



# Numerical Results via High-Order Iterative Scheme to a Nonlinear Wave Equation with Source Containing Two Unknown Boundary Values

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**Abstract.** In the present paper, we consider an initial-boundary value problem for nonlinear equation in which the source term contains two boundary values. First, the local existence and uniqueness of a weak solution to this problem is inferred directly from [12], in which a recurrent sequence via a  $N$ -order iterative scheme is established and then the  $N$ -order convergent rate of the sequence to the unique weak solution of the proposed model is also proved. As  $N = 2$ , the corresponding scheme called 2-order iterative scheme is considered, and a numerical algorithm to calculate approximate solutions via the 2-order iterative scheme is constructed by the finite-difference method. Finally, a numerical example is presented to evaluate the errors between the exact solution and the approximate solution. The numerical results show that the errors are decreasing as the fineness of meshes is increasing.

**Keywords:** Nonlinear wave equation · High-order iterative scheme · 2-order iterative scheme · Finite-difference scheme · Numerical results

**Classification:** 35L20 · 35L70 · 35Q72

## 1 Introduction

Our paper concerns with a Robin problem for a nonlinear equation as follows

$$u_{tt} - u_{xx} = f(x, t, u(x, t)), \quad u(0, t), \quad u(1, t), \quad 0 < x < 1, \quad 0 < t < T, \quad (1)$$

$$u_x(0, t) - h_0 u(0, t) = u_x(1, t) + h_1 u(1, t) = 0, \quad (2)$$

$$u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \tag{3}$$

where  $h_0 \geq 0, h_1 \geq 0$  are constants, with  $h_0 + h_1 > 0$  and  $f, \tilde{u}_0, \tilde{u}_1$  are given functions.

For a approach of mathematical model, due to the presence of the unknown values  $u(0, t)$  and  $u(1, t)$  in the nonlinear terms, we take into account the fact that the problem (1)–(3) is related to nonlocal problems arised naturally in some applied sciences such as fluid mechanics, heat transfer theory and control theory, see [4, 5]. The early studies of this type were considered by Agarwal [1], and later by Gupta [2] and Il'in and Moiseev [3], and the references cited therein.

We consider a recurrent sequence  $\{u^{(m)}\}$  associated with the Eq. (1) (see [12]) as follows

$$\begin{aligned} u_0 &\equiv 0, \\ \frac{\partial^2 u^{(m)}}{\partial t^2} - \Delta u^{(m)} &= \sum_{0 \leq i+j+s \leq N-1} D^{ijs} f[u^{(m-1)}] (u^{(m)} - u^{(m-1)})^i (u^{(m)}(0, t) - u^{(m-1)}(0, t))^j \\ &\quad \times (u^{(m)}(1, t) - u^{(m-1)}(1, t))^s, \quad 0 < x < 1, \quad 0 < t < T, \quad m \geq 1, \end{aligned} \tag{4}$$

where

$$D^{ijs} f[u^{(m-1)}](x, t) = \frac{1}{i!j!s!} D_3^i D_4^j D_5^s f(x, t, u^{(m-1)}(x, t), u^{(m-1)}(0, t), u^{(m-1)}(1, t)), \tag{5}$$

and  $u^{(m)}$  satisfies (2) and (3). The above sequence is called high-order iterative scheme if it converges to the weak solution  $u$  of (1)–(3) and satisfies the estimation  $\|u^{(m)} - u\|_X \leq C \|u^{(m-1)} - u\|_X^N$ , for some  $C > 0$ , all integer numbers  $N \geq 2$  and  $X$  is a certain Banach space. This definition is derived from the investigation of Newton-like methods in Banach spaces, see [13]. The high-order iterative scheme is also applied to some previous works, for example as in [9, 10, 12, 16]. Especially, when  $N = 2$ , we get from (4) and (5) the following 2-order iterative scheme

$$\begin{cases} u_{tt}^{(m)} - u_{xx}^{(m)} = F^{(m)}(x, t), \quad 0 < x < 1, \quad 0 < t < T, \\ u_x^{(m)}(0, t) - h_0 u^{(m)}(0, t) = u_x^{(m)}(1, t) + h_1 u^{(m)}(1, t) = 0, \\ u^{(m)}(x, 0) = \tilde{u}_0(x), \quad u_t^{(m)}(x, 0) = \tilde{u}_1(x), \end{cases} \tag{6}$$

where

$$\begin{aligned} F^{(m)}(x, t) &= f[u^{(m-1)}](x, t) + D_3 f[u^{(m-1)}](x, t) (u^{(m)}(x, t) - u^{(m-1)}(x, t)) \\ &\quad + D_4 f[u^{(m-1)}](x, t) (u^{(m)}(0, t) - u^{(m-1)}(0, t)) \\ &\quad + D_5 f[u^{(m-1)}](x, t) (u^{(m)}(1, t) - u^{(m-1)}(1, t)). \end{aligned} \tag{7}$$

In order to construct the numerical solutions of the 2-order iterative scheme (6)–(7), we shall use the spatial mesh  $x_i = ih, h = 1/N, i = 0, 1, \dots, N$ , and Pinder’s difference formulas [14] (pages 36, 43) to approximate the  $k^{th}$  derivatives. Then the problem (6)–(7) will be transferred to a second-order system of ordinary equations (in time variable) of unknown functions  $u_i^{(m)}(t) = u^{(m)}(x_i, t), i = 0, 1, \dots, N$ , this method were also used in [7, 8, 11, 15, 17]. Next, by using the discretization with time mesh  $t_j = j\Delta t, \Delta t = T/M, j = 0, 1, \dots, M$ , we will get an algorithm to determine the approximate solutions of  $u^{(m)}$  to the scheme (6)–(7), which is given by the following difference equation (see the formula (38))

$$\bar{u}_{j+1}^{(m)} = \left[ 2E - (\Delta t)^2 A_j^{(m)} \right] u_j^{(m)} - u_{j-1}^{(m)} + (\Delta t)^2 \bar{\delta}_j^{(m)}, j = 1, \dots, M - 1, \quad (8)$$

where  $A_j^{(m)}$  and  $\bar{\delta}_j^{(m)}$  are defined as in (37). Finally, we will show a numerical example to evaluate the errors between the approximate solution and the exact solution.

