Steps Towards Fuzzy Homotopy Based on Linguistic Variables

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Abstract. This paper studies on linguistic topological sapces which are generate from Hedge algebra. We also indicate homotopy classes of homotopic functions on this spaces as well as its equivalence relations.

Keywords: Linguistic variable · Linguistic topological sapce · Homotopic function

1 Introduction

Natural language processing (NLP) holds significant significance in the realm of artificial intelligence (AI) as it aids in the analysis, logical deduction, and decision-making processes. Within this context, "Computing with words" (CWW), a mathematical approach, addresses computational challenges framed in natural language. CWW draws from fuzzy set theory and fuzzy logic, initially proposed by L. A. Zadeh, offering an approximation technique within the range of values between 0 and 1. Notably, within the linguistic domain, linguistic hedges assume a crucial role in forming sets of linguistic variables.

An established utilization of fuzzy sets involves fuzzy graphs, fuzzy neural networks, and machine learning [2,8,11,12], which merge fuzzy sets with graph theory. Fuzzy graphs find numerous applications in modeling and deducing fuzzy knowledge, including scenarios like human trafficking, internet routing, and illegal immigration [10]. These applications operate within the range of values from 0 to 1, excluding linguistic values.

Nonetheless, numerous scenarios are not easily represented within the numerical context, such as linguistic summarization issues [9]. To address this challenge, the paper employs an abstract algebraic framework known as hedge algebra ($\mathbb{HA}$) to facilitate the analysis of linguistic content. The subsequent sections of the paper are structured as follows: Sect. 2 revisits key notions pertaining to word-based modeling using ($\mathbb{HA}$) and investigates the characteristics of topological

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linguistic spaces. In the primary Sect. 3, different categories of homotopic functions and equivalence equations are explored. Finally, Sect. 4 outlines the paper's conclusions and outlines directions for future research endeavors.

2 Preliminary

In this segment, fundamental principles of $\mathbb{HA}$ are introduced alongside crucial information utilized within this paper.

2.1 Hedge Algebra

First definition of a $\mathbb{HA}$ is 3-Tuple $\mathbb{HA} = (X, H, \leq\ \ )$ in [6]. In [5], to readily replicate fuzzy information, the 3-Tuple is augmented with the inclusion of two elements, denoted as G and C. So $\mathbb{HA} = (X, G, C, H, \leq)$ where $H \neq \emptyset$, $G = \{c^+,\ c^-\}$, $C = \{0, W, 1\}$. Domain of X is $\mathbb{L} = Dom(X) = \{\delta c|\ c \in G,\ \delta \in H^*(\text{hedge string over H})\}$, $\{\mathbb{L}, \leq\}$ is a POSET (partial order set) and $x = h_n h_{n-1} \ldots h_1 c$ Is referred to as the canonical string corresponding to the linguistic variable x.

Example 1. Fuzzy subset X *is Age*, $G = \{c^+ = young;\ c^- = old\}$, $H = \{less; more; very\}$ so term-set of linguistic variable $Age\ X$ is $\mathbb{L}$(X) or $\mathbb{L}$ for short: $\mathbb{L}$ = {less less young ; less more young ; young ; more more young ; very more young ; very very young ...}

Fuzziness properties of elements in $\mathbb{HA}$, specified by $\mathcal{F}$ (fuzziness measure) [5] as follows:

Definition 1. *A mapping* $\mathcal{F} : \mathbb{L} \to [0, 1]$ *is said to be the fuzziness measure of* $\mathbb{L}$ *if:*

1. $\sum_{c \in \{c^+, c^-\}} \mathcal{F}(c) = 1,\ \mathcal{F}(0) = \mathcal{F}(w) = \mathcal{F}(1) = 0.$
2. $\sum_{h_i \in H} \mathcal{F}(h_i x) = \mathcal{F}(x)$, $x = h_n h_{n-1} \ldots h_1 c$, *the canonical form.*
3. $\mathcal{F}(h_n h_{n-1} \ldots h_1 c) = \prod_{i=1}^{n} \mathcal{F}(h_i) \times \mu(x)$.

