# On a Class of Minihypers in the Geometries PG$(r, q)$

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**Abstract.** We characterize all minihypers with parameters $(v_3 + 2v_2, v_2 + 2v_1)$ in the geometries PG$(r, q)$. Apart from the trivial ones which are the sum of a plane and two lines, we construct several sporadic minihypers in the geometries PG$(r, q)$ with $q = 3$ and $q = 4$.

**Keywords:** Liear codes · Minihypers · Griesmer bound

## 1 Introduction

We begin by introducing two geometric objects that are equivalent to linear codes. Let PG$(r, q)$ be the $r$-dimensional projective geometry over $\mathbb{F}_q$ and denote the set of its points by $\mathcal{P}$. A multiset of points in PG$(r, q)$ is a mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$, i.e. a mapping which assigns to every point a non-negative integer called its multiplicity. This mapping is extended additively to the subsets of $\mathcal{P}$: for every $X \subseteq \mathcal{P}$ we set $\mathcal{K}(X) = \sum_{P \in X} \mathcal{K}(P)$.

A multiset $\mathcal{K}$ in PG$(r, q)$ is called an $(n, w)$-arc in PG$(r, q)$ if (i) $\mathcal{K}(\mathcal{P}) = n$, (ii) $\mathcal{K}(H) \le w$ for every hyperplane $H$ in PG$(r, q)$, and (iii) $\mathcal{K}(H_0) = w$ for some hyperplane $H_0$. Similarly, a multiset $\mathcal{K}$ in PG$(r, q)$ is called an $(n, w)$-minihyper (also $(n, w)$ blocking multiset) if it is a multiset with (i) $\mathcal{K}(\mathcal{P}) = n$, (ii) $\mathcal{K}(H) \ge w$ for every hyperplane $H$ in PG$(r, q)$, and (iii) $\mathcal{K}(H_0) = w$ for some hyperplane $H_0$. The notion of a minihyper was introduced by N. Hamada in connection with the so-called main problem in coding theory stated below. For more details one should consult the survey [1] and the references there.

Linear codes, arcs and minihypers are in some sense equivalent to linear codes.

**Theorem 1.** *The existence of the following objects is equivalent:*

*(1) a linear $[n, k, d]_q$-code with the property that in any generator matrix has at most $t$ identical columns;*
*(2) an $(n, n - d)$-arc in $PG(k - 1, q)$ with maximal point multiplicity $t$;*


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*(3) a $(tv_k - n, tv_{k-1} - n + d)$-minihyper in $PG(k-1, q)$ with maximal point multiplicity $\leq t$.*

We consider the so-called main problem in coding theory (cf. [2]) which is to optimize one of the three main parameters of a linear code given the other two. In this paper, we focus on the problem of determining the minimal length $n$ of a linear code of fixed dimension $k$ and fixed minimum distance $d$ over the field with $q$ elements. This value is commonly denoted by $n_q(k, d)$. There exists a natural lower bound on $n_q(k, d)$ – the so-called Griesmer bound:

$$n_q(k, d) \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil. \quad (1)$$

The RHS in the above inequality is denoted by $g_q(k, d)$. Linear codes with parameters $[n, k, d]_q$ of length $n = g_q(k, d)$ are called Griesmer codes. Arcs and minihypers associated with linear Griesmer codes are called Griesmer arcs, resp. Grismer minihypers.

In this paper, we adopt the geometric point of view and deal with minihypers. Our main result is the characterization of the minihypers with parametes $(v_3 + 2v_2, v_2 + 2v_1)$ in the geometries $\mathrm{PG}(r, q)$ for all $r$ and all prime powers $q$. Here $v_s$ denotes the Gaussian coefficient $v_s = \dfrac{v^s - 1}{v - 1}$. Minihypers with the above parameters are Griesmer minihypers. This can be shown by a straightforward calculation. This characterization we give is used further in the nonexistence proof for some hypothetical Griesmer codes [6]. The motivation for this investigation cam from the recent research on the exact values of $n_3(6, d)$ carried out in [8–11].

## 2 Preliminaries

In this section, we present some basic definitions and facts in the geometries $\mathrm{PG}(k-1, q)$. Since in coding theory the letter $k$ denotes the dimension of the linear codes, we consider the associated minihypers as multisets in the $(k-1)$-dimensional projective geometry $\mathrm{PG}(k-1, q)$.

