



Dynamics of a Chemostat-Type Model with Impulsive Effects in a Polluted Karst Environment

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Abstract. In this paper, we present a chemostat-type model with impulsive effects in a polluted karst environment. The globally asymptotically stable sufficient condition are gained for a microorganism-extinction periodic solution. System permanent condition are also presented. The results are illustrated by simulations.

Keywords: Chemostat-type model · Impulsive diffusing · Pulse inputting · Polluted karst environments · Extinction

1 Introduction

There are wide underground rivers in the underground rock gaps of karst areas. It easy for pollutants and nutrient of soils to come into aquifers for the thin soil layers of karst areas [1]. Many authors [2–4] employed SDE,PDE,ODE and IDE to study chemostat systems. References [5,6] devoted themselves to the effects of toxicants on population. But there are few authors devoted themselves to karst environmental chemostat system with impulsive diffusing and pulse inputting. We mark a notation as $N = nT, N + L = (n + l)T$.

2 The Model

In this paper, we present a chemostat-type model with impulsive effects in a polluted karst environment

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$$\left. \begin{array}{l}
\left. \begin{array}{l}
\frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\
\frac{dx_2}{dt} = D_2(x_2^0 - x_2) \\
\quad - \frac{1}{\delta} \times \frac{\eta x_2 Y}{\alpha + x_2}, \\
\frac{dY}{dt} = -D_2 \\
\quad + \frac{\eta x_2 Y}{\alpha + x_2} - \beta c_1 Y, \\
\frac{dc_1}{dt} = f_1 c_2 - (g_1 + m_1) c_1, \\
\frac{dc_2}{dt} = -h_1 c_2,
\end{array} \right\} t \neq (N+L), t \neq (N+1), \\
\left. \begin{array}{l}
x_1^+ = x_1, \\
x_2^+ = x_2, \\
Y^+ = Y, \\
c_1^+ = c_1, \\
c_2^+ = (1 - \theta_1) c_2,
\end{array} \right\} t = (N+L), \\
\left. \begin{array}{l}
x_1^+ = (1-d)x_1 + dx_2 + \theta_2, \\
x_2^+ = dx_1(1-d)x_2, \\
Y^+ = Y, \\
c_1^+ = (1-h_3)c_1, \\
c_2^+ = c_2 + \theta,
\end{array} \right\} t = (N+1),
\end{array} \right\} \quad (1)$$

where system (1) is constructed by two patches, patch (1) is a non-polluted environment and patch (2) is a polluted environment. The meanings of the variables and parameters can be consulted from reference [3-5].

3 The Foundation

If $Y = 0$, there are two subsystems of system (1) as

$$\left. \begin{array}{l}
\left. \begin{array}{l}
\frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\
\frac{dx_2}{dt} = D_2(x_2^0 - x_2),
\end{array} \right\} t \neq N, \\
\left. \begin{array}{l}
x_1^+ = (1+d)x_1 + dx_2 + \theta_1, \\
x_2^+ = dx_1 + (1-d)x_2,
\end{array} \right\} t = N,
\end{array} \right\} \quad (2)$$

and

$$\left\{ \begin{array}{l} \frac{dc_1}{dt} = f_1 c_2 - (g_1 + m_1) c_1, \\ \frac{dc_2}{dt} = -h_1 c_2, \end{array} \right\} t \neq N, t \neq (N + 1),$$

$$\left\{ \begin{array}{l} \Delta c_1 = 0, \\ \Delta c_2 = -\theta_1 c_2, \end{array} \right\} t = (N + L),$$

$$\left\{ \begin{array}{l} \Delta c_1 = -\theta_3 c_1, \\ \Delta c_2 = \theta, \end{array} \right\} t = (N + 1).$$
(3)

i) Integrating on system (2)

$$x_i(t) = x_i^0 - [x_i^0 - x_i(nT^+)]e^{-D_i(t-nT)}. \quad (4)$$

Then,

$$\left\{ \begin{array}{l} x_1((n+1)T^+) = (1-d)e^{-D_1T}x_1(nT^+) + de^{-D_2T}x_2(nT^+) \\ \quad + (1-d)(1-e^{-D_1T})x_1^0 + d(1-e^{-D_2T})x_2^0 + \theta_1, \\ x_2((n+1)T^+) = de^{-D_1T}x_1(nT^+) + (1-d)e^{-D_2T}x_2(nT^+) \\ \quad + d(1-e^{-D_1T})x_1^0 + (1-d)(1-e^{-D_2T})x_2^0. \end{array} \right. \quad (5)$$

