW-II, PA and our ESSW, we conduct the following experiments to observe that the basic characteristic parameters change with the increase of reconnection probability P. These parameters include average path length L, clustering coefficient C, maximum betweenness B^{max}, degree distribution variance σ^2, maximum degree d^{max} and diameter D, which can represent the characteristics of small-world network. The experimental results are shown in Fig. 6.

From Fig. 6(a), all L decreases monotonically as P increases. Therefore, the reduction of L may improve synchronization to a certain extent. On the other hand, the L of SOSW-II and ESSW almost coincide, but their synchronization capability is quite different, showing that L cannot determine synchronization alone. From Fig. 6(b), as P increases, all C decreases monotonically, so C may affect synchronization. However, referring to Fig. 6(a), when L and C are similar, synchronization performance is still quite different. From Fig. 6(c), when $P > 0.01$, all B^{max} gradually decreased with increasing P. The B^{max} of ESSW and SOSW-II are very similar, however, comparing with Figs. 4 and 5, when $P > 0.1$, B^{max} of ESSW and SOSW-II are almost the same, but their synchronization ability is quite different. Hence, synchronization cannot be judged by B^{max} only. From Figs. 6(d) and 6(e), σ^2 of WS and PA shows a monotonically increasing trend with the increase of P, d^{max} also increases gradually, indicating that node degree distribution is becoming more and more uneven. From Fig. 6(f), all D decreases monotonically with the increase of P. When P = 0.1, D of different networks is almost the same, However, it can be seen from Fig. 4 that synchronization capability varies greatly, which means that synchronization cannot be judged simply based on D.

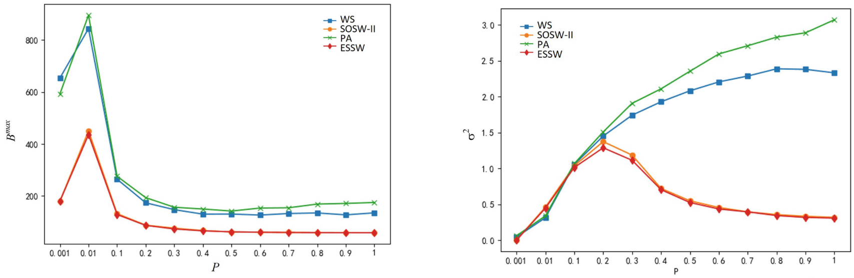
According to these results, we can conclude that the difference of topology structure will lead to the change of network synchronization capability. But this change is not absolute but the result of a combination of multiple parameters.

1) Each network parameters cannot independently determine network synchronization. Some characteristics are similar but may also represent completely different synchronization capability.
2) Maximum betweenness cannot be used as the only criterion for judging synchronization capability.
3) If the node degree distribution is too uneven, the inhibition effect is more obvious on network synchronization. Only a small average path length and a uniform network degree distribution can make small-world network more synchronous.



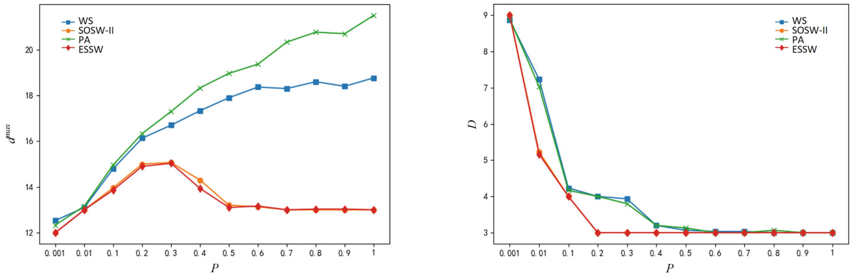
(a) Average path length L varies with reconnection probability P

(b) Clustering coefficient C varies with reconnection probability P



(c) Maximum betweenness B^{max} varies with reconnection probability P

(d) Degree distribution variance σ^2 varies with reconnection probability P



(e) Maximum degree of node dmax changes with reconnection probability P

(f) Diameter D changes with reconnection probability P

Fig. 6. Analysis of influence of network topology on synchronization capability

7 Conclusions

Our main work is summarized as follows.

1) By analyzing the influence of degree distribution on network synchronization, we define the variance of degree distribution of network, and propose the method of degree distribution connection. This method does not select nodes randomly, but those nodes with smaller degree according to the calculation weight to set the reconnection strategy. The experimental results show

that the increase of variance of the optimized network degree distribution is significantly slower than that of WS small-world, but the synchronization capability is stronger than that of WS, which it is proved that the method of making the network degree distribution uniform is effective for solving complex network synchronization problem.

2) Combining degree distribution with eigenvector criterion, we propose the method of building ESSW network to improve the synchronization capability of small-world network. The method preferentially selects nodes with larger degrees when deleting network edges, and reconnect edges to network according to eigenvector criterion. The experimental results show that the synchronization of ESSW is stronger than that of WS, SOSW-II and PA. Which it is proved that network construction method combining degree distribution and eigenvector criterion is effective.
3) We analyze the influence of different algorithms on synchronization through a series of comparative experiments. The results show that, (1) No single characteristic parameter can independently determine the strength of network synchronization; (2) The network betweenness has certain limitations as a parameter to judge synchronization. In some cases, network synchronization with the same network betweenness may still be quite different; (3) When reconnection probability is small, average path length will decrease rapidly, and degree distribution will have little effect on network synchronization. However, when variation of average path length decreases and gradually maintains a certain constant, excessively large degree distribution variance will inhibit network synchronization ability.

To sum up, network synchronization may change due to the slight topology change, which is the result of the joint interaction of multiple topological properties. In the paper, we propose a method of combining degree distribution and eigenvector criterion from the viewpoint of network topology, which provides a new solution to the network synchronization problem.

Our further work includes: 1) In real systems, nodes in network may often interfered by environment, causing failure sometimes, which will affect synchronization capability of network. So, we shall improve our algorithms to enhance synchronization robustness, so that complex network can adapt to changes of environment. 2) The methods proposed in the paper are modeled from the mathematical description. We shall improve our algorithms to fit the real applications.

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