The outline of the paper is as follows. In Sect. 2, we introduce some notations and modified lemmas. In Sect. 3, by using the finite-difference method, we establish a numerical algorithm to determine the finite-difference approximate solutions of  $u^{(m)}$  to the iterative scheme (6)–(7). Moreover, a numerical example is presented to evaluate the errors between the approximate solution  $u^{(m)}$  and the exact solution  $u_{ex}$ . In Sect. 4, the conclusion is presented to describe the main results of the paper.

## 2 Preliminaries

In this section, we present some materials that we shall use in order to present our results.

Let  $\Omega = (0, 1), Q_T = (0, 1) \times (0, T)$  and  $\langle \cdot, \cdot \rangle$  be the scalar product in  $L^2$ , i.e.,

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx,$$

and the corresponding norm  $\|\cdot\|$ , i.e.,  $\|u\|^2 = \langle u, u \rangle$ . Also, we denote  $\|\cdot\|_X$  is the norm in the Banach space  $X$ , and  $L^p(0, T; X), 1 \leq p \leq \infty$  for the Banach space of real functions  $u : (0, T) \rightarrow X$  measurable with the corresponding norm  $\|\cdot\|_{L^p(0, T; X)}$  defined by

$$\|u\|_{L^p(0, T; X)} = \left( \int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 < t < T} \|u(t)\|_X \text{ for } p = \infty.$$

Let  $u(t)$ ,  $u'(t) = u_t(t) = \dot{u}(t)$ ,  $u''(t) = u_{tt}(t) = \ddot{u}(t)$ ,  $u_x(t) = \nabla u(t)$ ,  $u_{xx}(t) = \Delta u(t)$ , denote  $u(x, t)$ ,  $\frac{\partial u}{\partial t}(x, t)$ ,  $\frac{\partial^2 u}{\partial t^2}(x, t)$ ,  $\frac{\partial u}{\partial x}(x, t)$ ,  $\frac{\partial^2 u}{\partial x^2}(x, t)$ , respectively.

For  $f \in C^k([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ ,  $f = f(x, t, y_1, y_2, y_3)$ , we put  $D_1 f = \frac{\partial f}{\partial x}$ ,  $D_2 f = \frac{\partial f}{\partial t}$ ,  $D_{2+i} f = \frac{\partial f}{\partial y_i}$ ,  $i = 1, 2, 3$  and  $D^\alpha f = D_1^{\alpha_1} \dots D_5^{\alpha_5} f$ ;  $\alpha = (\alpha_1, \dots, \alpha_5) \in \mathbb{Z}_+^5$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_5 = k$ ,  $D^{(0, \dots, 0)} f = D^{(0)} f = f$ .

We consider the symmetric bilinear form  $a(\cdot, \cdot)$  defined by

$$a(u, v) = \int_0^1 u_x(x)v_x(x)dx + h_0 u(0)v(0) + h_1 u(1)v(1), \text{ with } u, v \in H^1, \quad (9)$$

and the corresponding norm  $\|v\|_a = \sqrt{a(v, v)}$ .

Then, two norms  $\|v\|_{H^1}$ , and  $\|v\|_a$  are equivalent norms on  $H^1$ . Furthermore, the following inequalities are valid, see [12] (Lemma 2.1 and 2.2)

- (i)  $\|v\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|v\|_{H^1}$ ,
  - (ii)  $\sqrt{a_0} \|v\|_{H^1} \leq \|v\|_a \leq \sqrt{a_1} \|v\|_{H^1}$ ,
  - (iii)  $|a(u, v)| \leq a_1 \|u\|_{H^1} \|v\|_{H^1}$ , for all  $u, v \in H^1$ ,
  - (iv)  $a(v, v) \geq a_0 \|v\|_{H^1}^2$ , for all  $v \in H^1$ ,
- (10)

where  $a_0 = \frac{1}{3} \min\{1, \max\{h_0, h_1\}\}$  and  $a_1 = 1 + 2(h_0 + h_1)$ .

**Definition 1.** A function  $u = u(x, t)$  is a weak solution of the problem (1)–(3) if

$$u \in L^\infty(0, T; H^2), \quad u' \in L^\infty(0, T; H^1), \quad u'' \in L^\infty(0, T; L^2),$$

and  $u$  satisfies the following variational equation

$$\langle u''(t), w \rangle + a(u(t), w) = \langle f[u](t), w \rangle, \forall w \in H^1, \text{ and a.e., } t \in (0, T), \quad (11)$$

where  $f[u](x, t) = f(x, t, u(x, t), u(0, t), u(1, t))$ , together with the initial conditions

$$u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1. \quad (12)$$

For our study of problem (1)–(3) we will need the following assumptions

(H<sub>1</sub>)  $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$ ;

(H<sub>2</sub>)  $f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$  such that

- (i)  $D_3^i D_4^j D_5^s f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ , with  $0 \leq i + j + s \leq 2$ ,
- (ii)  $D_1 D_3^i D_4^j D_5^s f, D_3^{i+1} D_4^j D_5^s f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ , with  $i + j + s = 1$ .

Let fixed positive constant  $T^*$  and positive constant  $M$ . For every  $T \in (0, T^*]$ , we put

$$W_T = \{v \in L^\infty(0, T; H^2) : v_t \in L^\infty(0, T; H^1), \quad v_{tt} \in L^2(Q_T)\}. \quad (13)$$

Note that  $W_T$  is a Banach space with respect to the norm

$$\|v\|_{W_T} = \max\{\|v\|_{L^\infty(0,T;H^2)}, \|v_t\|_{L^\infty(0,T;H^1)}, \|v_{tt}\|_{L^2(Q_T)}\} \tag{14}$$

(see Lions [6]). We also put

$$\begin{cases} W(M, T) = \{v \in W_T : \|v\|_{W_T} \leq M\}, \\ W_1(M, T) = \{v \in W(M, T) : v_{tt} \in L^\infty(0, T; L^2)\}. \end{cases} \tag{15}$$

To prove the existence and the uniqueness of solution to the problem (1)–(3), we cite the results given in [12] where the authors used the high-order iterative method to construct a recurrence sequence converging, at  $N$ -order rate, to the weak solutions of the problem (1)–(3). Therefore, in case of  $N = 2$ , the existence and the convergence of the recurrence  $\{u^{(m)}\}$  defined by (6)–(7) are claimed the following theorems.