Truth and significance hold pivotal roles in fuzzy logic, artificial intelligence, and machine learning. Within RCT (restriction-centered theory) in [9], truth values are structured hierarchically, encompassing ground-level or first-order truth values as well as second-order ones. While first-order truth values adopt numerical expressions, second-order truth values assume the form of linguistic interpretations. A linguistic truth value, designated as "ℓ," constitutes a fuzzy set. We study linguistic truth values on POSET $\mathbb{L}$ whose elements are comparable [3].

Definition 2. *A* $\mathfrak{L}$ *STRUCT*$[\rho]$ *on relational signature* ρ *is a tuple:*

$$\mathfrak{L} = \langle \mathbb{L},\ f_{a_i}^{\mathfrak{L}},\ c_j^{\mathfrak{L}} \rangle \tag{1}$$

Consists of a universe $\mathbb{L} \neq \emptyset$ *together with an interpretation of:*

- *each constant symbol c_j from ρ as an element $c_j^{\mathfrak{L}} \in \mathbb{L}$*
- *each a_i-ary function symbol f_{a_i} from ρ as a function:*

$$f_i^{\mathfrak{L}} : \mathbb{L}^{a_i} \to \mathbb{L} \tag{2}$$

In $\mathbb{HA}$, $\ell \in \mathbb{L}$ and there are order properties:

Theorem 1. *In [6] let $\ell_1 = h_n \dots h_1 u$ and $\ell_2 = k_m \dots k_1 u$ be two arbitrary canonical representations of ℓ_1 and ℓ_2, then there exists an index $j \le \bigwedge\{m, n\}+1$ such that $h_i = k_j$, for $\forall i < j$, and:*

1. *$\ell_1 < \ell_2$ iff $h_j x_j < k_j x_j$ where $x_j = h_{j-1} \dots h_1 u$;*
2. *$\ell_1 = \ell_2$ iff $m = n = j$ and $h_j x_j = k_j x_j$;*
3. *ℓ_1 and ℓ_2 are incomparable iff $h_j x_j$ and $k_j x_j$ are incomparable;*

Example 2. Take into linguistic variables: $\{\mathscr{V}$high, $\mathscr{P}$high, $\mathscr{L}$high$\}$ belonging to the set $\mathbb{L}$, where $\{\mathscr{V}$high, $\mathscr{P}$high, $\mathscr{L}$high$\}$ correspond to linguistic truth values representing "very true," "possible true," and "less true," derived from the underlying truth of a variable pressure. Suppose there are propositions p = "Lucie is young is $\mathscr{V}$high" and q = "Lucie is smart is $\mathscr{P}$high." The interpretations over $\mathbb{L}$ are as follows:

- pressure$(p) = \mathscr{V}$high $\in \mathbb{L}$, pressure is a unary function.
- $p \wedge q = \mathscr{V}$high $\wedge\ \mathscr{P}$high $= \mathscr{P}$high $\in$. $\wedge$ is a binary function.
- $p \vee q = \mathscr{V}$high $\vee\ \mathscr{P}$high $= \mathscr{V}$high $\in$. $\vee$ is a binary function.

2.2 Linguistic Topological Spaces

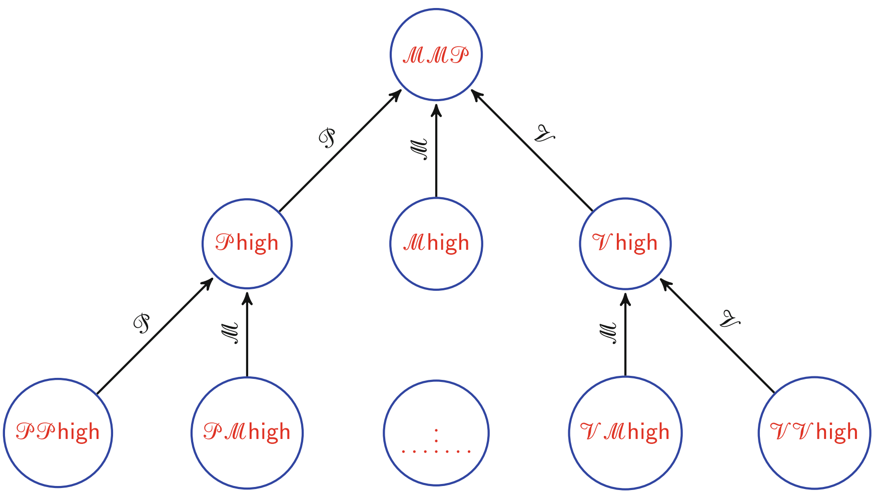

Fig. 1. Tree of hedges

Linguistic topological spaces (LTS), as presented in [4], can be regarded as a particular instance of a fuzzy topological space [1].

