



Dynamics of a Population-Hibernation Model with Genic Mutation and Impulsive Effects in a Polluted Environment

Jianjun Jiao^(✉) 

School of Mathematics and Statistics,
Guizhou University of Finance and Economics,
Guiyang 550025, People's Republic of China
jiaojianjun05@126.com

Abstract. In this work, we present a population-hibernation system with genic mutation and impulsive effects in a polluted environment. The $(0, 0, \tilde{c}_o, \tilde{c}_e)$ of (1) is globally asymptotically stable. (1) is also proved to be permanent.

Keywords: Population-hibernation model · Mutation · Impulses

1 Introduction

Many authors [1,2] considered that the environmental pollutants cause many diseases. Hibernation constitutes an effective strategy of animals [3,4]. Population dynamics were investigated by theories of impulsive differential equations lately [5,6]. Jiao et al. [7] investigated an impulsive predator-prey system with periodic switches and hibernation. Jiao and Chen [8] considered a predator-prey system with mutation and impulses. However, theories of impulsive differential equations introducing into system with genic mutation and hibernation have been considered very little. For convenience, we make notations as $N = n\tau$ and $N + l = (n + l)\tau$.

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2 The Model

In this work, we present a single population-hibernation model with mutation and impulsive effects in a polluted environment

$$\left. \begin{array}{l} \left. \begin{array}{l} \frac{dx}{dt} = -ax - cx^2 - \beta_1 xc_o, \\ \frac{dy}{dt} = -d_1 y, \\ \frac{dc_o}{dt} = fc_e - (g+m)c_o, \\ \frac{dc_e}{dt} = -hc_e, \\ \Delta x = (1-\theta_1)b_1x + \theta_2b_2y, \\ \Delta y = \theta_1b_1x + (1-\theta_2)b_2y, \\ \Delta c_o = 0, \\ \Delta c_e = 0, \end{array} \right\} t \in (N, (N+l)], \\ \left. \begin{array}{l} \frac{dx}{dt} = -d_2x - \beta_2xc_o, \\ \frac{dy}{dt} = -d_3y, \\ \frac{dc_o}{dt} = fc_e - (g+m)c_o, \\ \frac{dc_e}{dt} = -hc_e, \\ \Delta x = 0, \\ \Delta y = -\mu_1y, \\ \Delta c_o = 0, \\ \Delta c_e = \mu, \end{array} \right\} t = (N+l), \end{array} \right\} t \in ((N+l), (N+1)], \quad (1)$$

the biological meanings of the variables and the parameters of (1) can reference [6–8].

3 The Dynamics

For (1), the subsystem of (1) is as

$$\left. \begin{array}{l} \left. \begin{array}{l} \frac{dc_o}{dt} = fc_e - (g+m)c_o, \\ \frac{dc_e}{dt} = -hc_e, \end{array} \right\} t \neq N, \\ \left. \begin{array}{l} \Delta c_o = 0, \\ \Delta c_e = \mu, \end{array} \right\} t = N. \end{array} \right\} \quad (2)$$

Then, we give an important property of system (2) as following.

Lemma 1. [6] (2) has a globally asymptotically stable solution $(\widetilde{c}_o, \widetilde{c}_e)$ as

$$\begin{cases} \widetilde{c}_o = \widetilde{c}_o(0)e^{-(g+m)(t-N)} + \frac{\mu f(e^{-(g+m)(t-N)} - e^{-h(t-N)})}{(h-g-m)(1-e^{-h\tau})}, \\ \widetilde{c}_e = \frac{\mu e^{-h(t-N)}}{1-e^{-h\tau}}, \\ \widetilde{c}_o(0) = \frac{\mu f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})}, \\ \widetilde{c}_e(0) = \frac{\mu}{1-e^{-h\tau}}, \end{cases} \quad (3)$$

Remark 2. $m_o - \varepsilon \leq c_o(t) \leq M_o + \varepsilon$ and $m_e - \varepsilon \leq c_e(t) \leq M_e + \varepsilon$ hold for t large enough, where $m_o = \frac{\mu f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(e^{(g+m)\tau} - 1)(1-e^{-h\tau})} > 0$, $M_o = \frac{\mu f(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})} + \frac{\mu f}{|h-g-m|(1-e^{-h\tau})} > 0$, $m_e = \frac{\mu e^{-h\tau}}{1-e^{-h\tau}} > 0$ and $M_e = \frac{\mu}{1-e^{-h\tau}} > 0$ for $\varepsilon > 0$.