From system (5), we gain

$$\left\{ \begin{array}{l} x_1^* = \frac{K_1 + K'_1}{K_2 + K'_2}, \\ x_2^* = \frac{K_3 + K'_3}{K_4 + K'_4}, \end{array} \right. \quad (6)$$

where $K_1 = [(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-D_2T})e^{-D_1T}]x_1^0 - d(1-e^{-D_2T})x_2^0 + (1-e^{-D_2T})x_1^0$, $K'_1 = de^{-D_2T}\theta_1$, $K_2 = 1 - (1-d)e^{-D_1T} - (1-d)e^{-D_2T}$, $K'_2 = -(1-2d)e^{-(D_1+D_2)T}$, $K_3 = -d(1-e^{-D_1T})x_1^0 + [(1-d)(1-e^{-D_2T}) - (1-2d)(1-e^{-D_1T})e^{-D_2T}]x_2^0$, $K'_3 = de^{-D_1T}\theta_1$, and $K_4 = 1 - (1-d)e^{-D_2T} - (1-d)e^{-D_1T}$, $K'_4 = -(1-2d)e^{-(D_2+D_1)T}$.

The map $G : R_+^2 \rightarrow R_+^2$ coming from system (5) is presented as

$$\left\{ \begin{array}{l} G_1(x) = G_{11}x_1 + G_{12}x_2 + G_{13}, \\ G_2(x) = G_{21}x_1 + G_{22}x_2, \end{array} \right. \quad (7)$$

where $x = (x_1, x_2) \in R_+^2$, and $G_{11} = (1-d)e^{-D_1T}$, $G_{12} = de^{-D_2T}$, $G_{13} = (1-d)(1-e^{-D_1T})x_1^0 + d(1-e^{-D_2T})x_2^0 + \theta_1$, $G_{21} = de^{-D_1T}$, $G_{22} = (1-d)e^{-D_2T}$, $G_{23} = -d(1-e^{-D_1T})x_1^0 + (1-d)(1-e^{-D_2T})x_2^0$.

Being similar to reference [7], two lemmas are gotten.

Lemma 1. $G^n(x) \rightarrow (x_1^*, x_2^*)$ (as $n \rightarrow \infty$) holds for (x_1, x_2) of system (7).

Lemma 2. The solution $(\widetilde{x}_1, \widetilde{x}_2)$, which is defined as

$$\begin{cases} \widetilde{x}_1 = x_1^0 - (x_1^0 - x_1^*)e^{-D_1(t-nT)}, N < t \leq (N+1), \\ \widetilde{x}_2 = x_2^0 - (x_2^0 - x_2^*)e^{-D_2(t-nT)}, N < t \leq (N+1), \end{cases} \quad (8)$$

is globally stable.

ii) Integrating on system (3), we have

$$\begin{cases} c_1((n+1)T^+) = c_1(nT^+)e^{-(g_1+m_1)T} \\ \quad + \frac{c_2(nT^+)f_1(e^{-(g_1+m_1)(1-l)T} - e^{-(h_1-g_1-m_1)lT-(g_1+m_1)(1-l)T})}{(h_1-g_1-m_1)} \\ \quad + \frac{(1-\theta_1)c_2(nT^+)f_1(e^{-h_1lT} - e^{-(h_1-g_1-m_1)T})}{(h_1-g_1-m_1)}, \\ c_2((n+1)T^+) = (1-\theta_1)e^{-h_1T}c_2(nT^+) + \theta. \end{cases} \quad (9)$$

From (9), we have c_1^* and c_2^* with

$$\begin{cases} c_1^* = \frac{1-\theta_3}{1-(1-\theta_3)e^{-(g_1+m_1)T}} \\ \quad \times \left[\frac{\theta f_1(e^{-(g_1+m_1)(1-l)T} - e^{-(h_1-g_1-m_1)lT-(g_1+m_1)(1-l)T})}{(h_1-g_1-m_1)(1-(1-\theta_1)e^{-h_1T})(1-e^{-(g_1+m_1)T})} \right. \\ \quad \left. + \frac{(1-\theta_1)\theta f_1(e^{-h_1lT} - e^{-(h_1-g_1-m_1)T})}{(h_1-g_1-m_1)(1-(1-\theta_1)e^{-h_1T})(1-e^{-(g_1+m_1)T})} \right], \\ c_2^* = \frac{1-\theta_3}{1-(1-\theta_1)e^{-h_1T}}. \end{cases} \quad (10)$$

Obviously, for (9), (c_1^*, c_2^*) is globally asymptotically stable.

Lemma 3. The globally asymptotically stable $(\widetilde{c}_1, \widetilde{c}_2)$ of (3) exists. and $\widetilde{c}_1, \widetilde{c}_2$ are in reference [8], c_1^*, c_2^* are as (11), and $c_1^{**} = c_1^*e^{-(g_1+m_1)lT} + \frac{f_1c_2^*(1-e^{-(h_1-g_1-m_1)lT})}{(h_1-g_1-m_1)}$, $c_2^{**} = (1-\theta_1)e^{-h_1lT}c_2^*$.

Remark 4. There exist positive constants m_0, M_0, m_e, M_e , it is easy to get $m_0 \leq c_o(t) \leq M_0$ and $m_e \leq c_e(t) \leq M_e$.