**Theorem 1.** *Let  $(H_1)$ – $(H_2)$  hold. Then there are two positive constants  $M$  and  $T$  depending on  $\tilde{u}_0, \tilde{u}_1, f, h_0, h_1$ , such that, for  $u_0 \equiv 0$ , the recurrent sequence  $\{u^{(m)}\}$  defined by (6)–(7) exists and contains in  $W_1(M, T)$ .*

Using Theorem 1 and some arguments of weak convergence, the following theorem is also confirmed.

**Theorem 2.** *Let  $(H_1)$  –  $(H_2)$  hold. Then, there are two positive constants  $M$  and  $T$  such that the problem (1)–(3) has a unique weak solution  $u \in W_1(M, T)$  and the recurrent sequence  $\{u^{(m)}\}$  defined by (6)–(7) converges, at 2-order rate, to  $u$  strongly in the space  $W_1(T)$  in sense*

$$\|u^{(m)} - u\|_{W_1(T)} \leq C \|u^{(m-1)} - u\|_{W_1(T)}^2, \tag{16}$$

for all  $m \geq 1$ , where  $C$  is a suitable constant. On the other hand, the following estimate is fulfilled

$$\|u^{(m)} - u\|_{W_1(T)} \leq C_T (k_T)^{2^m}, \text{ for all } m \in \mathbb{N}, \tag{17}$$

where  $C_T > 0$  and  $0 < k_T < 1$  are constants depending only on  $T$ .

### 3 Numerical Results

Consider the following problem

$$\begin{cases} u_{tt} - u_{xx} = f(x, t, u(x, t), u(0, t), u(1, t)), & 0 < x < 1, \ 0 < t < T, \\ u_x(0, t) - h_0 u(0, t) = u_x(1, t) + h_1 u(1, t) = 0, \\ u(x, 0) = \tilde{u}_0(x), \ u_t(x, 0) = \tilde{u}_1(x), \end{cases} \tag{18}$$

where

$$\left\{ \begin{array}{l} f(x, t, u(x, t), u(0, t), u(1, t)) = \frac{1}{16}|u|u + \frac{25}{16}u^2(0, t) + \frac{125}{27}u^3(1, t) + G(x, t), \\ h_0 = 1, h_1 = 2, \\ G(x, t) = -e^{-\frac{3t}{2}} + \frac{1}{4} \left( -x^2 + \frac{4}{5}x + \frac{44}{5} \right) e^{-\frac{t}{2}} \\ \qquad \qquad \qquad - \frac{1}{16} \left( x^4 - \frac{8}{5}x^3 - \frac{24}{25}x^2 + \frac{32}{25}x + \frac{416}{25} \right) e^{-t} \\ \tilde{u}_0(x) = -x^2 + \frac{4}{5}x + \frac{4}{5}, \quad \tilde{u}_1(x) = \frac{1}{2}x^2 - \frac{2}{5}x - \frac{2}{5} \end{array} \right. \quad (19)$$

Then  $u_{ex}(x, t) = \left( -x^2 + \frac{4}{5}x + \frac{4}{5} \right) e^{-\frac{1}{2}t}$  is the exact solution of (18) corresponding to the constants  $h_0 = 1, h_1 = 2$  and the given functions  $f(x, t, u(x, t), u(0, t), u(1, t)), \tilde{u}_0, \tilde{u}_1$  as in (19).

In this section, we first construct a difference scheme to approximate the solution  $u$  of (18) via approximating  $u^{(m)}$  in the 2-order iterative scheme as follows

$$\left\{ \begin{array}{l} u_{tt}^{(m)} - u_{xx}^{(m)} = F^{(m)}(x, t), \quad 0 < x < 1, \quad 0 < t < T, \\ u_x^{(m)}(0, t) - h_0 u^{(m)}(0, t) = u_x^{(m)}(1, t) + h_1 u^{(m)}(1, t) = 0, \\ u^{(m)}(x, 0) = \tilde{u}_0(x), \quad u_t^{(m)}(x, 0) = \tilde{u}_1(x), \end{array} \right. \quad (20)$$

where

$$\begin{aligned} F^{(m)}(x, t) &= f[u^{(m-1)}](x, t) + D_3 f[u^{(m-1)}](x, t) \left( u^{(m)}(x, t) - u^{(m-1)}(x, t) \right) \\ &\quad + D_4 f[u^{(m-1)}](x, t) \left( u^{(m)}(0, t) - u^{(m-1)}(0, t) \right) \\ &\quad + D_5 f[u^{(m-1)}](x, t) \left( u^{(m)}(1, t) - u^{(m-1)}(1, t) \right) \\ &= \alpha^{(m)}(x, t)u^{(m)}(x, t) + \beta^{(m)}(x, t)u^{(m)}(0, t) \\ &\quad + \gamma^{(m)}(x, t)u^{(m)}(1, t) + \delta^{(m)}(x, t), \end{aligned} \quad (21)$$

and

$$\begin{aligned} \alpha^{(m)}(x, t) &= D_3 f[u^{(m-1)}](x, t), \\ \beta^{(m)}(x, t) &= D_4 f[u^{(m-1)}](x, t), \\ \gamma^{(m)}(x, t) &= D_5 f[u^{(m-1)}](x, t), \\ \delta^{(m)}(x, t) &= f[u^{(m-1)}](x, t) - \alpha^{(m)}(x, t)u^{(m-1)}(x, t) \\ &\quad - \beta^{(m)}(x, t)u^{(m-1)}(0, t) - \gamma^{(m)}(x, t)u^{(m-1)}(1, t), \end{aligned} \quad (22)$$

with the notations

$$\begin{aligned} f[u](x, t) &= f(x, t, u(x, t), u(0, t), u(1, t)), \\ D_i f[u](x, t) &= D_i f(x, t, u(x, t), u(0, t), u(1, t)), \quad i = 3, 4, 5. \end{aligned} \tag{23}$$

