On Codeword Bits Disparity in Polar Codes

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Abstract. Polar codes, the codes contributed based on channel polarization, are the first class of error-correction codes which can be proved to achieve the channel capacity of binary-input discrete memoryless channels. A set of bit-channels are generated after the channel combining and channel splitting operations. These bit-channels have different capacities. The idea of polar coding is to transmit information bits through bit-channels with higher capacities, while transmitting frozen bits through the ones with lower capacities. The generator matrix of polar codes is invertible. The input bits can also be represented as the codeword bits multiplied by the generator matrix. Based on these observations, we make an assumption that there are significance disparities existing in the codeword bits of polar codes and the significance level of codeword bits are relevant to their bit indices. To analyze the significance level of codeword bits, the codeword bits are transmitted over non-uniform channels. Some of the codeword bits are transmitted through pure noisy channels, which can be equalized as punctured bits. Simulation results show that there exist significance disparities in the codeword bits.

Keywords: Polar codes · Significance disparity · Codeword bits · SC decoding

1 Introduction

Polar codes proposed by Arikan are the first class of error-correction codes which can be proved to achieve the capacity of binary-input discrete memoryless channels (B-DMC) [1]. They have recursive structures and simple encoding and decoding complexity . They can achieve good error-correction performance even with short code lengths. Due to these characteristics, polar codes have been adopted as coding schemes for the 5G communication systems [2].

The main idea of polar codes is to generate a set of coordinate channels called bit-channels and transmit information through the bit-channels with

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higher capacity. For a polar code with code length N and code dimension k, the construction of polar codes is to select k good bit-channels out of all the N bit-channels. In [3,4], density evolution (DE) is employed to construct polar codes. In [5], the authors use upgraded and degraded channels to approximate the bit-channels. In [6], Gaussian approximation (GA) is employed to decrease the computation complexity of DE. In [7], fast construction of polar codes based on polarization weight (PW) is proposed.

Many efficient decoding algorithms of polar codes have been studied. In [8], successive cancellation list (SCL) decoding algorithm is proposed to improve the error-correction performance of polar codes. The cyclic redundancy check (CRC) aided SCL decoding can substantially improve the performances of polar codes [8,9]. An adaptive CRC aided SCL decoder which enlarges the list size progressively is proposed in [10]. Simplified-SC (SSC) decoder [11] and fast-simplified-SC (FSSC) [12] decoder are proposed to reduce decoding latency. Belief propagation (BP) decoding is proposed in [13] for pipelined implementations.

The code lengths of polar codes are the power of 2 with a kernel of $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. To achieve arbitrary code lengths, many puncturing and shortening methods have been proposed. Some of the coded bits are not transmitted. For punctured polar codes, these bits are unknown bits. In the receiving end, the log-likelihood ratios (LLRs) of the punctured bits are set to zeros. For shortened polar codes, these bits are known bits and the LLRs of the shortened bits are set to infinity. In [14], quasi-uniform puncturing (QUP) algorithm is proposed. The first N_p coded bits are punctured. In [15], the last N_p coded bits are shortened. The information sets are reselected after puncturing and shortening in [14,15]. In [16], punctured bits are selected according to the information and frozen sets.

The generator matrix of polar codes is invertible. There may be some similarity between the codeword bits and the input bits. In [17], the authors make a conjecture that the codeword bits can be sorted according to the reliability of information bits. The various performances of these puncturing and shortening methods also indicate that codeword bits of different positions have different effects on the performance of polar codes. In [18], the importance differences of input bits have been studied. This paper will study the importance differences of codeword bits.

The remainder of the paper is organized as follows. Section 2 presents a brief review of polar codes. In Sect. 3, the various importance levels of codeword bits of polar codes are studied. Simulation results are presented in Sect. 4. Finally, concluding remarks are drawn in Sect. 5.