Similar to reference [7], we can obtain

Lemma 5. There exists a positive constant $\lambda > 0$, we can easily have $x_1(t) \leq [\delta(D_1x_1^0 + D_2x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}]e^{\lambda T_1}$, $x_2(t) \leq [\delta(D_1x_1^0 + D_2x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}]e^{\lambda T_1}$, $Y(t) \leq \delta(D_1x_1^0 + D_2x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$, $c_0(t) \leq \delta(D_1x_1^0 + D_2x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$, and $c_e(t) \leq \delta(D_1x_1^0 + D_2x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$.

4 Dynamical Analysis

Theorem 1. Suppose

$$d > \frac{1}{2}, \quad (11)$$

and

$$\begin{aligned}
 & (\eta - D_2)T - \frac{\alpha\eta}{D_2(\alpha + x_2^0)} \ln \frac{\alpha e^{-D_2T} + (e^{-D_2T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\
 & + \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{\frac{g_1 + m_1}{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})} + \frac{f_1 c_2^* l T}{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}} + \frac{f_1 c_2^{**}(1-l)T}{\frac{h_1 - g_1 - m_1}{f_1 c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})} + \frac{(h_1 - g_1 - m_1)^2}{(h_1 - g_1 - m_1)^2}} \\
 & - \frac{f_1 c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} < 0,
 \end{aligned} \tag{12}$$

hold, the globally asymptotically stable $(\widetilde{x}_1, \widetilde{x}_2, 0, \widetilde{c}_1, \widetilde{c}_2)$ exists, and x_2^*, x_2^{**} are with (6), and $c_1^*, c_1^{**}, c_2^*, c_2^{**}$ are with in (10).

Proof. Doing it as $y_1 = x_1 - \widetilde{x}_1, y_2 = x_2 - \widetilde{x}_2, Y = Y, z_1 = c_1 - \widetilde{c}_1, z_2 = c_2 - \widetilde{c}_2$, then, linear system with considering for one periodic solution $(\widetilde{x}_1, \widetilde{x}_2, 0, \widetilde{c}_1, \widetilde{c}_2)$ is presented as

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dY}{dt} \\ \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \end{pmatrix} = \begin{pmatrix} -D_1 & 0 & 0 & 0 & 0 \\ 0 & -D_2 & K_5 & 0 & 0 \\ 0 & 0 & K'_5 & 0 & 0 \\ 0 & 0 & 0 & -(g_1 + m_1) & f_1 \\ 0 & 0 & 0 & 0 & -h_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ Y \\ z_1 \\ z_2 \end{pmatrix},$$

where $K_5 = -\frac{1}{\delta} \times \frac{\eta x_2(t)}{\alpha + x_2(t)}, K'_5 = -[D_2 + \beta c_1(t) - \frac{\eta x_2(t)}{\alpha + x_2(t)}]$. Then, the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} e^{-D_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{-D_2 t} & \dagger & 0 & 0 \\ 0 & 0 & K_6 & 0 & 0 \\ 0 & 0 & 0 & K'_6 & \ddagger \\ 0 & 0 & 0 & 0 & e^{-h_1 t} \end{pmatrix},$$

where $K_6 = e^{\int_0^t (-D_2 + \frac{\eta x_2(\xi)}{\alpha + x_2(\xi)} - \beta c_1(\xi)) d\xi}, K'_6 = e^{-(g_1+m_1)t}$. There is no need for computing \dagger, \ddagger .

When $t = (n + l)T$, we get

$$\begin{pmatrix} y_1(nT^+) \\ y_2(nT^+) \\ Y(nT^+) \\ z_1(nT^+) \\ z_2(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 - \theta_1 \end{pmatrix} \begin{pmatrix} y_1(nT) \\ y_2(nT) \\ Y(nT) \\ z_1(nT) \\ z_2(nT) \end{pmatrix}.$$

When $t = (n + 1)T$, we also get

$$\begin{pmatrix} y_1(nT^+) \\ y_2(nT^+) \\ Y(nT^+) \\ z_1(nT^+) \\ z_2(nT^+) \end{pmatrix} = \begin{pmatrix} 1 - d & d & 0 & 0 & 0 \\ d & 1 - d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 - \theta_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(nT) \\ y_2(nT) \\ Y(nT) \\ z_1(nT) \\ z_2(nT) \end{pmatrix}.$$

The stability of $(\widetilde{x}_1, \widetilde{x}_1, 0, \widetilde{c}_1, \widetilde{c}_2)$ is decided by eigenvalues of

$$M = \begin{pmatrix} 1 - d & d & 0 & 0 & 0 \\ d & 1 - d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 - \theta_3 & 0 \\ 0 & 0 & 0 & 0 & 1 - \theta_1 \end{pmatrix} \Phi(\tau).$$