From (1) another subsystem of (1) is also as

$$\begin{cases} \left. \begin{aligned} \frac{dx}{dt} &= -ax - cx^2, \\ \frac{dy}{dt} &= -d_1y, \end{aligned} \right\} t \in (N, (N+l)], \\ \left. \begin{aligned} \Delta x &= (1-\theta_1)b_1x + \theta_2b_2y, \\ \Delta y &= \theta_1b_1x + (1-\theta_2)b_2y, \end{aligned} \right\} t = (N+l), \\ \left. \begin{aligned} \frac{dx}{dt} &= -d_2x, \\ \frac{dy}{dt} &= -d_3y, \end{aligned} \right\} t \in ((N+l), (N+1)], \\ \left. \begin{aligned} \Delta x &= 0, \\ \Delta y &= -\mu_1y, \end{aligned} \right\} t = (N+1). \end{cases} \quad (4)$$

Integrating (4), we get

$$\begin{cases} x = \begin{cases} \frac{ax(N^+)e^{-a(t-N)}}{a + cx(N^+)(1-e^{-a(t-N)})}, t \in (N, (N+1)N], \\ x((N+l)^+)e^{-d_2(t-N+l)}, t \in ((N+l), (N+1)] \end{cases} \\ y = \begin{cases} y(N^+)e^{-d_1(t-N)}, t \in (N, (N+l)N], \\ y((N+l)^+)e^{-d_3(t-N)}, t \in ((N+l)N, (N+1)N]. \end{cases} \end{cases} \quad (5)$$

The stroboscopic map of (4) is

$$\begin{cases} x((N+1)^+) = [1 + (1 - \theta_1)b_1] \times \frac{ax(N^+)e^{-[al+d_2(1-l)]\tau}}{a + cx(N^+)(1 - e^{-al\tau})} \\ \quad + \theta_2 b_2 e^{-[d_1l+d_2(1-l)]\tau} y(N^+), \\ y((N+1)^+) = \theta_1 b_1 (1 - \mu_1) \times \frac{ax(N^+)e^{-[al+d_3(1-l)]\tau}}{a + cx(N^+)(1 - e^{-al\tau})} \\ \quad + [1 + (1 - \theta_2)b_2](1 - \mu_1) e^{-[d_1l+d_3(1-l)]\tau} y(N^+). \end{cases} \quad (6)$$

$G_1(0, 0)$ and $G_2(x^*, y^*)$ are obtained and

$$\begin{cases} x^* = By^*, \quad Be^{-al\tau} > A, \\ y^* = \frac{a(Be^{-al\tau} - A)}{cAB(1 - e^{-al\tau})}, \quad Be^{-al\tau} > A, \end{cases} \quad (7)$$

here $A = \frac{1 - [1 + (1 - \theta_2)b_2](1 - \mu_1)e^{-[d_1l+d_3(1-l)]\tau}}{\theta_1 b_1 (1 - \mu_1) e^{-d_3(1-l)\tau}}$, and $B = [1 + (1 - \theta_1)b_1]e^{-d_2(1-l)\tau} A + \theta_2 b_2 e^{-[d_1l+d_3(1-l)]\tau}$.

Similarly with reference [6], we have the following two theorems.

Theorem 3. *i)* Suppose $Be^{-al\tau} < A$ holds, $G_1(0, 0)$ of (6) is globally asymptotically stable;

ii) Suppose $Be^{-al\tau} > A$ holds, $G_2(x^*, y^*)$ of (6) is globally asymptotically stable.