2 Preliminaries

2.1 Polar Codes

Given a B-DMC channel $(\mathcal{X}, \mathcal{Y}, W(y \mid x))$, where $\mathcal{X} \in \{0, 1\}$ is the input alphabet, $\mathcal{Y}$ is the output alphabet, and $W(y \mid x)$ is the transition probabilities of channel W. Use a_i^j to denote $(a_i, a_{i+1}, \cdots, a_j)$. When $j < i$, $a_i^j = \varnothing$. Let W^N denote N independent uses of W, we have

$$W^N\left(y_1^N \mid x_1^N\right) = \prod_{i=1}^{N} W\left(y_i \mid x_i\right), \tag{1}$$

These N independent channels can be combined into a single synthesized channel W_N. The relationship between W_N and W^N can be represented as

$$W_N\left(y_1^N \mid u_1^N\right) = W^N\left(y_1^N \mid u_1^N G_N\right), \tag{2}$$

where u_1^N are the input bits of W_N.

The channel W_N can be split into a series of bit-channels $W_N^{(i)} : \mathcal{X} \to \mathcal{Y}^N \times \mathcal{X}^{i-1}, 1 \le i \le N$, each channel $W_N^{(i)}$ with single input bits u_i and its channel transmission probabilities are given by

$$W_N^{(i)}\left(y_1^N, u_1^{i-1} \mid u_i\right) \triangleq \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N\left(y_1^N \mid u_1^N\right). \tag{3}$$

As N tends to infinity, the capacity of the bit-channels $W_N^{(i)}$ will tend to 0 or 1. In other words, this set of bit-channels will be polarized into noiseless and noisy channels. This phenomenon is called *channel polarization*. The idea of polar coding is to transmit the information bits through the noiseless bit-channels and assign fixed bits over the noisy bit-channels.

Let u_1^N and x_1^N denote the input bits and codeword bits of polar codes. The encoding process of polar codes can be written as

$$x_1^N = u_1^N G_N, \tag{4}$$

where G_N is the generator matrix of order N, and

$$G_N = B_N F^{\otimes n}, \tag{5}$$

where B_N is the bit-reversal matrix, $\otimes n$ is the Kronecker power and $F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Let $\mathcal{A}$ denote the information set, and $\mathcal{A}^c$ be its complementary set. The codeword x_1^N can be rewritten as

$$x_1^N = u_{\mathcal{A}} G_{\mathcal{A}} + u_{\mathcal{A}^c} G_{\mathcal{A}^c}, \tag{6}$$

where $u_{\mathcal{A}}$ denote the input bits with indices corresponding to the information set, $G_{\mathcal{A}}$ denote the submatrix of G_N formed by the rows with indices in $\mathcal{A}$ and $u_{\mathcal{A}^c}$ are referred as frozen bits.

2.2 Successive Cancellation Decoding Algorithm

The successive cancellation (SC) decoding algorithm of polar codes decodes input bits sequentially from u_1 to u_N. Define the log-likelihood ratio (LLR) as

$$L_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}\right) \triangleq \log \frac{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1} \mid u_i = 0\right)}{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1} \mid u_i = 1\right)}, \tag{7}$$

The LLRs can be calculated recursively as follows

$$\begin{aligned} &L_N^{(2i-1)}\left(y_1^N, \hat{u}_1^{2i-2}\right) \\ &= 2\tanh^{-1}\left(\tanh\left(L_{N/2}^{(i)}\left(y_1^{N/2}, \hat{u}_{1,e}^{2i-2} \oplus \hat{u}_{1,o}^{2i-2}\right)/2\right)\right. \\ &\quad \left.\times \tanh\left(L_{N/2}^{(i)}\left(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}\right)/2\right)\right), \end{aligned} \tag{8}$$

$$\begin{aligned} L_N^{(2i)}\left(y_1^N, \hat{u}_1^{2i-1}\right) &= L_{N/2}^{(i)}\left(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}\right) \\ &\quad + (-1)^{\hat{u}_{2i-1}} L_{N/2}^{(i)}\left(y_1^{N/2}, \hat{u}_{1,e}^{2i-2} \oplus \hat{u}_{1,o}^{2i-2}\right). \end{aligned} \tag{9}$$