Replacing (21) and (22) into (20), we get

$$\begin{cases} u_{tt}^{(m)} - u_{xx}^{(m)} - \alpha^{(m)}(x, t)u^{(m)} - \beta^{(m)}(x, t)u^{(m)}(0, t) - \gamma^{(m)}(x, t)u^{(m)}(1, t) \\ \qquad \qquad \qquad = \delta^{(m)}(x, t), \quad 0 < x < 1, \quad 0 < t < T, \\ u_x^{(m)}(0, t) - h_0 u^{(m)}(0, t) = u_x^{(m)}(1, t) + h_1 u^{(m)}(1, t) = 0, \\ u^{(m)}(x, 0) = \tilde{u}_0(x), \quad u_t^{(m)}(x, 0) = \tilde{u}_1(x). \end{cases} \tag{24}$$

Putting

$$u_i^{(m)}(t) = u^{(m)}(x_i, t), \quad x_i = ih, \quad i = 0, 1, \dots, N, \quad h = \frac{1}{N}. \tag{25}$$

Rewriting (24) at the node  $x = x_i$  :

$$\begin{cases} \ddot{u}_i^{(m)}(t) - \Delta u_i^{(m)}(t) - \alpha_i^{(m)}(t)u_i^{(m)}(t) - \beta_i^{(m)}(t)u_0^{(m)}(t) - \gamma_i^{(m)}(t)u_N^{(m)}(t) \\ \qquad \qquad \qquad = \delta_i^{(m)}(t), \quad 0 \leq i \leq N, \quad 0 \leq t \leq T, \\ u_x^{(m)}(0, t) - h_0 u_0^{(m)}(t) = u_x^{(m)}(1, t) + h_1 u_N^{(m)}(t) = 0, \\ u_i^{(m)}(0) = \tilde{u}_0(x_i) = \tilde{u}_{0i}, \quad \dot{u}_i^{(m)}(0) = \tilde{u}_1(x_i) = \tilde{u}_{1i}, \quad 0 \leq i \leq N, \end{cases} \tag{26}$$

in which

$$\begin{aligned} \alpha_i^{(m)}(t) &= \alpha^{(m)}(x_i, t) = D_3 f[u^{(m-1)}](x_i, t) \\ &= D_3 f(x_i, t, u_i^{(m-1)}(t), u_0^{(m-1)}(t), u_N^{(m-1)}(t)), \\ \beta_i^{(m)}(t) &= \beta^{(m)}(x_i, t) = D_4 f[u^{(m-1)}](x_i, t) \\ &= D_4 f(x_i, t, u_i^{(m-1)}(t), u_0^{(m-1)}(t), u_N^{(m-1)}(t)), \\ \gamma_i^{(m)}(t) &= \gamma^{(m)}(x_i, t) = D_5 f[u^{(m-1)}](x_i, t) \\ &= D_5 f(x_i, t, u_i^{(m-1)}(t), u_0^{(m-1)}(t), u_N^{(m-1)}(t)), \\ \delta_i^{(m)}(t) &= \delta^{(m)}(x_i, t) \\ &= f(x_i, t, u_i^{(m-1)}(t), u_0^{(m-1)}(t), u_N^{(m-1)}(t)) - \alpha_i^{(m)}(t)u_i^{(m-1)}(t) \\ &\quad - \beta_i^{(m)}(t)u_0^{(m-1)}(t) - \gamma_i^{(m)}(t)u_N^{(m-1)}(t). \end{aligned} \tag{27}$$

Replacing the derivatives in spatial variable  $x$  of (26) by the following approximations

$$\begin{aligned}
 u_x^{(m)}(x_i, t) &\simeq \frac{u_i^{(m)}(t) - u_{i-1}^{(m)}(t)}{h}, \\
 u_x^{(m)}(0, t) &\simeq \frac{u_1^{(m)}(t) - u_0^{(m)}(t)}{h}, \\
 u_x^{(m)}(1, t) &\simeq \frac{u_N^{(m)}(t) - u_{N-1}^{(m)}(t)}{h}, \\
 u_{xx}^{(m)}(x_i, t) &\simeq \frac{u_{i-1}^{(m)}(t) - 2u_i^{(m)}(t) + u_{i+1}^{(m)}(t)}{h^2},
 \end{aligned} \tag{28}$$

then we obtain

$$\begin{aligned}
 \ddot{u}_i^{(m)}(t) - \frac{u_{i-1}^{(m)}(t) - 2u_i^{(m)}(t) + u_{i+1}^{(m)}(t)}{h^2} - \alpha_i^{(m)}(t)u_i^{(m)}(t) - \beta_i^{(m)}(t)u_0^{(m)}(t) \\
 - \gamma_i^{(m)}(t)u_N^{(m)}(t) = \delta_i^{(m)}(t), \quad 1 \leq i \leq N - 1, \\
 \frac{u_1^{(m)}(t) - u_0^{(m)}(t)}{h} - h_0u_0^{(m)}(t) = \frac{u_N^{(m)}(t) - u_{N-1}^{(m)}(t)}{h} + h_1u_N^{(m)}(t) = 0, \\
 u_0^{(m)}(t) = \frac{u_1^{(m)}(t)}{1 + hh_0}, \quad u_N^{(m)}(t) = \frac{u_{N-1}^{(m)}(t)}{1 + hh_1}, \\
 u_i^{(m)}(0) = \tilde{u}_0(x_i) = \tilde{u}_{0i}, \quad \dot{u}_i^{(m)}(0) = \tilde{u}_1(x_i) = \tilde{u}_{1i}, \quad 0 \leq i \leq N,
 \end{aligned} \tag{29}$$

Using the boundary conditions (29)<sub>2</sub> with  $u_0^{(m)}(t) = \frac{u_1^{(m)}(t)}{1 + hh_0}$ ,