Theorem 4. *i)* Suppose $Be^{-al\tau} < A$ holds, periodic solution $(0, 0)$ of (2) is globally asymptotically stable;

ii) Suppose $Be^{-al\tau} > A$ holds, periodic solution (\tilde{x}, \tilde{y}) of (2) is globally asymptotically stable, and

$$\begin{cases} \tilde{x} = \begin{cases} \frac{ax^* e^{-a(t-N)}}{a + cx^*(1 - e^{-a(t-N)})}, & t \in (N, (N+1)N], \\ x^{**} e^{-d_2(t-(N+l))}, & t \in ((N+l), (N+1)], \end{cases} \\ \tilde{y} = \begin{cases} y^* e^{-d_1(t-N)}, & t \in (N, (N+l)], \\ y^{**} e^{-d_3(t-N)}, & t \in ((N+l), (N+1)], \end{cases} \end{cases} \quad (8)$$

here x^*, y^* are by (7), and x^{**}, y^{**} are by

$$\begin{cases} x^{**} = [1 + (1 - \theta_1)b_1] \times \frac{ax^* e^{-al\tau}}{a + cx^*(1 - e^{-al\tau})} + \theta_1 b_1 y^* e^{-d_1l\tau}, \\ y^{**} = \theta_1 b_1 \times \frac{ax^* e^{-al\tau}}{a + cx^*(1 - e^{-al\tau})} + [1 + (1 - \theta_1)b_1] y^* e^{-d_1l\tau}. \end{cases} \quad (9)$$

From (1) and Remark 3., we get

$$-(a + \beta M_o)x - cx^2 \leq \frac{dx}{dt} \leq -ax - cx^2, \quad (10)$$

and

$$-(d_2 + \beta M_o)x \leq \frac{dx}{dt} \leq -d_2x. \quad (11)$$

Then, $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are respectively the solutions of

$$\left. \begin{array}{l} \left. \begin{array}{l} \frac{dx_1}{dt} = -(a + \beta_1 M_o)x_1 - cx_1^2, \\ \frac{dy_1}{dt} = -d_1y_1, \\ \Delta x_1 = (1 - \theta_1)b_1x_1 + \theta_2b_2y_1, \\ \Delta y_1 = \theta_1b_1x_1 + (1 - \theta_2)b_2y_1, \end{array} \right\} t \in (N, (N + l)], \\ \left. \begin{array}{l} \frac{dx_1}{dt} = -(d_2 + \beta_2 M_o)x_1, \\ \frac{dy_1}{dt} = -d_3y_1, \\ \Delta x_1 = 0, \\ \Delta y_1 = -\mu_1y_1, \end{array} \right\} t = (N + l), \\ \left. \begin{array}{l} \frac{dx_1}{dt} = -(d_2 + \beta_2 M_o)x_1, \\ \frac{dy_1}{dt} = -d_3y_1, \\ \Delta x_1 = 0, \\ \Delta y_1 = -\mu_1y_1, \end{array} \right\} t \in ((N + l), (N + 1)], \\ \left. \begin{array}{l} \Delta x_1 = 0, \\ \Delta y_1 = -\mu_1y_1, \end{array} \right\} t = (N + 1), \end{array} \right\} \quad (12)$$

and

$$\left. \begin{array}{l} \left. \begin{array}{l} \frac{dx_2}{dt} = -ax_2 - cx_2^2, \\ \frac{dy_2}{dt} = -d_1y_2, \\ \Delta x_2 = (1 - \theta_1)b_1x_2 + \theta_2b_2y_2, \\ \Delta y_2 = \theta_1b_1x_2 + (1 - \theta_2)b_2y_2, \end{array} \right\} t \in (N, (N + l)], \\ \left. \begin{array}{l} \frac{dx_2}{dt} = -d_2x_2, \\ \frac{dy_2}{dt} = -d_3y_2, \\ \Delta x_2 = 0, \\ \Delta y_2 = -\mu_1y_2, \end{array} \right\} t = (N + l), \\ \left. \begin{array}{l} \frac{dx_2}{dt} = -d_2x_2, \\ \frac{dy_2}{dt} = -d_3y_2, \\ \Delta x_2 = 0, \\ \Delta y_2 = -\mu_1y_2, \end{array} \right\} t \in ((N + l), (N + 1)], \\ \left. \begin{array}{l} \Delta x_2 = 0, \\ \Delta y_2 = -\mu_1y_2, \end{array} \right\} t = (N + 1), \end{array} \right\} \quad (13)$$