Definition 3. *[1] A fuzzy topological is a family T of fuzzy sets in X which satisfies the following condition:*

1. $\phi, X \in T$,
2. *If* $A, B \in T$, *then* $A \cap B \in T$,
3. *If* $A_i \in T$ *for each* $i \in I$ *then* $\bigcup_I A_i \in T$.

T is called a fuzzy topoloty for X and the pair $\langle X, T\rangle$ is a fuzzy topological space (FTS).

Definition 4. *[4] A linguistic fuzzy topological is a family T of linguistic term sets in L which the following condition holds:*

1. $\phi, L \in T$,
2. *If* $A, B \in T$, *then* $A \cap B \in T$,
3. *If* $A_i \in T$ *for each* $i \in I$ *then* $\bigcup_I A_i \in T$.

T is called a linguistic fuzzy topoloty for L.

Let L be a language which is generate from a $\mathbb{HA}$ with m hedges and T be a set of subset of leaf nodes on a complete m-ary tree as Fig. 1 then we have property

Property 1. [4]

1. T is a linguistic topology on L
2. Couple (L, T) is a linguistic topological space.

Example 3. Give a $\mathbb{HA}$:

$$\mathbb{HA} = \langle \mathcal{X} = \mathsf{pressure}; c^+ = \mathsf{high}; \mathcal{H} = \{\mathcal{P}, \mathcal{M}, \mathcal{V}\}\rangle \tag{3}$$

be an $\mathbb{HA}$ with order as $\mathcal{P} < \mathcal{M} < \mathcal{V}$ ($\mathcal{P}$ for possible, $\mathcal{M}$ for more and $\mathcal{V}$ for very are hedges). Let $\{h_i, h_j, h_i \in \mathcal{H} \cup W\}$ in which W is the neutral element, that is W$c^+ = c^+$ ...then language L which generated from linguistic variable $\mathcal{X}$ is as follow Fig. 2:

$$\begin{aligned}
L =& \{h_i h_j h_k \mathsf{high} | h_i \neq h_j \wedge h_i \neq h_k \wedge h_j \neq h_k\}\\
=& \{\mathcal{P}\mathsf{high}, \mathcal{M}\mathsf{high}, \mathcal{V}\mathsf{high},\\
& \mathcal{P}\mathcal{M}\mathsf{high}, \mathcal{M}\mathcal{P}\mathsf{high}, \mathcal{P}\mathcal{V}\mathsf{high},\\
& \mathcal{V}\mathcal{P}\mathsf{high}, \mathcal{M}\mathcal{V}\mathsf{high}, \mathcal{V}\mathcal{M}\mathsf{high},\\
& \mathcal{P}\mathcal{M}\mathcal{V}\mathsf{high}, \mathcal{V}\mathcal{P}\mathcal{M}\mathsf{high}, \mathcal{P}\mathcal{V}\mathcal{M}\mathsf{high},\\
& \mathcal{V}\mathcal{M}\mathcal{P}\mathsf{high}, \mathcal{M}\mathcal{V}\mathcal{P}\mathsf{high}, \mathcal{M}\mathcal{P}\mathcal{V}\mathsf{high}\}
\end{aligned}$$