$u_N^{(m)}(t) = \frac{u_{N-1}^{(m)}(t)}{1 + hh_1}$ , and after eliminating the unknown functions  $u_0(t)$  and  $u_N(t)$  in the equations  $i = 1, 2, N - 2, N - 1$ , then the system (29) is rewritten as follows

$$\left\{ \begin{aligned}
 \ddot{u}_1^{(m)}(t) + \hat{a}_1^{(m)}(t)u_1^{(m)}(t) - \frac{1}{h^2}u_2^{(m)}(t) + \bar{a}_1^{(m)}(t)u_{N-1}^{(m)}(t) &= \delta_1^{(m)}(t), \\
 \ddot{u}_2^{(m)}(t) + \hat{a}_2^{(m)}(t)u_1^{(m)}(t) + b_2^{(m)}(t)u_2^{(m)}(t) - \frac{1}{h^2}u_3^{(m)}(t) \\
 + \bar{a}_2^{(m)}(t)u_{N-1}^{(m)}(t) &= \delta_2^{(m)}(t), \\
 &\vdots \\
 \ddot{u}_i^{(m)}(t) + \hat{a}_i^{(m)}(t)u_1^{(m)}(t) - \frac{1}{h^2}u_{i-1}^{(m)}(t) + b_i^{(m)}(t)u_i^{(m)}(t) - \frac{1}{h^2}u_{i+1}^{(m)}(t) \\
 + \bar{a}_i^{(m)}(t)u_{N-1}^{(m)}(t) &= \delta_i^{(m)}(t), \quad 3 \leq i \leq N - 3, \\
 &\vdots \\
 \ddot{u}_{N-2}^{(m)}(t) + \hat{a}_{N-2}^{(m)}(t)u_1^{(m)}(t) - \frac{1}{h^2}u_{N-3}^{(m)}(t) + b_{N-2}^{(m)}(t)u_{N-2}^{(m)}(t) \\
 + \bar{a}_{N-2}^{(m)}(t)u_{N-1}^{(m)}(t) &= \delta_{N-2}^{(m)}(t), \\
 \ddot{u}_{N-1}^{(m)}(t) + \hat{a}_{N-1}^{(m)}(t)u_1^{(m)}(t) - \frac{1}{h^2}u_{N-2}^{(m)}(t) + \bar{a}_{N-1}^{(m)}(t)u_{N-1}^{(m)}(t) &= \delta_{N-1}^{(m)}(t),
 \end{aligned} \right. \tag{30}$$

where

$$\begin{aligned} \hat{a}_i^{(m)}(t) &= \begin{cases} -\frac{\beta_1^{(m)}(t)}{1+hh_0} - \alpha_1^{(m)}(t) - \frac{1}{h^2(1+hh_0)} + \frac{2}{h^2}, & i = 1, \\ -\frac{\beta_2^{(m)}(t)}{1+hh_0} - \frac{1}{h^2}, & i = 2, \\ -\frac{\beta_i^{(m)}(t)}{1+hh_0}, & 3 \leq i \leq N-1, \end{cases} \\ \bar{a}_i^{(m)}(t) &= \begin{cases} -\frac{\gamma_i^{(m)}(t)}{1+hh_1}, & 1 \leq i \leq N-3, \\ -\frac{\gamma_{N-2}^{(m)}(t)}{1+hh_1} - \frac{1}{h^2}, & i = N-2, \\ -\frac{\gamma_{N-1}^{(m)}(t)}{1+hh_1} - \alpha_{N-1}^{(m)}(t) - \frac{1}{h^2(1+hh_1)} + \frac{2}{h^2}, & i = N-1, \end{cases} \quad (31) \\ b_i^{(m)}(t) &= \frac{2}{h^2} - \alpha_i^{(m)}(t), \quad 2 \leq i \leq N-2. \end{aligned}$$

We rewrite (30) into a vector equation as follows

$$\begin{cases} \ddot{X}^{(m)}(t) + A^{(m)}(t)X^{(m)}(t) = \bar{\delta}^{(m)}(t), \\ X^{(m)}(0) = X_0, \dot{X}^{(m)}(0) = X_1, \end{cases} \quad (32)$$

where

$$\begin{cases} X^{(m)}(t) = \left( u_1^{(m)}(t), \dots, u_{N-1}^{(m)}(t) \right)^T, \\ \bar{\delta}^{(m)}(t) = \left( \delta_1^{(m)}(t), \dots, \delta_{N-1}^{(m)}(t) \right)^T, \\ X_0 = \left( \tilde{u}_0(x_1), \dots, \tilde{u}_0(x_{N-1}) \right)^T, \\ X_1 = \left( \tilde{u}_0(x_1), \dots, \tilde{u}_0(x_{N-1}) \right)^T, \end{cases} \quad (33)$$

and  $A^{(m)}(t) \in \mathfrak{M}_{N-1}$  ( $\mathfrak{M}_{N-1}$  is the set of real  $N-1$ -order matrices) is defined by

$$A^{(m)}(t) = \begin{matrix} \begin{matrix} \hat{a}_1^{(m)}(t) & \frac{-1}{h^2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \bar{a}_1^{(m)}(t) \\ \hat{a}_2^{(m)}(t) & b_2^{(m)}(t) & \frac{-1}{h^2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \bar{a}_2^{(m)}(t) \\ \hat{a}_3^{(m)}(t) & \frac{-1}{h^2} & b_3^{(m)}(t) & \frac{-1}{h^2} & 0 & \cdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \hat{a}_i^{(m)}(t) & 0 & 0 & \frac{-1}{h^2} & b_i^{(m)}(t) & \frac{-1}{h^2} & 0 & \cdots & \vdots & \bar{a}_i^{(m)}(t) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \hat{a}_{N-3}^{(m)}(t) & \vdots & \vdots & \cdots & \cdots & 0 & \frac{-1}{h^2} & b_{N-3}^{(m)}(t) & \frac{-1}{h^2} & \bar{a}_{N-3}^{(m)}(t) \\ \hat{a}_{N-2}^{(m)}(t) & \vdots & \vdots & \cdots & \cdots & \cdots & 0 & \frac{-1}{h^2} & b_{N-2}^{(m)}(t) & \bar{a}_{N-2}^{(m)}(t) \\ \hat{a}_{N-1}^{(m)}(t) & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{-1}{h^2} & \bar{a}_{N-1}^{(m)}(t) \end{matrix} \end{matrix} \quad (34)$$