Obviously,

$$\begin{cases} x_1 \leq x \leq x_2, \\ y_1 \leq y \leq y_2. \end{cases} \quad (14)$$

Similar to theorem (4), we have

Theorem 5. *i)* Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} < A_1$ holds, triviality solution $(0, 0)$ of (13) is globally asymptotically stable;

ii) Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1$ holds, the periodic solution $(\widetilde{x}_1, \widetilde{y}_1)$ of (13) is globally asymptotically stable, and

$$\begin{cases} \widetilde{x}_1 = \begin{cases} \frac{(a + \beta_1 M_o)x_1^* e^{-(a+\beta_1 M_o)(t-N)}}{(a + \beta_1 M_o) + cx_1^*(1 - e^{-(a+\beta_1 M_o)(t-N)})}, & t \in (N, (N + 1)], \\ x^{**} e^{-d_2(t-(N+l))}, & t \in ((N + l), (N + 1)], \end{cases} \\ \widetilde{y}_1 = \begin{cases} y^* e^{-d_1(t-N)}, & t \in (N, (N + l)], \\ y^{**} e^{-d_3(t-N)}, & t \in ((N + l), (N + 1)], \end{cases} \end{cases} \quad (15)$$

here x_1^*, y_1^* are by

$$\begin{cases} x_1^* = B y_1^*, \quad B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1, \\ y_1^* = \frac{(a + \beta_1 M_o)(B_1 e^{-(a+\beta_1 M_o)l\tau} - A_1)}{c A_1 B_1 (1 - e^{-(a+\beta_1 M_o)l\tau})}, \quad B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1, \end{cases} \quad (16)$$

and x_1^{**}, y_1^{**} are by

$$\begin{cases} x_1^{**} = [1 + (1 - \theta_1)b_1] \\ \quad \times \frac{(a + \beta_1 M_o)x_1^* e^{-(a+\beta_1 M_o)l\tau}}{(a + \beta_1 M_o) + cx_1^*(1 - e^{-(a+\beta_1 M_o)l\tau})} + \theta_1 b_1 y_1^* e^{-d_1 l\tau}, \\ y_1^{**} = \theta_1 b_1 \\ \quad \times \frac{(a + \beta_1 M_o)x_1^* e^{-(a+\beta_1 M_o)l\tau}}{(a + \beta_1 M_o) + cx_1^*(1 - e^{-(a+\beta_1 M_o)l\tau})} + [1 + (1 - \theta_1)b_1] y_1^* e^{-d_1 l\tau}. \end{cases} \quad (17)$$

here $A_1 = \frac{1 - [1 + (1 - \theta_2)b_2](1 - \mu_1)e^{-[d_1 l + d_3(1-l)]\tau}}{\theta_1 b_1 (1 - \mu_1)e^{-d_3(1-l)\tau}}$, and $B_1 = [1 + (1 - \theta_1)b_1]e^{-d_2(1-l)\tau} A_1 + \theta_2 b_2 e^{-[d_1 l + d_3(1-l)]\tau}$.

Remark 6. Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1$ holds, for $\varepsilon_2 > 0$, there exists a T_2 such that $x_1 \geq \widetilde{x}_1 - \varepsilon_2$ and $y_1 \geq \widetilde{y}_1 - \varepsilon_2$ for $t > T_2$.

From theorem (4),(4) and remark (6), we gets

Theorem 7. *i)* Suppose $B e^{-al\tau} < A$ holds, the population of (1) will go extinct.

ii) Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1$ and $B e^{-al\tau} > A$ hold, (1) is permanent.