From condition (12), $e^{-D_1\tau} < 1$, and $e^{-D_2\tau} < 1$, the eigenvalues of M are presented as

$$\begin{aligned} \gamma_1 &= \frac{K_7 + \sqrt{K_7'}}{2} << 1, \\ \gamma_2 &= \frac{K_7 - \sqrt{K_7'}}{2} < (1 - d)e^{-D_2T} < 1, \\ \gamma_3 &= e^{\int_0^T (-D_2 + \frac{\eta x_2(\xi)}{\alpha + x_2(\xi)} - \beta c_1(\xi)) d\xi}, \\ \gamma_4 &= (1 - \theta_3)e^{-(g_1 + m_1)\tau} < 1, \\ \gamma_5 &= (1 - \theta_1)e^{-h_1\tau} < 1. \end{aligned}$$

where $K_7 = (1 - d)(e^{-D_1T} + e^{-D_2T})$, $K_7' = (1 - d)^2(e^{-D_1T} + e^{-D_2T})^2 - 4(1 - 2d)e^{-(D_1 + D_2)T}$. For the Floquet theory [6] and condition (14), it is easily to have $\gamma_3 < 1$, then, the locally stable $(\widetilde{x}_1, \widetilde{x}_2, 0, \widetilde{c}_1, \widetilde{c}_2)$ exists.

Choosing $\varepsilon > 0$, we have

$$\rho_1 = \exp\left[\int_0^\tau K_8 - \beta(\widetilde{c_1(t)} - \varepsilon)dt\right] < 1.$$

where $K_8 = -D_2 + \frac{\eta(\widetilde{x_2+\varepsilon})}{\alpha+(\widetilde{x_2+\varepsilon})}$. We get $\frac{dx_2}{dt} \leq D_2(x_2^0 - x_2)$ with considering system (1). So

$$\left\{ \begin{array}{l} \frac{dm_1}{dt} = D_1(x_1^0 - m_1), \\ \frac{dm_2}{dt} = D_2(x_2^0 - m_2), \end{array} \right\} t \neq (n+1)T, \tag{13}$$

$$\left\{ \begin{array}{l} \Delta m_1 = d(m_2 - m_1) + \theta_2, \\ \Delta m_2 = d(m_1 - m_2), \end{array} \right\} t = (n+1)T.$$

From lemma (2), and [8], we get

$$\left\{ \begin{array}{l} x_1 \leq m_1 \leq \widetilde{x_1} + \varepsilon, \\ x_2 \leq m_2 \leq \widetilde{x_2} + \varepsilon. \end{array} \right. \tag{14}$$

and

$$\left\{ \begin{array}{l} \widetilde{c_1} - \varepsilon \leq c_1 \leq \widetilde{c_1} + \varepsilon, \\ \widetilde{c_2} - \varepsilon \leq c_2 \leq \widetilde{c_2} + \varepsilon. \end{array} \right. \tag{15}$$

Therefore,

$$\frac{dY}{dt} \leq Y\left[-D_2 + \frac{\eta(\widetilde{x_2} + \varepsilon)}{\alpha + (\widetilde{x_2} + \varepsilon)} - \beta(\widetilde{c_1} - \varepsilon)\right]. \tag{16}$$

Then, $Y(t) \leq Y(0^+) \exp\left[\int_0^t \left(-D_2 + \frac{\eta(\widetilde{x_2+\varepsilon})}{\alpha+(\widetilde{x_2+\varepsilon})} - \beta(\widetilde{c_1} - \varepsilon)\right)ds\right]$, thus

$$Y((n+1)T) \leq Y(nT^+) \times \exp\left[\int_{nT}^{(n+1)T} \left(-D_2 + \frac{\eta(\widetilde{x_2(s)} + \varepsilon)}{\alpha + (\widetilde{x_2(s)} + \varepsilon)} - \beta(\widetilde{c_1(s)} - \varepsilon)\right)ds\right]. \tag{17}$$

Hence $Y(n\tau) \leq Y(0^+)\rho_1^n$ and $Y(nT) \rightarrow 0$ as $n \rightarrow \infty$. So $Y(t) \rightarrow 0$ as $t \rightarrow \infty$.

For $\varepsilon > 0$, it has a $t_0 > 0$ such that $0 < Y < \varepsilon$ for all $t \geq t_0$. It is no difficulty to gain

$$\frac{\delta\alpha D_2 + \eta\varepsilon}{\delta(\alpha + M)} \left[\frac{\delta(\alpha + M)D_2x_2^0}{\delta(\alpha + M)D_2 + \eta\varepsilon} - x_2 \right] \leq \frac{dx_2}{dt} \leq D_2(x_2^0 - x_2). \tag{18}$$