Approximating the derivatives  $\dot{X}^{(m)}(t_j)$ ,  $\ddot{X}^{(m)}(t_j)$  by the differences in time variable and the following partition

$$\begin{aligned}
 \dot{X}^{(m)}(t_j) &\simeq \frac{X_{j+1}^{(m)} - X_j^{(m)}}{\Delta t}, \\
 \ddot{X}^{(m)}(t_j) &\simeq \frac{X_{j-1}^{(m)} - 2X_j^{(m)} + X_{j+1}^{(m)}}{(\Delta t)^2},
 \end{aligned} \quad (35)$$

$$X_j^{(m)} = X^{(m)}(t_j) = \left( u_1^{(m)}(t_j), \dots, u_{N-1}^{(m)}(t_j) \right)^T,$$

with  $t_j = j\Delta t$ ,  $j = 0, \dots, M$ ,  $\Delta t = \frac{T}{M}$ ,

$$X_1 = \dot{X}^{(m)}(0) \simeq \frac{X_1^{(m)} - X_0^{(m)}}{\Delta t},$$

then the Eq. (32) was rewritten as follows

$$\begin{cases} \frac{X_{j-1}^{(m)} - 2X_j^{(m)} + X_{j+1}^{(m)}}{(\Delta t)^2} + A_j^{(m)}X_j^{(m)} = \bar{\delta}_j^{(m)}, \\ X_0^{(m)} = X_0, \\ X_1^{(m)} = X_0^{(m)} + \Delta t X_1, \end{cases} \quad (36)$$

where

$$\begin{aligned}
 \bar{\delta}_j^{(m)} &= \bar{\delta}^{(m)}(t_j) = \left( \delta_1^{(m)}(t_j), \dots, \delta_{N-1}^{(m)}(t_j) \right)^T, \\
 A_j^{(m)} &= A^{(m)}(t_j).
 \end{aligned} \quad (37)$$

Hence, the Eq. (36) can be rewritten as below

$$\begin{cases} X_{j+1}^{(m)} = \left[ 2E - (\Delta t)^2 A_j^{(m)} \right] X_j^{(m)} - X_{j-1}^{(m)} + (\Delta t)^2 \bar{\delta}_j^{(m)}, j = 1, \dots, M-1, \\ X_0^{(m)} = X_0, \\ X_1^{(m)} = X_0 + \Delta t X_1, \end{cases} \quad (38)$$

in which

$$\begin{cases} X^{(m)}(t) = \left( u_1^{(m)}(t), \dots, u_{N-1}^{(m)}(t) \right)^T = \left( u^{(m)}(x_1, t), \dots, u^{(m)}(x_{N-1}, t) \right)^T, \\ X_j^{(m)} = X^{(m)}(t_j) = \left( u^{(m)}(x_1, t_j), \dots, u^{(m)}(x_{N-1}, t_j) \right)^T, \\ X_0 = \left( \tilde{u}_0(x_1), \dots, \tilde{u}_0(x_{N-1}) \right)^T, \\ X_1 = \left( \tilde{u}_1(x_1), \dots, \tilde{u}_1(x_{N-1}) \right)^T, \\ \bar{\delta}^{(m)}(t) = \left( \delta_1^{(m)}(t), \dots, \delta_{N-1}^{(m)}(t) \right)^T, \\ \bar{\delta}_j^{(m)} = \bar{\delta}^{(m)}(t_j) = \left( \delta_1^{(m)}(t_j), \dots, \delta_{N-1}^{(m)}(t_j) \right)^T, \\ \delta_i^{(m)}(t_j) = \delta^{(m)}(x_i, t_j) \\ \quad = f(x_i, t_j, u_i^{(m-1)}(t_j), u_0^{(m-1)}(t_j), u_N^{(m-1)}(t_j)) - \alpha_i^{(m)}(t_j) u_i^{(m-1)}(t_j) \\ \quad \quad - \beta_i^{(m)}(t_j) u_0^{(m-1)}(t_j) - \gamma_i^{(m)}(t_j) u_N^{(m-1)}(t_j), \\ u_0^{(m-1)}(t) = \frac{u_1^{(m-1)}(t)}{1 + hh_0}, u_N^{(m-1)}(t) = \frac{u_{N-1}^{(m-1)}(t)}{1 + hh_1}, \end{cases} \quad (39)$$

We describe the scheme (38) as follows.

**A.** For fixed constants  $M, N$ . With  $m = 0$ , we give the first vector

$$X_j^{(0)} = \left( u_1^{(0)}(t_j), \dots, u_{N-1}^{(0)}(t_j) \right)^T = \left( u^{(0)}(x_1, t_j), \dots, u^{(0)}(x_{N-1}, t_j) \right)^T \equiv 0, j = 1, \dots, M.$$

**B.** At the  $(m-1)^{\text{th}}$  iterative step, suppose that we know

$$X_j^{(m-1)} = X^{(m-1)}(t_j) = \left( u_1^{(m-1)}(t_j), \dots, u_{N-1}^{(m-1)}(t_j) \right)^T, j = 1, \dots, M.$$

**C.** Then, we compute consecutively the vectors  $X_j^{(m)} = \left( u_1^{(m)}(t_j), \dots, u_{N-1}^{(m)}(t_j) \right)^T, j = 1, \dots, M$  by the steps as below

**C1.** The vector  $X_1^{(m)}$  is defined by  $X_1^{(m)} = \left( u_1^{(m)}(t_1), \dots, u_{N-1}^{(m)}(t_1) \right)^T = X_0 + \Delta t X_1$ .