Proof. *i)* According to the condition $B e^{-al\tau} < A$, theorem (5), and (14) for any sufficient small $\varepsilon_1 > 0$, there exists a $T_1 > 0$ such that $x \leq x_2 \leq \varepsilon_1$ and $y \leq y_2 \leq \varepsilon_1$ for $t > T_1$. That is to say, for any sufficient small $\varepsilon_1 > 0$, there exists a $T_1 > 0$ such that $x \leq \varepsilon_1$ and $y \leq \varepsilon_1$ for $t > T_1$. These show that population of (1) will go extinct.

ii) According to the condition $B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1$, Remark (7) and (14), for any sufficient small $\varepsilon_2 > 0$, there exists a $T_2 > 0$ such that $x \geq x_1 \geq \widetilde{x}_1 - \varepsilon_2 \geq \frac{(a+\beta_1 M_o)x_1^* e^{-(a+\beta_1 M_o)l\tau}}{(a+\beta_1 M_o)+cx_1^*(1-e^{-(a+\beta_1 M_o)l\tau})} + x^{**} e^{-d_2(1-l)\tau} - \varepsilon_1 \triangleq m_1$ and $y \geq y_1 \geq \widetilde{y}_1 - \varepsilon_1 \geq y_1^* e^{-d_1 l\tau} + y_1^{**} e^{-d_3(1-l)\tau} - \varepsilon_1 \triangleq m_2$ for $t > T_2$, where x_1^* , y_1^* and x_1^{**} , y_1^{**} are determined as (16) and (17) respectively. From $Be^{-al\tau} > A$, and (14), for any sufficient small $\varepsilon_2 > 0$, there exists a $T_3 > 0$ such that $x \leq \widetilde{x}_2 - \varepsilon_2 \leq x_2^* + x_2^{**} + \varepsilon_2 \triangleq m_3$ and $y \leq y_2 \leq y_2^* + y_2^{**} + \varepsilon_2 \triangleq m_4$. From the above discussion, we know $m_1 < x < m_3$ and $m_2 < y(t) < m_4$ for $t > \max\{T_2, T_3\}$. That is to say, the normal population $x(t)$ and genic mutation population $y(t)$ of system (1) will be permanent.

Therefore,

Remark 8. i) Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} < A_1$, and $Be^{-al\tau} < A$ holds, $(0, 0, \widetilde{c}_o, \widetilde{c}_e)$ of (1) is globally asymptotically stable.

ii) Suppose $B_1 e^{-(a+\beta_1 M_o)l\tau} > A_1$, and $Be^{-al\tau} > A$ hold, (1) are permanent.

4 Discussion

In this work, we consider a single population-hibernation model with genic mutation, birth pulse and impulsive input toxin in a polluted environment. If it is supposed that the variables are shown in the table below:

$x(0)$	$y(0)$	$c_o(0)$	$c_e(0)$	a	c	d_1	d_2	d_3	β_1	β_2	f	g	m	h	θ_1	θ_2	b_1	b_2	μ_1	μ	l	τ
0.3	0.1	0.3	0.3	0.2	0.1	0.1	0.2	0.1	0.1	0.1	0.3	0.4	0.1	0.4	0.1	0.5	0.3	0.3	0.6	0.2	0.5	1

there exists a globally asymptotically stable solution $(0, 0, \widetilde{c}_o, \widetilde{c}_e)$ of (1) (one can see Fig. 1). If it is supposed that another variables are shown in the table below:

$x(0)$	$y(0)$	$c_o(0)$	$c_e(0)$	a	c	d_1	d_2	d_3	β_1	β_2	f	g	m	h	θ_1	θ_2	b_1	b_2	μ_1	μ	l	τ
0.3	0.1	0.3	0.3	0.2	0.1	0.1	0.2	0.1	0.1	0.1	0.3	0.4	0.1	0.4	0.1	0.5	0.3	0.3	0.2	0.2	0.5	1

then, (1) is permanent (one can see Fig. 2). If it is supposed that variables are shown in the table below:

$x(0)$	$y(0)$	$c_o(0)$	$c_e(0)$	a	c	d_1	d_2	d_3	β_1	β_2	f	g	m	h	θ_1	θ_2	b_1	b_2	μ_1	μ	l	τ
0.3	0.1	0.3	0.3	0.2	0.1	0.1	0.2	0.1	0.1	0.1	0.3	0.4	0.1	0.4	0.1	0.5	0.3	0.3	0.9	0.4	0.5	1

there exists a globally asymptotically stable solution $(0, 0, \tilde{c}_o, \tilde{c}_e)$ of system (1) (one can see Fig. 3).