$(v_1, v_2), (n_1, n_2)$ are of

$$\left. \begin{cases} \frac{dv_1}{dt} = D_1(x_1^0 - v_1), \\ \frac{dv_2}{dt} = \frac{\delta(\alpha + M)D_2 + \eta\varepsilon}{\delta(\alpha + M)} \left[\frac{\delta(\alpha + M)D_2x_2^0}{\delta(\alpha + M)D_2 + \eta\varepsilon} - v_2 \right], \\ \Delta v_1 = d(v_2 - v_1) + \theta_2, \\ \Delta v_2 = d(v_1 - v_2), \end{cases} \right\} \begin{array}{l} t \neq nT, \\ t = nT, \end{array} \quad (19)$$

and

$$\left. \begin{cases} \frac{dn_1}{dt} = D_1(x_1^0 - n_1), \\ \frac{dn_2}{dt} = D_2(x_2^0 - n_2), \\ \Delta n_1 = d(n_2 - n_1) + \theta_2, \\ \Delta n_2 = d(n_1 - n_2), \end{cases} \right\} \begin{array}{l} t \neq nT, \\ t = nT, \end{array} \quad (20)$$

respectively, while

$$\left\{ \begin{array}{l} \tilde{v}_1 = x_1^0 - (x_1^0 - v_1^*)e^{-D_1(t-nT)}, \\ \tilde{v}_2 = \frac{\delta(\alpha + M)D_2x_2^0}{\delta(\alpha + M)D_2 + \eta\varepsilon} \\ \quad - \left(\frac{\delta(\alpha + M)D_2x_2^0}{\delta(\alpha + M)D_2 + \eta\varepsilon} - v_2^* \right) e^{-\frac{\delta(\alpha + M)D_2 + \eta\varepsilon}{\delta(\alpha + M)}(t-nT)}, \end{array} \right. \quad (21)$$

with

$$\left\{ \begin{array}{l} v_1^* \\ = \frac{[(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T})e^{-D_1T}]x_1^0 - d(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T})\frac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon}}{1 - (1-d)e^{-D_1T} - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-2d)e^{-(D_1+\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)})T}} \\ \quad + \frac{(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} + de^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau})\theta_2}{1 - (1-d)e^{-D_1\tau} - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-2d)e^{-(D_1+\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)})T}}, \\ v_2^* \\ = \frac{-d(1-e^{-D_1\tau})x_1^0 + [(1-d)(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T}) - (1-2d)(1-e^{-D_1T})e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau}]\frac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon}}{1 - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-d)e^{-D_1T} - (1-2d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T} + D_1T} \\ \quad + \frac{de^{-D_1T}\theta_2}{1 - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T} - (1-d)e^{-D_1T} - (1-2d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T} + D_1T}, \end{array} \right. \quad (22)$$

For $\varepsilon_1 > 0$, existing a $t_1, t > t_1$ such that

$$\tilde{v}_1 - \varepsilon_1 < x_1 < \tilde{n}_1 + \varepsilon_1,$$

and

$$\tilde{v}_2 - \varepsilon_1 < x_2 < \tilde{n}_2 + \varepsilon_1.$$

One will gain the followings with considering $\varepsilon \rightarrow 0$,

$$\widetilde{x}_1 - \varepsilon_1 < x_1 < \widetilde{x}_1 + \varepsilon_1,$$

and

$$\widetilde{x}_2 - \varepsilon_1 < x_2 < \widetilde{x}_2 + \varepsilon_1.$$

This completes the proofs.

Theorem 2. Suppose

$$d > \frac{1}{2}, \quad (23)$$

and

$$\begin{aligned} & (\eta - D_2)T - \frac{\alpha\eta}{D_2(\alpha + x_2^0)} \ln \frac{\alpha e^{-D_2T} + (e^{-D_2T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\ & + \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{g_1 + m_1} + \frac{f_1 c_2^* l T}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ & + \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})} + \frac{f_1 c_2^{**}(1 - l)T}{h_1 - g_1 - m_1} \\ & - \frac{f_2 c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0, \end{aligned} \quad (24)$$

hold, system permanence, and x_2^* and x_2^{**} are with (6), $c_1^*, c_1^{**}, c_2^*, c_2^{**}$ are with (10).

Proof. Owing to remark (4), and lemma (5), we have obtain that (x_1, x_2, Y, c_1, c_2) is bounded. It can be easily obtained that $c_1 \geq m_o$ and $c_2 \geq m_e$ with considering with remark (4).

Therefore,

$$\left\{ \begin{array}{l} \frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\ \frac{dx_2}{dt} \geq D_2(x_2^0 - x_2) - \frac{\eta M}{\delta \alpha} x_2, \\ \Delta x_1 = d(-x_1 + x_2) + \theta_2, \\ \Delta x_2 = d(x_1 - x_2), \end{array} \right\} \begin{array}{l} t \neq nT, \\ t = nT, \end{array} \quad (25)$$

with its comparison system

$$\left\{ \begin{array}{l} \frac{dw_1}{dt} = D_1(x_1^0 - w_1), \\ \frac{dw_2}{dt} = \frac{\delta \alpha D_2 + \eta M}{\delta \alpha} \left[\frac{\delta \alpha D_2 x_2^0}{\delta \alpha D_2 + \eta M} - w_2 \right], \\ \Delta w_1 = d(w_2 - w_1) + \theta_2, \\ \Delta w_2 = d(w_1 - w_2), \end{array} \right\} \begin{array}{l} t \neq nT, \\ t = nT. \end{array} \quad (26)$$