**C2.** The vector  $X_2^{(m)}$  is defined by  $X_2^{(m)} = \left(u_1^{(m)}(t_2), \dots, u_{N-1}^{(m)}(t_2)\right)^T$ .

(i) For the vectors  $X_0 = (\tilde{u}_0(x_1), \dots, \tilde{u}_0(x_{N-1}))^T$ ,  $X_1 = (\tilde{u}_1(x_1), \dots, \tilde{u}_1(x_{N-1}))^T$ , we compute the values  $\hat{a}_i^{(m)}(t_1)$ , ( $2 \leq i \leq N - 2$ ),  $\bar{a}_i^{(m)}(t_1)$ ,  $b_i^{(m)}(t_1)$  ( $1 \leq i \leq N - 1$ ),  $\alpha_i^{(m)}(t_1)$ ,  $\beta_i^{(m)}(t_1)$ ,  $\gamma_i^{(m)}(t_1)$  and the matrix

$$A_1^{(m)} = A^{(m)}(t_1),$$

and the vector  $\vec{\delta}_1^{(m)} = \vec{\delta}^{(m)}(t_1) = \left(\delta_1^{(m)}(t_1), \dots, \delta_{N-1}^{(m)}(t_1)\right)^T$ .

(ii) Then, the vector  $X_2^{(m)} = \left(u_1^{(m)}(t_2), \dots, u_{N-1}^{(m)}(t_2)\right)^T$  can be given by the following formula

$$X_2^{(m)} = \left[2E - (\Delta t)^2 A_1^{(m)}\right] (X_0 + \Delta t X_1) - X_0 + (\Delta t)^2 \vec{\delta}_1^{(m)}. \quad (40)$$

**C3.** The vector  $X_3^{(m)} = \left(u_1^{(m)}(t_3), \dots, u_{N-1}^{(m)}(t_3)\right)^T$  is computed as follows.

(i) Computing the values  $\hat{a}_i^{(m)}(t_2)$ , ( $2 \leq i \leq N - 2$ ),  $\bar{a}_i^{(m)}(t_2)$ ,  $b_i^{(m)}(t_1)$  ( $1 \leq i \leq N - 1$ ),  $\alpha_i^{(m)}(t_2)$ ,  $\beta_i^{(m)}(t_2)$ ,  $\gamma_i^{(m)}(t_2)$  and the matrix

$$A_2^{(m)} = A^{(m)}(t_2),$$

and the vector

$$\vec{\delta}_2^{(m)} = \vec{\delta}^{(m)}(t_2) = \left(\delta_1^{(m)}(t_2), \dots, \delta_{N-1}^{(m)}(t_2)\right)^T.$$

(ii) Then, the vector  $X_3^{(m)} = \left(u_1^{(m)}(t_3), \dots, u_{N-1}^{(m)}(t_3)\right)^T$  is defined by

$$X_3^{(m)} = \left[2E - (\Delta t)^2 A_2^{(m)}\right] X_2^{(m)} - (X_0 + \Delta t X_1) + (\Delta t)^2 \vec{\delta}_2^{(m)}. \quad (41)$$

**C4.** The vector  $X_{j+1}^{(m)} = \left(u_1^{(m)}(t_{j+1}), \dots, u_{N-1}^{(m)}(t_{j+1})\right)^T$  is computed as follows.

Suppose that we have calculated the vectors  $X_1^{(m)}$ ,  $X_2^{(m)}$ ,  $\dots$ ,  $X_j^{(m)}$ , then the vector  $X_{j+1}^{(m)} = \left(u_1^{(m)}(t_{j+1}), \dots, u_{N-1}^{(m)}(t_{j+1})\right)^T$  can be determined by recurrence as follows

(i) Computing the values  $\hat{a}_i^{(m)}(t_j)$ , ( $2 \leq i \leq N - 2$ ),  $\bar{a}_i^{(m)}(t_j)$ ,  $b_i^{(m)}(t_j)$  ( $1 \leq i \leq N - 1$ ),  $\alpha_i^{(m)}(t_j)$ ,  $\beta_i^{(m)}(t_j)$ ,  $\gamma_i^{(m)}(t_j)$  and the matrix

$$A_j^{(m)} = A^{(m)}(t_j),$$

and the vector

$$\vec{\delta}_j^{(m)} = \vec{\delta}^{(m)}(t_j) = \left(\delta_1^{(m)}(t_j), \dots, \delta_{N-1}^{(m)}(t_j)\right)^T.$$

(ii) Then, the vector  $X_{j+1}^{(m)} = \left(u_1^{(m)}(t_{j+1}), \dots, u_{N-1}^{(m)}(t_{j+1})\right)^T$  can be determined by the following formula

$$X_{j+1}^{(m)} = \left[2E - (\Delta t)^2 A_j^{(m)}\right] X_j^{(m)} - X_{j-1}^{(m)} + (\Delta t)^2 \vec{\delta}_j^{(m)}. \tag{42}$$

We do the process of computation until  $j = M - 1$ , then we can determine  $X_j^{(m)}$  by

$$X_j^{(m)} = \left(u^{(m)}(x_1, t_j), \dots, u^{(m)}(x_{N-1}, t_j)\right)^T, \quad 1 \leq j \leq M. \tag{43}$$

**C5.** The error of two iterative steps, the  $m^{\text{th}}$  step and the  $(m - 1)^{\text{th}}$  step, is defined by

$$\left\|u^{(m)} - u^{(m-1)}\right\|_{M,N} = \max_{1 \leq j \leq M} \max_{1 \leq i \leq N-1} \left|u^{(m)}(x_i, t_j) - u^{(m-1)}(x_i, t_j)\right|. \tag{44}$$

The iterative process will be stopped at the  $m^{\text{th}}$  step when the error of two steps is satisfied

$$\left\|u^{(m)} - u^{(m-1)}\right\|_{M,N} < 10^{-4}. \tag{45}$$

**C6.** The error of the approximate solution  $u^{(m)}(x, t)$  and the exact solution  $u_{ex}(x, t)$  is defined by

$$E_{M,N} = \left\|u^{(m)} - u_{ex}\right\|_{M,N} = \max_{1 \leq j \leq M} \max_{1 \leq i \leq N-1} \left|u^{(m)}(x_i, t_j) - u_{ex}(x_i, t_j)\right|. \tag{46}$$

All computations were carried out using MATLAB software.