If it is supposed that variables are shown in the table below:

$x(0)$	$y(0)$	$c_o(0)$	$c_e(0)$	a	c	d_1	d_2	d_3	β_1	β_2	f	g	m	h	θ_1	θ_2	b_1	b_2	μ_1	μ	l	τ
0.3	0.1	0.3	0.3	0.2	0.1	0.1	0.2	0.1	0.1	0.1	0.3	0.4	0.1	0.4	0.1	0.5	0.3	0.3	0.9	0.2	0.5	1

then, (1) is permanent (see Fig. 4). The results show that the amount of genic mutation rate of the normal population also plays important role for the permanence of system (1). Our results show that the impulsive environmental toxin will reduce biological diversity of the nature world.

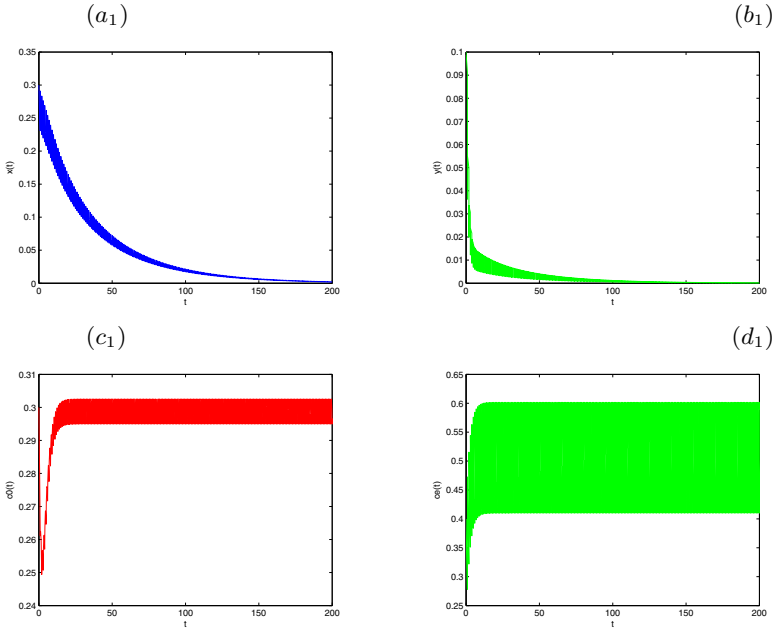


Fig. 1. The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the second table.