With considering (2) and (8), $(\widetilde{w}_1, \widetilde{w}_2)$ of (26) is

$$\begin{cases} \widetilde{w}_1 = x_1^0 - (x_1^0 - w_1^*)e^{-D_1(t-nT)}, nT < t \leq (n+1)T, \\ \widetilde{w}_2 = \frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M} - \left(\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M} - w_2^* \right) e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha}(t-nT)}, \\ nT < t \leq (n+1)T, \end{cases} \quad (27)$$

with

$$\begin{cases} w_1^* = \frac{[(1-d)(1-e^{-D_1 T}) - (1-2d)(1-e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T})e^{-D_1 T}]x_1^0 - d(1-e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T})\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M}}{1 - (1-d)e^{-D_1 T} - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T} - (1-2d)e^{-(D_1 + \frac{\delta\alpha D_2 + \eta M}{\delta\alpha})T} + (1-e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T} + de^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T})\theta_2} \\ + \frac{1 - (1-d)e^{-D_1 T} - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T} - (1-2d)e^{-(D_1 + \frac{\delta\alpha D_2 + \eta M}{\delta\alpha})T}}{1 - (1-d)e^{-D_1 T} - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T} - (1-2d)e^{-(D_1 + \frac{\delta\alpha D_2 + \eta M}{\delta\alpha})T}}, \\ w_2^* = \frac{-d(1-e^{-D_1 T})x_1^0 + [(1-d)(1-e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T}) - (1-2d)(1-e^{-D_1 T})e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T}]\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M}}{1 - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} T} - (1-d)e^{-D_1 T} - (1-2d)e^{-(\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} + D_1)\tau} + de^{-D_1 \tau}\theta_2} \\ + \frac{1 - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} \tau} - (1-d)e^{-D_1 T} - (1-2d)e^{-(\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} + D_1)T}}{1 - (1-d)e^{-\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} \tau} - (1-d)e^{-D_1 T} - (1-2d)e^{-(\frac{\delta\alpha D_2 + \eta M}{\delta\alpha} + D_1)T}}. \end{cases} \quad (28)$$

Furthermore, $(\widetilde{w}_1, \widetilde{w}_2)$ of (27) is globally asymptotic stable. Then, it exists a $\varepsilon > 0$ such that $x_1 \geq w_1 \geq \widetilde{w}_1 - \varepsilon \geq w_1^* - \varepsilon = k_1$ and $x_2 \geq w_2 \geq \widetilde{w}_2 - \varepsilon \geq w_2^* - \varepsilon = k_2$.

Since

$$\begin{aligned} & (\eta - D_2)T + \frac{\eta\alpha}{D_2(\alpha + x_2^*)} \ln \frac{\alpha e^{-D_2 T} + (e^{-D_2 T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\ & + \frac{c_1^*(1 - e^{-(g_1+m_1)l\tau})}{g_1 + m_1} + \frac{f_1 c_2^* l T}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ & + \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1-l)T}{h_1 - g_1 - m_1} \\ & - \frac{f_1 c_2^{**}(e^{-(h_2-g_2-m_2)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0, \end{aligned} \quad (29)$$

$m_3 > 0$ and $\varepsilon_1 > 0$ can be selected to do as

$$\begin{aligned} \sigma &= (\eta - \frac{\delta\alpha D_2 + \eta m_3}{\delta\alpha} - \varepsilon)T - \frac{\eta\alpha}{D_2(\alpha - \varepsilon + \frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3} x_2^0)} \\ & \times \ln \frac{(\alpha - \varepsilon + \frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3})e^{-\frac{\delta\alpha D_2 + \eta m_3}{\delta\alpha} T} - \frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3} x_2^0 + k_2^*}{\alpha - \varepsilon + k_2^*} \\ & + \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{g_1 + m_1} + \frac{f_1 c_2^* l \tau}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ & + \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1-l)T}{h_1 - g_1 - m_1} \\ & - \frac{f_1 c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0, \end{aligned} \quad (30)$$

where k_2^* is defined as (34)

Let $K_9 \in N^+$ and $K_9\tau > T_1$, integrating (36) on $(nT, (n + 1)T), n \geq K_9$, we have

$$\begin{aligned}
 Y((n + 1)T) &\geq Y(nT^+) \exp\left(\int_{nT}^{(n+1)T} \left[-D_2 + \frac{\eta(\bar{k}_1 - \varepsilon_1)}{\alpha + (\bar{k}_1 - \varepsilon_1)} - \beta(\bar{k}_2 + \varepsilon_1)\right] dt\right) \\
 &= x(nT)e^\sigma,
 \end{aligned}$$

then $Y((K_9 + k)T) \geq Y(K_9T^+)e^{k\sigma} \rightarrow \infty$, as $k \rightarrow \infty$, it is an illogicality with the bounded Y . Hence $Y \geq m_3$.