The ith decision element $\hat{u}_i$ can be represented as

$$\hat{u}_i \triangleq \begin{cases} u_i, & \text{if } i \in \mathcal{A}^c \\ h_i\left(y_1^N, \hat{u}_1^{i-1}\right), & \text{if } i \in \mathcal{A} \end{cases} \tag{10}$$

where $h_i : \mathcal{Y}^N \times \mathcal{X}^{i-1} \to \mathcal{X}, i \in \mathcal{A}$ is the decision function, and

$$h_i\left(y_1^N, \hat{u}_1^{i-1}\right) \triangleq \begin{cases} 0, & \text{if } L_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}\right) \geq 0 \\ 1, & \text{otherwise} \end{cases} \tag{11}$$

2.3 Construction of Polar Codes

The construction of polar codes focuses on choosing the bit-channels with low error probabilities to transmit the information bits. However, the exact value of error probabilities of bit-channels $P_e(W_N^{(i)})$ are not easy to calculate. Many approximation methods have been proposed to calculate these error probabilities, including density evolution [3,4], Tal and Vardy method [5], Gaussian approximation (GA) method [6] and the construction method based on polarization weight [7]. In this subsection, the GA method of polar codes is introduced.

When the codeword bits of polar codes are transmitted over additive white Gaussian noise (AWGN) channels, the intermediate values of SC decoding can be seen as Gaussian values [6]. Assume that all zero bits are transmitted. The signals are BPSK modulated with $s_i = 1 - 2x_i, i = 1, \cdots, N$, where s_i is the signal after modulation, and x_i is the codeword bit. Let y_i denote the received signal. Then $y_i = s_i + n_i, i = 1, \cdots, N$, where $(n_1, n_2, \cdots, n_N)$ is an i.i.d. set of Gaussian random variables with mean zero and variance σ^2. Let $L_1^{(i)}(y_i)$ denote the LLRs of y_i. Then $L_1^{(i)}(y_i)$ is a Gaussian value with the distribution function of $\mathcal{N}\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right)$.

The LLR values given by (8)–(9) can also be considered as Gaussian random variables with $\mathbf{D}\left[L_N^{(i)}\right] = 2\mathbf{E}\left[L_N^{(i)}\right]$, where $\mathbf{E}$ and $\mathbf{D}$ are the mean and variance, respectively. Calculate the expected values of both sides of (8)–(9), the following expressions can be derived

$$\mathbf{E}\left[L_N^{(2i-1)}\right] = \phi^{-1}\left(1 - \left(1 - \phi\left(\mathbf{E}\left[L_{N/2}^{(i)}\right]\right)\right)^2\right), \tag{12}$$

$$\mathbf{E}\left[L_N^{(2i)}\right] = 2\mathbf{E}\left[L_{N/2}^{(i)}\right], \tag{13}$$

where

$$\phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} \tanh\left(\frac{u}{2}\right) \mathrm{e}^{-\frac{(u-x)^2}{4x}}\, \mathrm{d}u, & x > 0 \\ 1, & x = 0. \end{cases} \tag{14}$$

The error probabilities of bit-channels [19] are given by

$$P_e(W_N^{(i)}) = Q\left(\sqrt{\mathbf{E}\left[L_N^{(i)}\right]/2}\right), \quad 1 < i < N. \tag{15}$$

3 The Various Important Levels of Different Codeword Bits in Polar Codes

3.1 BLER Performance of Polar Codes over Non-uniform Channels

When the coreword bits are transmitted over non-uniform channels, the Gaussian approximation method can be generalized. For the basic unit of encoding and decoding in [1] with input bits (u_1, u_2) and received signals (y_1, y_2), we have

$$\mathbf{E}\left[L\left(u_1\right)\right] = \phi^{-1}\left(1 - \left(1 - \phi\left(\mathbf{E}\left[L\left(y_1\right)\right]\right)\right)\left(1 - \phi\left(\mathbf{E}\left[L\left(y_2\right)\right]\right)\right)\right), \tag{16}$$

and

$$\mathbf{E}\left[L\left(u_2\right)\right] = \mathbf{E}\left[L\left(y_1\right)\right] + \mathbf{E}\left[L\left(y_2\right)\right]. \tag{17}$$

$\mathbf{E}\left[L_N^{(i)}\right]$ can be calculated recursively using (16) and (17), and $P_e(W_N^{(i)})$ can be calculated accordingly.