For a given $(n, w)$-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$, we denote by $\gamma_i(\mathcal{K})$ the maximal multiplicity of an $i$-dimensional flat in $\mathrm{PG}(k-1, q)$, i.e. $\gamma_i(\mathcal{K}) = \max_\delta \mathcal{K}(\delta)$, $i = 0, \ldots, k-1$, where $\delta$ runs over all $i$-dimensional flats in $\mathrm{PG}(k-1, q)$. If $\mathcal{K}$ is clear from the context we shall skip the name of the arc and shall write simply $\gamma_i$.

For an $(n, w)$-minihyper $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ we denote by $\beta_i(\mathcal{K})$ the minimal multipliciuty of an $i$-dimensional flat in $\mathrm{PG}(k-1, q)$: $\beta_i(\mathcal{K}) = \min_\delta \mathcal{B}(\delta)$, where $\delta$ runs over all $i$-dimensional flats.

Let $C$ be a Griesmer $[n, k, d]_q$-code. Denote by $\mathcal{K}$ the $(n, n-d)$-arc associated with $C$. Then using a similar argument as in the geometric proof of the Griesmer bound we get that

$$\gamma_i = \sum_{j=k-1-i}^{k-1} \lceil \frac{d}{q^j} \rceil. \quad (2)$$

If $\mathcal{B} = \gamma_0 - \mathcal{K}$ then $\beta_i = \gamma_0 v_{i+1} - \gamma_i$.

Below we give without proof several classical results for linear codes, and th related arcs and minihypers.

**Theorem 1** *(H. N. Ward, [12]).*

*(1) Let $C$ be an $[n, k, d]$ Griesmer code over $\mathbb{F}_p$, $p$ a prime. If $p^e$ divides $d$, then $p^e$ is a divisor of $C$.*
*(2) Let $\mathcal{K}$ be a Griesmer $(n, w)$-arc in $PG(k-1, p)$, $p$ a prime, and let $w \equiv n \pmod{p^e}$. Then for every hyperplane $\mathcal{K}(H) \equiv n \pmod{p^e}$, that is $p^e$ is a divisor of $\mathcal{K}$.*
*(3) Let $\mathcal{B}$ be a Griesmer $(n, w)$-minihyper in $PG(k-1, p)$, $p$ a prime, and let $w \equiv n \pmod{p^e}$. Then for every hyperplane $\mathcal{B}(H) \equiv n \pmod{p^e}$, that is $p^e$ is a divisor of $\mathcal{B}$.*

An $(n, w)$-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ is called extendable if there exists an $(n+1, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P$ of $\mathrm{PG}(k-1, q)$. Similarly, an $(n, w)$-miihyper $\mathcal{B}$ in $\mathrm{PG}(k-1, q)$ is called reducible if there exists an $(n-1, w)$-minihyper $\mathcal{B}'$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{B}'(P) \leq \mathcal{B}(P)$ for every point $P$ of $\mathrm{PG}(k-1, q)$. The next statement is the geometric version of Hill and Lizak's extension result [3,4]. Below we state the exetnsion theorem of Hill and Lizak in several formulations.

**Theorem 2** *(Hill-Lizak, [3,4]).*

*(1) Let $C$ be an $[n, k, d]_q$-code with $\gcd(d, q) = 1$ and with all weights congruent to $0$ or $d$ mod $q$. Then $C$ is extendeable to an $[n+1, k, d+1]_q$-code.*
*(2) Let $\mathcal{K}$ be an $(n, w)$-arc in $PG(k-1, q)$ with $(n-w, q) = 1$, such that the multiplicities of all hyperplanes are $n$ or $w$ modulo $q$. Then $\mathcal{K}$ is extendable to an $(n+1, w)$-arc.*
*(3) Let $\mathcal{B}$ be an $(n, w)$-minihyper in $PG(k-1, q)$ with $(n-w, q) = 1$, such that the multiplicities of all hyperplanes are $n$ or $w$ modulo $q$. Then $\mathcal{B}$ can be reduced to an $(n-1, w)$-minihyper.*

Further, we give a more elaborate extension (reducibility) condition found by Hitoshi Kanda [5] which applies only for codes over the field with three elements.