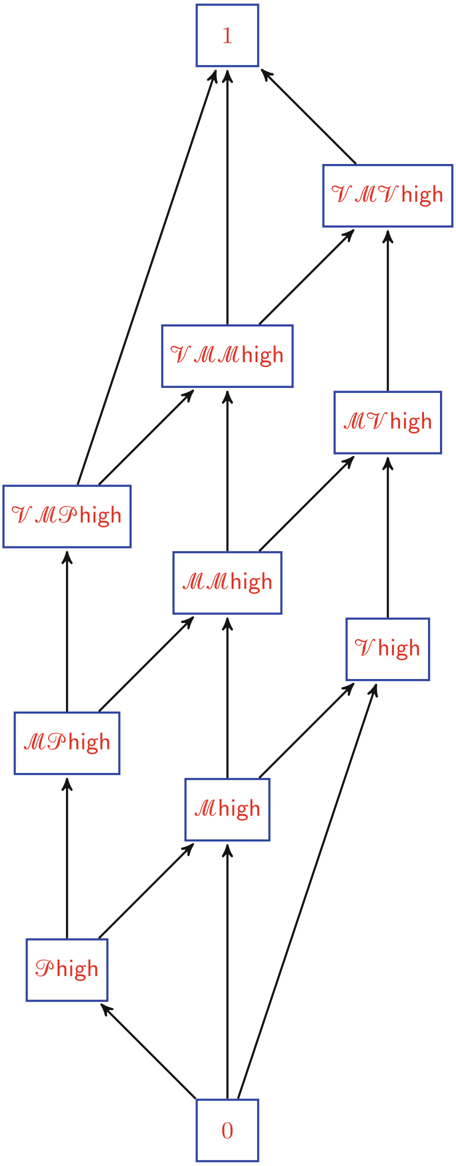

Fig. 2. Hasse diagram

3 Linguistic Homotopic Ralations

Fuzzy homotopy theory on numerical domain [0, 1] was presented in [7]. This paper study homotopic relations on linguistic domain $\mathbb{L}$ of linguistic topological space [4].

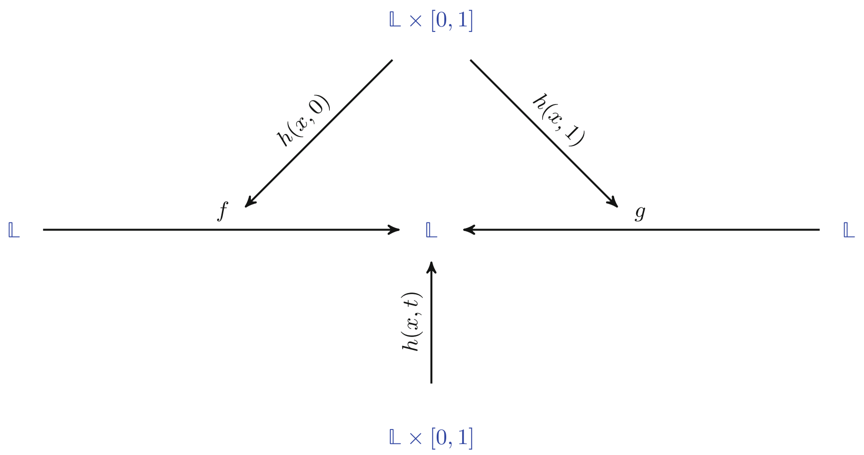

Fig. 3. Homotopic ralation diagram

Definition 5. *Given two functions $f : \mathbb{L} \longrightarrow \mathbb{L}$ and $g : \mathbb{L} \longrightarrow \mathbb{L}$. f and g are said to be homotopic if there is a function h exists:*

$$h : \mathbb{L} \times [0,\ 1] \longrightarrow \mathbb{L} \tag{4}$$

such that:

$$\begin{gathered} h(x, t) \in \mathbb{L} \\ h(x, 0) = f(x) \quad \textit{and} \quad h(x, 1) = g(x) \end{gathered} \tag{5}$$

With functions $f,\ g,\ h$ are illustrated as in the Fig. 3.

Example 4. With the parameter $t \in [0,\ 1]$ and relationship between f and g as follows:

$$h(x, t) = f(x) + t \times [g(x) - f(x)]$$

Suppose f is the identity function

$$f(x) = x$$

and $g(x)$ denote a constant function:

$$g(x) = (\mathbb{W})$$

Let $M = (x_M)$ represent a point on line segment of fuzziness measure in Definition 1. Point $h(M, t) = N$ with $N = x_N$ is the point on the line so that $\mathcal{F}(x_N) = t \times \mathcal{F}(x_M)$. Then, we have:

$$h(M, 0) = M$$
$$h(M, 1) = \mathbb{W}$$

Property 2. The relation on $\mathbb{L}$, $f \sim g$ in which f is homotopic to g is an equivalence equation.