The block error rate of polar codes is calculated as

$$P(\varepsilon) = 1 - \prod_{i \in \mathcal{A}}\left(1 - P_e(W_N^{(i)})\right). \tag{18}$$

3.2 The Proposed Scheme About Significance Disparity of Codeword Bits in Polar Codes.

The generator matrix G_N of polar codes is invertible, and $G_N^{-1} = G_N$, so that (4) can be rewritten as $u_1^N = x_1^N G_N$. This equality shows that there may be some similarity between input bit u_i and codeword bit x_i. As is known to us, u_i has different error probabilities for different bit indices. To research the significance disparity of the codeword bits with different positions, we transmit the codeword bits through non-uniform channels. Some of the codeword bits are transmitted through the pure noisy channels, which can be seen as punctured bits, and the other codeword bits will be transmitted over AWGN channels.

Consider a codeword bit x_i which is over the pure noisy channel. The LLR of x_i can be calculated as

$$L(y_i) = \log \frac{\Pr(y_i \mid x_i = 0)}{\Pr(y_i \mid x_i = 1)} = 0. \tag{19}$$

It is the same as when x_i is punctured. Note that for puncturing schemes of polar codes, the information sets are often redesigned after a puncturing pattern is set. In this paper, the information sets are fixed, and we study the effects of different codeword bits on the BLER performances.

4 Simulation Results

To illustrate the various significance disparity of codeword bits in polar codes, we use the punctured bits to equalize the codeword bits transmitted over the pure noisy channels.

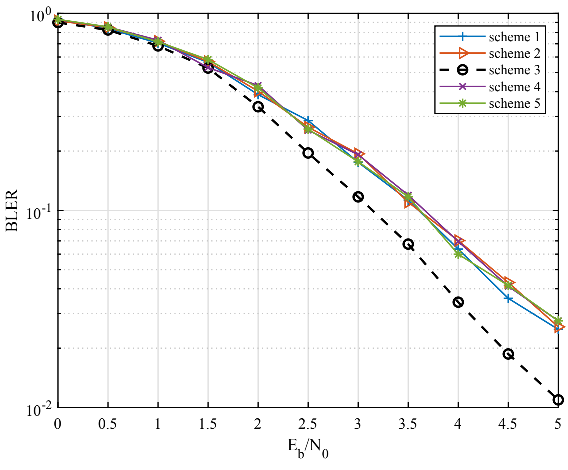

Fig. 1. The BLER performances of polar codes with 30 punctured codeword bits.

Firstly, we consider the polar codes with code length 256, code dimension 93. Figure 1 illustrates the BLER performances of polar codes by randomly puncturing 30 codeword bits in different sets. For puncturing schemes 1 to 5, the punctured codeword bits are randomly selected from the sets $[1, 64]$, $[65, 128]$, $[97, 160]$, $[129, 192]$ and $[193, 256]$, respectively. It can be found that the BLER

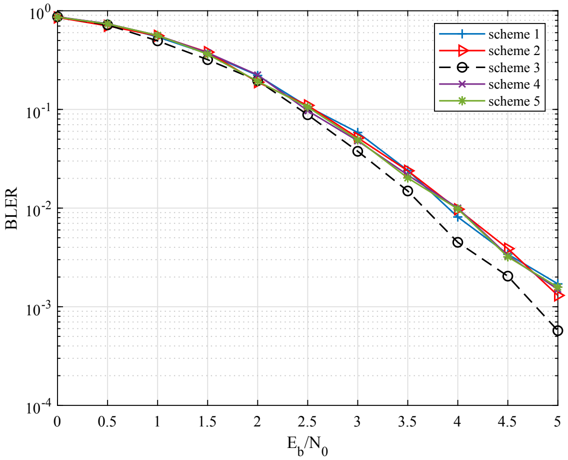

Fig. 2. The BLER performances of polar codes with 20 punctured codeword bits.