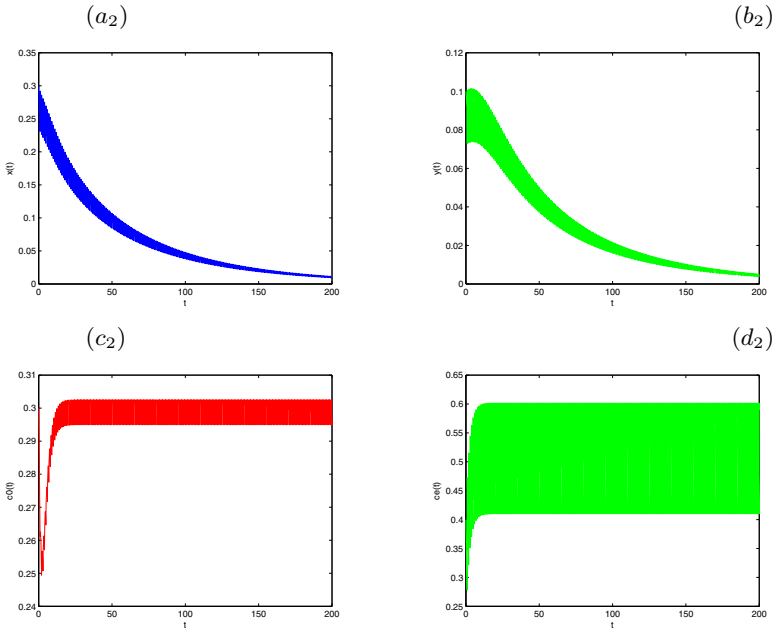


Fig. 2. The permanence of system (1) with parameters in the second table.

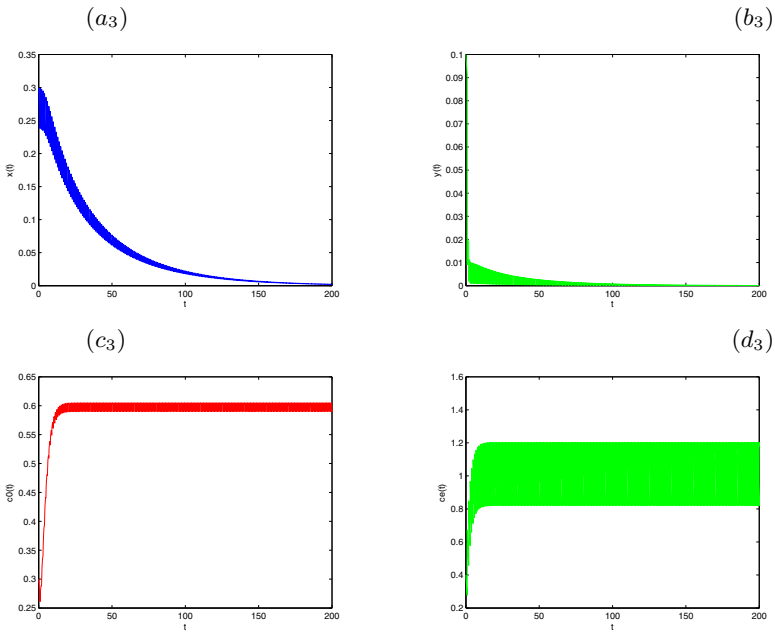


Fig. 3. The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the third table.

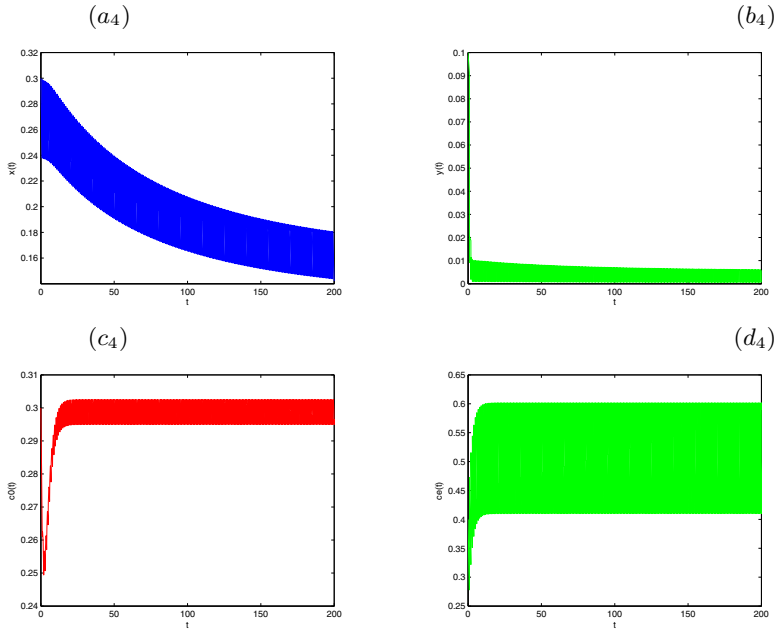


Fig. 4. The permanence of system (1) with parameters in the fourth table.

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