**Theorem 3** *(H. Kanda, [5]).*

*(1) Let $\mathcal{C}$ be an $[n, k, d]_3$ code with $(d, 3) = 1$ such that $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod 9$. Then $\mathcal{C}$ is doubly-extendable.*
*(2) Let $\mathcal{K}$ be an $(n, w)$-arc in $PG(k-1, 3)$. Assume that for every hyperplane $H$ $\mathcal{K}(H) \equiv n, n+1, n+2 \pmod 9$. Then $\mathcal{K}$ can be extended to an $(n+2, w)$-arc.*
*(3) Let $\mathcal{B}$ be an $(n, w)$-minihyper in $PG(k-1, 3)$. Assume that for every hyperplane $\mathcal{B}(H) \equiv n, n+1, n+2 \pmod 9$. Then $\mathcal{B}$ can be reduced to an $(n-2, w)$-minihyper.*

The following argument will be used several times in this paper. Let $\mathcal{B}$ be an $(n, w)$-minihyper in $\mathrm{PG}(k-1, q)$. Fix an $i$-dimensional flat $\delta$ in $\mathrm{PG}(k-1, q)$, with $\mathcal{B}(\delta) = t$. Let further $\pi$ be a $j$-dimensional flat in $\mathrm{PG}(k-1, q)$ of complementary dimension, i.e. $i + j = k - 2$ and $\delta \cap \pi = \varnothing$. Define the projection $\varphi = \varphi_{\delta,\pi}$ from $\delta$ onto $\pi$ by

$$\varphi\colon \begin{cases} \mathcal{P} \setminus \delta \to \pi \\ Q \qquad \to \pi \cap \langle \delta, Q \rangle. \end{cases} \tag{3}$$

This means that every point $Q$ of $\mathrm{PG}(k-1, q)$, which is not in $\delta$, has as an image the point which is the intersection of $\pi$ and the subspace generated by $\delta$ and $Q$. Here $\mathcal{P}$ denotes the pointset of $\mathrm{PG}(k-1, q)$. Note that $\varphi$ maps $(i + s)$-flats containing $\delta$ into $(s-1)$-flats in $\pi$. Given a set of points $F \subset \pi$, we define the induced minihyper $\mathcal{B}^{\varphi}$ by

$$\mathcal{B}^{\varphi}(F) = \sum_{\varphi_{\delta,\pi}(P) \in \mathcal{F}} \mathcal{B}(P).$$

We shall exploit the observation that if $F$ is an $f$-dimensional flat in $\pi$ then $\mathcal{B}^{\varphi}(F) \geq \beta_{f+i+1} - t$.

## 3 The Characterization of $(v_3 + 2v_2, v_2 + 2v_1)$-minihypers in $\mathrm{PG}(r, q)$, $q \geq 5$

Let $\mathcal{B}$ be a $(v_3 + 2v_2, v_2 + 2v_1)$-minihyper in $\mathrm{PG}(3, q)$, $q \geq 5$. Note that $(v_3 + 2v_2, v_2 + 2v_1) = (q^2 + 3q + 3q, q + 3)$. The restriction $\mathcal{B}|_H$ of $\mathcal{B}$ to a minimal plane is a $(v_2 + 2v_1, v_1) = (q + 3, 1)$-plane blocking set. For $q \geq 5$, blocking sets with these parameters are the sum of a line and two points.

Fix a minimal plane $\pi_0$ and a 1-line $L$ in it. Denote by $\pi_i$, $i = 0, \ldots, q$, all planes through $L$. The planes $\pi_i$ are all minimal since $1 + (q+1)(q+2) = q^2 + 3q + 3$. Consider a projection $\varphi$ from the 1-point on $L$. Set $L_i = \varphi(\pi_i)$. Clearly, the lines $L_i$ are of type $(0, q + \epsilon_0^{(i)}, \epsilon_1^{(i)}, \ldots, \epsilon_q^{(i)})$ with $\sum_j \epsilon_j^{(i)} = 2$.

Denote by $X$ the set of points in the projection plane of multiplicity $q + \epsilon$. First we shall assume that not all points of $X$ in the projection plane are collinear. Note that no four of these points are collinear. Otherwise, there exists a point $P$ and three lines $M_1, M_2, M_3$ through $P$ that do not contain points of multiplicity $q + \varepsilon$. Now we have

$$3 \cdot 2 + 2q \geq \sum \mathcal{B}^{\varphi}(M_i) \geq 3(q + 2),$$

a contradiction.