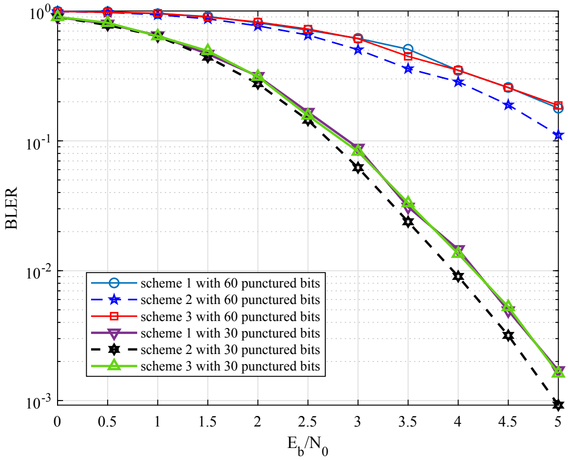

Fig. 3. The BLER performances of polar codes with randomly punctured codeword bits in different sets.

performances of scheme 3 are better than the other schemes, which implies that the average significance levels of codeword bits in the 3rd set are lower than the codeword bits in other sets. We also provide the BLER performances of polar codes by randomly puncturing 20 codeword bits in the aforementioned sets, as shown in Fig. 2. It can be seen in Fig. 2 that scheme 3 also has better BLER performances than the other schemes when there are 20 punctured codeword bits. It can be shown that there are various significance levels between the codeword bits, and the significance levels of the codeword bits will be affected by their positions.

Following the phenomena shown in Fig. 1 and Fig. 2, we make an assumption that the codeword bits in the middle set are with the lower average significance levels than the others. We define three puncturing schemes, in which the codeword bits are punctured from the sets [1, 128], [65, 192] and [129, 256], respectively. The BLER performances of the three schemes are given in Fig. 3 with 60 and 30 punctured codeword bits in each set. It can be seen that scheme 2 has better performances than the other schemes both with 60 and 30 punctured codeword bits. The puncture set of scheme 2 [65, 192] is still in the middle sets.

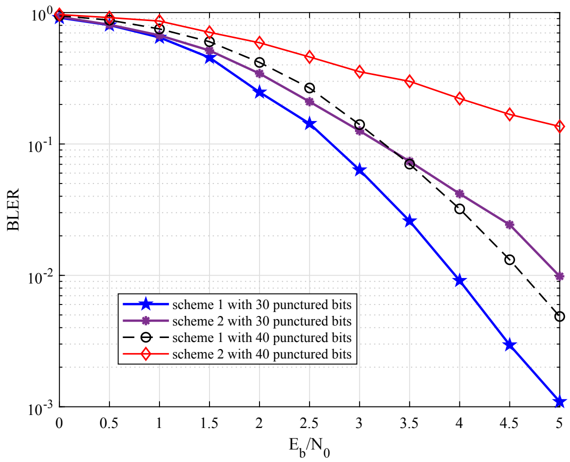

Fig. 4. The BLER performances of polar codes with randomly punctured bits in the middle part of bit indices.

Observing the BLER curves in the previous figures, all these results imply that the codeword bits in the middle of the index are with lower average significance levels than the others. Then we focus on the schemes with better performances than the others in the previous results, which means we focus on the puncturing range [65, 192] and [97, 160]. For simplicity, the scheme with puncture sets [65, 192] and [97, 160] are termed as scheme 1 and 2, respectively. From Fig. 4, it can be seen that although there are the same number of codeword bits being punctured, scheme 1 has better BLER performances than that of scheme 2. As both the schemes are puncturing codeword bits from the middle set, and the puncturing set of scheme 1 is larger than that of scheme 2, we conclude that the continuous impulse noise would lead to much more negative affections on BLER performances of polar codes than that of the discrete impulse noise.