Proof. Prove the relation satisfying the properties $\mathbb{L}$, $f \sim g$: Reflexivity, symmetry and transitivity.

- Symmetry: Every function is homotopic to itself
  Define a function $h : \mathbb{L} \times [0,\ 1] \longrightarrow \mathbb{L}$ as

$$h(x,t) = t \times f(x) + (1-t) \times f(x)$$

  This function satisfies the conditions for homotopy since $h(x,0) = f(x)$ and $h(x,1) = f(x)$
- Symmetry: If f is homotopic to g, then g is homotopic to f. Let h_1 be the homotopy between f and g.
  Define a function $h_1 : \mathbb{L} \times [0,\ 1] \longrightarrow \mathbb{L}$ as

$$h_1(x,t) = h(x, 1-t)$$

  $h_1(.)$ satisfies the conditions for homotopy since:

$$h_1(x,0) = h(x,1) = g(x) \text{ and } h_1(x,1) = h(x,0) = f(x)$$

- Transitivity: If f is homotopic to g and g is homotopic to h, then f is homotopic to h.
  Let h be the homotopy between f and g, and let h_1 be the homotopy between g and h. Define a function $h_2 : \mathbb{L} \times [0,\ 1] \longrightarrow \mathbb{L}$ as

$$h_2(x,t) = h(x,t) + (1-t) \times (h_1(x,t) - h(x,t))$$

  h_2 satisfies the conditions for homotopy since

$$h_2(x,0) = h_1(x,0) = f(x) \text{ and } h_2(x,1) = h_1(x,1) = h(x)$$

Since the relation $f \sim g$ of homotopy satisfies reflexivity, symmetry, and transitivity, it is indeed an equivalence relation.

4 Conclusion and Forthcoming Study

The paper study properties on linguistic topological space based on hedge algebra and fuzziness measure.

- Study homotopic functions on linguistic domain $\mathbb{L}$
- Show that the relationship between homotopic functions on $\mathbb{L}$ is an equivalence relation.

In the future, two studies will be:

- Research on homotopy classes.
- Apply linguistic homotopy in quantum logic and homotopy data analysis.

References

1. Chang, C.L.: Fuzzy topological spaces. Math. Anal. **24**(1), 182–190 (1968)
2. Glykas, M.: Fuzzy Cognitive Maps, Advances in Theory, Tools and Applications. Springer Heidelberg (2010). https://doi.org/10.1007/978-3-642-03220-2
3. Van Han, N., Cong Vinh, P.: Reasoning with words: a hedge algebra linguistic cognitive map approach. Concurrency Comput. Pract. Experience **33**(2), e5711 (2020)
4. Van Han, N., Cong Vinh, P.: Towards linguistic fuzzy topological spaces based on hedge algebra. EAI Endorsed Trans. Context-aware Syst. Appl. **8**(1), e5711 (2022)
5. Ho, N.C., Son, T.T., Khang, T.D., Viet, L.X.: Fuzziness measure, quantified semantic mapping and interpolative method of approximate reasoning in medical expert systems. J. Comput. Sci. Cybern. **18**(3), 237–252 (2002)
6. Ho, N.C., Wechler, W.: Hedge algebras: an algebraic approach to structure of sets of linguistic truth values. Fuzzy Sets Syst. **35**(3), 281–293 (1990)
7. Kandemir, M.B.: The foundations of homotopic fuzzy sets. Konuralp J. Math. **10**(1), 34–39 (2022)
8. Kosko, B.: Fuzzy cognitive maps. Int. J. Man-Machine Stud. **24**, 65–75 (1986)
9. Zadeh, L.A.: Computing with words - principal concepts and ideas. In: Studies in Fuzziness and Soft Computing. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-27473-2
10. John, N., Mordeson, J.N., Mathew, S.: Advanced Topics in Fuzzy Graph Theory, Springer, Cham (2019). https://doi.org/10.1007/978-3-030-04215-8
11. Papageorgiou, E.I.: Fuzzy Cognitive Maps for Applied Science and Engineering From Fundamentals to Extensions and Learning Algorithms. Springer-Verlag, Heidelberg (2014). https://doi.org/10.1007/978-3-642-39739-4
12. Rosenfeld, A.: Fuzzy graphs. Fuzzy Sets Appl. 77–95 (1975)