If at most three of the points from $X$ are collinear (call this line $M$) then the same argument gives that all the remaining points on $M$ are 2-points. Consider a 2-point $P$ on $M$ and call the two lines without points from $X$ by $M_1$ and $M_2$. Now for the lines $M_i$ we have

$$2 \cdot 2 + 2 \cdot 3 + (q - 3) \cdot 0 \geq \mathcal{B}^{\varphi}(M_1) + \mathcal{B}^{\varphi}(M_2) \geq 2(q + 2),$$

a contradiction.

If $q$ is odd, there exist three collinear points in $X$. Therefore all the points from $X$ are collinear. If $q \geq 8$ is even it is possible that the points in $X$ and the common point of the lines $L_i$ form a hyperoval. Then an external point to the hyperoval is incident with $q/2 \geq 4$ external lines and for the external lines through an external point we get again $2 \cdot q/2 + 2q \geq q/2(q+2)$, which is a contradiction. Hence in all cases the points from $X$ are collinear. This implies that $\mathcal{B}$ is the sum of a plane and a $(2v_2, 2v_1)$ which is known to be the sum of two lines (cf. [7]).

The above argument proves the following result.

**Theorem 4.** *Every* $(v_3 + 2v_2, v_2 + 2v_1)$*-minihyper in* $PG(3,q)$*,* $q \geq 5$ *is the sum of a plane and two lines.*

The same theorem is true for minihypers with the same parameters in geometries of larger dimension.

**Theorem 5.** *Every* $(v_3 + 2v_2, v_2 + 2v_1)$*-minihyper in* $PG(r,q)$*,* $r \geq 3$*,* $q \geq 5$ *is the sum of a plane and two lines.*

*Proof.* We shall prove this result by induction on $r$. The first step in the induction is provided by Theorem 4.

Denote by $\mathcal{B}$ a minihyper in with parameters $(v_3 + 2v_2, v_2 + 2v_1)$ in $\mathrm{PG}(r,q)$. Since the projection from a 0-point is again a minihyper with the same parameters in $\mathrm{PG}(r-1,q)$, and since such a minihyper is the sum of a plane and two lines we have that the admissible hyperplane multiplicities for $\mathcal{B}$ are contained in the set

$$\{3, q+3, 2q+3, 3q+3, q^2+q+3, q^2+2q+3, q^2+3q+3\}. \tag{4}$$

Consider a hyperline $S$ in $\mathrm{PG}(r,q)$ of multiplicity 1. Let the point of multiplicity 1 in $S$ be denoted by $P$. Denote by $H_i$ the hyperplanes through $S$. The restriction of $\mathcal{B}$ to each $H_i$ is the sum of a line $L_i$ and two further points (which also might lie on $L_i$).

First let us assume that the lines $L_i$ are not coplanar. In such case there exists a hyperline $T$ in $\mathrm{PG}(r,q)$ which is not blocked by the set $\cup L_i$. Denote by $F_i$ the hyperpalnes through $T$. One of them meets $\cup L_i$ in one point and the remaining $q$ meet $\cup L_i$ in $q+1$ points. Since each hyperplane is to be blocked $q+3$ times we need $(q+2) + q \cdot 2 = 3q+2$ additional points. But outside $\cup L_i$ we have just $2q+2$ points which is a contradidiction. Hence the lines $L_i$ are coplanar and $\cup L_i$ is a plane.

Since the admissible multiplicities of hyperplanes are in the list (4), we have that $\mathcal{B}$ is the sum of a line and a $(2v_2, 2v_1)$-minihyper. This proves the result since the latter is the sum of two planes.

For $(v_3+2v_2, v_2+2v_1)$-minihypers in $\mathrm{PG}(3,q)$, where $q = 3$ or 4, the situation is more complicated. It is studied in the next two sections.

## 4 $(21, 6)$-minihypers in $\mathrm{PG}(3, 3)$

Each line in $\mathrm{PG}(3, 3)$ has to blocked at least once. This implies that the restriction of a $(21, 6)$-minihyper to a plane is a line plus two points, or a projective triangle (a quadrangle plus two of its diagonal points). Moreover the maximal point multiplicity is 3.