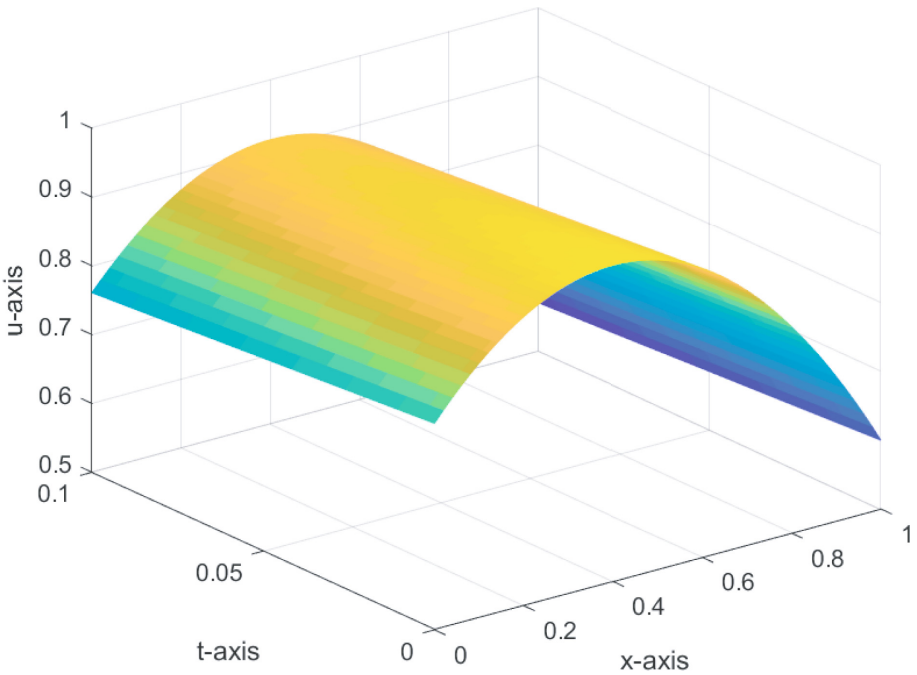
With  $T = 0.1$ ,  $N = 10, 20, 30, 40, 50$  and  $M = N^2$ , Table 1 describes the errors  $E_{N,M}$  between the approximate solution  $u^{(m)}(x, t)$  and the exact solution  $u_{ex}(x, t)$ , corresponding to the different meshes. The following figures describe the curved surfaces of approximate solution  $u^{(m)}(x, t)$  corresponding to mesh of  $N = 50, M = 2500$ , and the curved surface of exact solution  $u_{ex}(x, t)$  corresponding to the mesh of  $N = 50$  and  $M = 2500$ . Specifically, we have some remarks as follows.

(i) At the stop of iteration process, Table 1 presents the errors  $E_{N,M}$  between the approximate solution  $u^{(m)}(x, t)$  and the exact solution  $u_{ex}(x, t)$ , corresponding to the different meshes. It is clear that the errors will be decreasing when the fineness of meshes is increasing.

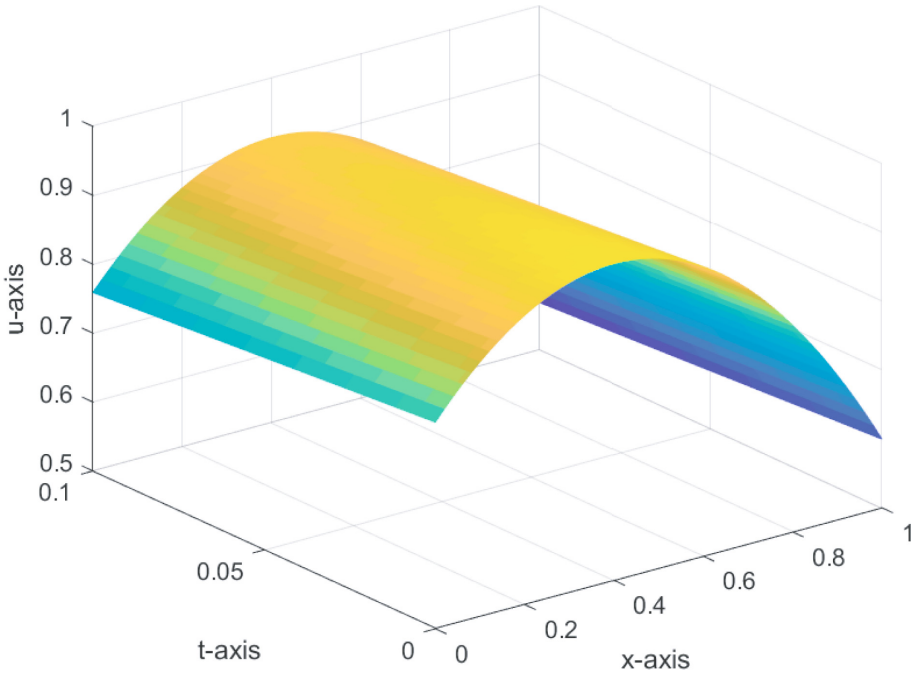
**Table 1.** Errors of approximate solution  $u^{(m)}(x, t)$  and exact solution  $u_{ex}(x, t)$  corresponding to different meshes

$N$	$M$	$E_{N,M}$
10	100	0,004304964136035
20	400	0,003423100759947
30	900	0,002560797440241
40	1600	0,001966395627641
50	2500	0,001602844628803

(ii) Fig. 1 and Fig. 2, respectively, describe the curved surface of exact solution  $u_{ex}(x, t)$  and of approximate solution  $u^{(m)}(x, t)$  corresponding to mesh of  $N = 50$  and  $M = 2500$ .



**Fig. 1.** The curved surface of exact solution  $u_{ex}(x, t)$  corresponding to mesh of  $N = 50$ ,  $M = 2500$



**Fig. 2.** The curved surface of approximate solution  $u^{(m)}(x, t)$  corresponding to mesh of  $N = 50, M = 2500$

### 4 Conclusions

In the present paper, we consider an initial-boundary value problem for a non-linear wave equation in which the source contains two boundary values. First, the existence of a recurrent sequence via the 2-order iterative scheme and its convergence to the unique weak solution of the proposed model are directly claimed by the results given in [12]. Next, a numerical algorithm for finding the approximate solutions corresponding to this scheme is constructed by the finite-difference method. To close the paper, a concrete example is numerically considered to evaluate the errors between the approximate solution and the exact solution.

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