5 Discussion

In this work, we present a chemostat-type model with impulsive effects in a polluted karst environment. If it is supposed that the variables are shown in the table below:

$x_1(0)$	$x_2(0)$	$Y(0)$	$c_1(0)$	$c_2(0)$	D_1	D_2	x_1^0	x_2^0	δ	η	α	β	θ_1	θ_2	θ_3	θ	f_1	g_1	m_1	h_1	d	T
2	2	2	0.13	0.15	2	0.2	2	3	1	0.5	2	0.5	0.5	0.8	0.6	0.01	0.4	0.3	0.1	0.4	0.2	1

system (1) is permanent (one can see Fig.1). If it is supposed that another variables are shown in the table below:

$x_1(0)$	$x_2(0)$	$Y(0)$	$c_1(0)$	$c_2(0)$	D_1	D_2	x_1^0	x_2^0	δ	η	α	β	θ_1	θ_2	θ_3	θ	f_1	g_1	m_1	h_1	d	T
2	2	2	0.13	0.15	2	0.2	2	3	1	0.5	2	0.5	0.5	0.8	0.1	0.01	0.4	0.3	0.1	0.4	0.2	1

there exists a globally asymptotically stable solution $(\widetilde{x_1(t)}, \widetilde{x_2(t)}, 0, \widetilde{c_1(t)}, \widetilde{c_2(t)})$ of system (1) (one can see Fig.2). If it is supposed that another variables are shown in the table below:

$x_1(0)$	$x_2(0)$	$Y(0)$	$c_1(0)$	$c_2(0)$	D_1	D_2	x_1^0	x_2^0	δ	η	α	β	θ_1	θ_2	θ_3	θ	f_1	g_1	m_1	h_1	d	T
2	2	2	0.13	0.15	2	0.16	2	3	1	0.5	2	0.5	0.5	0.8	0.1	0.01	0.4	0.3	0.1	0.4	0.2	1

then, system (1) is permanent (see Fig.3).

The simulations show that parameters $0 < \theta_3 < 1$ and D_2 are very important for system (1). The parameters $D_1, \theta_1, \theta_2, \theta$ and d of system (1) can also be discussed. The results will guide us how to manage the source of water management in karst areas.

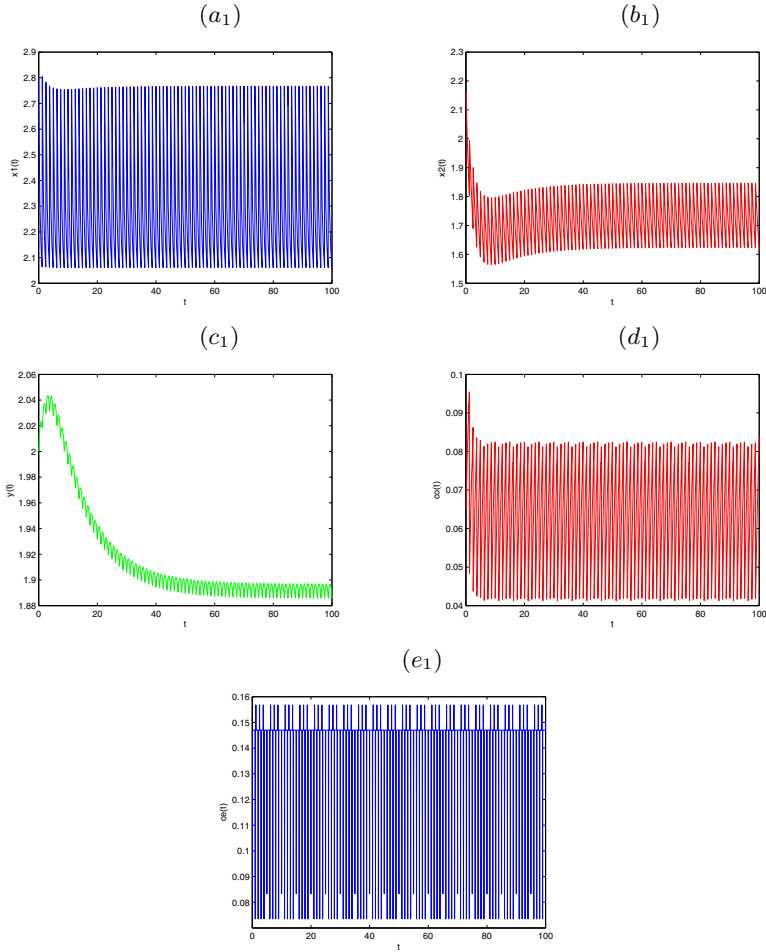


Fig. 1. The permanence of system (1) with parameters in the first table.