Figure 5 illustrates the BLER performances of puncturing codeword bits from different sets with different probabilities. The code length of polar codes is 512 and code rate is 0.375. The codeword bits are divided into 2 sets, the first set is $[1, 128] \cup [385, 512]$, and the second set is [129, 384]. For all the schemes 1 to 6, there are 128 codeword bits being punctured. For scheme 1 to 6, there are 16, 32,

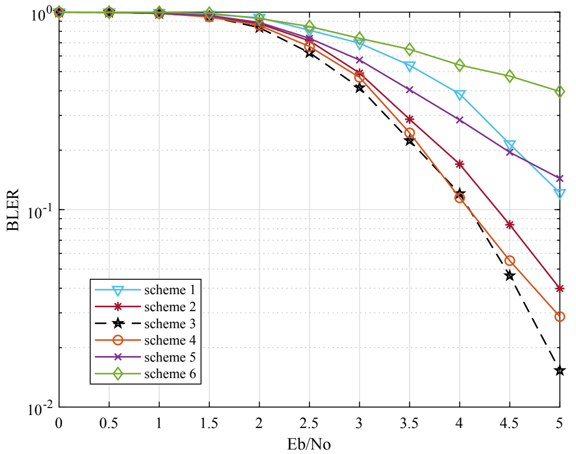

Fig. 5. The BLER performances of polar codes with punctured bits from different sets.

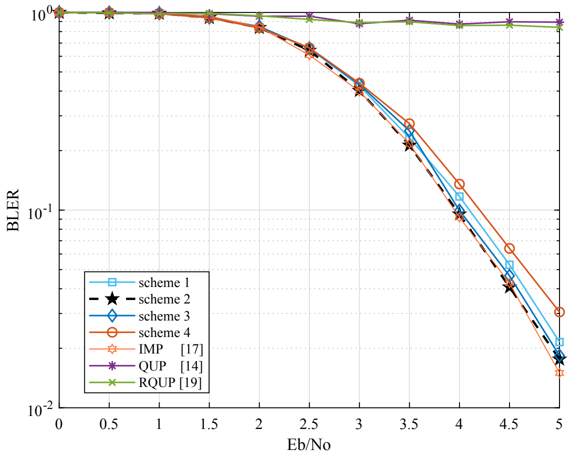

Fig. 6. The BLER performance comparison between the proposed schemes and other existing schemes.

48, 64, 80 and 96 codeword bits randomly punctured from the first set, and the remaining codeword bits are randomly punctured from the second set. It can be found that scheme 4 can be equalized as randomly puncturing 128 bits without dividing sets, and scheme 3 has better BLER performance than scheme 4, which means the randomly puncturing cannot be optimal. Figure 5 also illustrates that the codeword bits in the middle positions are with the lower average significance levels than the codeword bits in other positions.

Finally, we compare the proposed schemes with other existing puncturing schemes, which are shown in Fig. 6. The proposed schemes are with the same divided sets as shown in Fig. 5. For scheme 1 to 4, there are 40, 48 and 56 and 64 bits being punctured from the first set, respectively. It can be found that the proposed scheme 1 to 3 have better BLER performances than scheme 4, which is equalized to randomly puncturing. The performances of the proposed schemes are similar to that of the method in [17], where the importance levels are sorted according to the error probabilities of input bits with the same indices. The QUP [14] and RQUP [20] schemes are also compared, and it can be found that the QUP and RQUP schemes cannot provide satisfactory performance with fixed information sets.

5 Conclusion

In this paper, to research the significance disparity between the codeword bits of polar codes, we adopted non-uniform channels to transmit the codeword bits. By analyzing the log-likelihood ratio of a codeword bit affected by pure noisy channel, we equalized this bit as a punctured bit. By studying the previous research, we made an assumption that the significance disparity of codeword bits would follow a symmetrically distributed law based on their positions. The simulation results proved that the significance disparities of codeword bits exist.

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