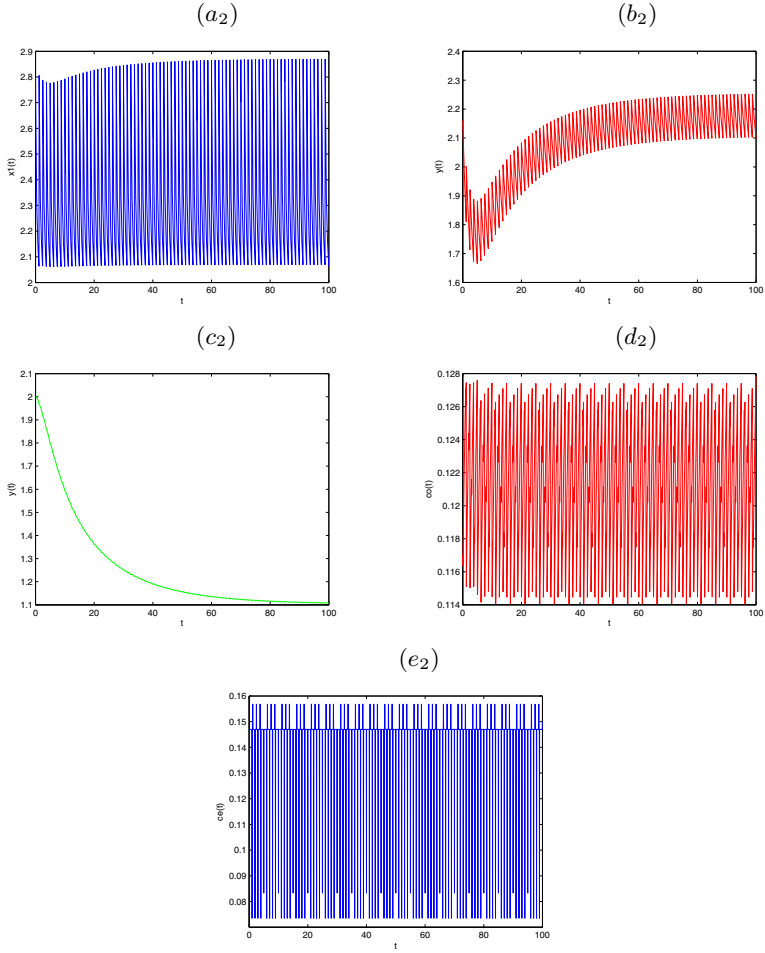


Fig. 2. The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the second table.

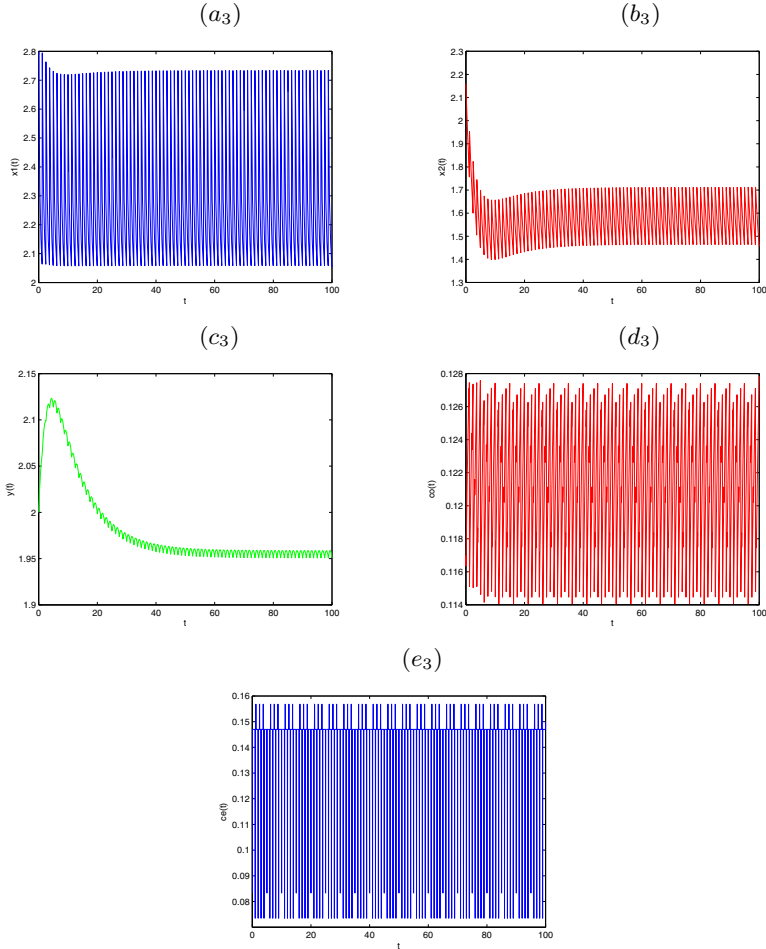


Fig. 3. The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the third table.

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