Let $\mathcal{B}$ be a $(21, 6)$-minihyper (w.r.t. hyperplanes) in $\mathrm{PG}(3, 3)$. By Theorem 1 the possible multiplicity of a hyperplane is 0 (mod 3), i.e. these belong to $\{6, 9, 12, 15, 18, 21\}$. Moreover, a hyperplane of multilicity $\geq 15$ does not have 0-points. Otherwise, an easy counting gives $|\mathcal{B}| \geq 15 + 9 \cdot 1 = 24 > 21$ (since each line through the 0-point outside the plane should be blocked). Hence if $\mathcal{B}$ has a hyperplane of multiplicity at least 15 then it is the sum of a plane and a $(8, 2)$-minihyper which in turn is the sum of two lines [1,7].

It remains to consider $(21, 6)$-minihypers with hyperlanes of multiplicity $6, 9, 12$. Note that 9- and 12-planes cannot have points of multiplicity 3. Consequently, $\mathcal{B}$ also does not have a point of multiplicity 3. For the spectrum of $\mathcal{B}$ we have

$$\begin{aligned} a_6 + \quad a_9 + \quad a_{12} &= \quad 40 \\ 6a_6 + \quad 9a_9 + 12a_{12} &= \quad 273 \\ 15a_6 + 36a_9 + 66a_{12} &= 840 + 9\lambda_2 \end{aligned}$$

whence

$$a_6 = 30 + \lambda_2,\ a_9 = 9 - 2\lambda_2,\ a_{12} = 1 + \lambda_2.$$

Note that there cannot be a point-plane pair $(P, \pi)$ with $P$ a 2-point, $\pi$ – a 12-plane, and $P \notin \pi$. Indeed, all lines through a 0-point in $\pi$ must be blocked exacly once, while one of them is blocked twice (because of the 2-point). Hence all 2-ponts are in the intersection of all 12-planes. This implies that $\lambda_2 \leq 3$. The cases $\lambda_2 = 3$ or 2 are ruled out by an easy countnig of the multiplicities of the four (resp. three) 12-planes through the common line containing the 2-points. In the case $\lambda_2 = 1$, an easy counting gives that the two 12-planes meet in a 5-line, and that the other two planes through this 5-line are 6-planes.

Let us first consider the case $\lambda_2 = 1$. The two 12-planes have four 0-points in common: two in each of the 12-planes. The two 0-points in each of the 12-planes are collinear with the 2 point. Moreover the plane defined by the four 0-points in the 12-planes is a 6-plane consisting of four 1-points and one 2-point; the 1-points in this plane are collinear. This determines $\mathcal{B}$ uniquely.

Indeed, select five points $P_1, P_2, P_3, P_4, Q$ in general position. The point $R = P_1P_2 \cap P_3P_4$ is the 2 -point; the line $QR$ is the 5-line which is the common line of the 12-planes $\pi_0 = \langle P_1, P_2, Q\rangle$, $\pi_1 = \langle P_3, P_4, Q\rangle$. Let the other two planes through $QR$ are $\pi_2$ and $\pi_3$. They are 6-planes. The 1-points off $QR$ in $\pi_2$ and $\pi_3$ define the 4-line in the plane $\langle Q, P_1, P_2\rangle$.

It remains to consider the case when $\lambda_2 = 0, a_{12} = 1$. Denote the unique 12-plane by $\pi_0$, and by $P$ the 0-point in $\pi_0$. There are nine 1-points outside $\pi_0$. Assume these nine points and the point $P$ form a cap. Then a tangent plane to this cap in a point different from $P$ must be a 5-plane, which is impossible.

Hence there exist three collinear 1-points outside $\pi_0$ , $P_1, P_2, P_3$ say. Denote the fourth point on this line by $Q_1$; clearly $Q_1 \in \pi_0$.

Set $\pi_1 = \langle P, P_1, Q_1 \rangle$, and let $\pi_2$ and $\pi_3$ be the other other planes through $PQ_1$. Denote the other two points on $PQ_1$ by $Q_2$ and $Q_3$. It is easily checked that the remining six points of $\mathcal{B}$ are on two lines in $\pi_2$ and $\pi_3$, respectively, that meet $\pi_0$ in $Q_2$ and $Q_3$, respectively.

It is easily checked that this configuration is unique. We can sum up these observation in the following theorem.

**Theorem 6.** *A* $(21, 6)$*-minihyper in* $PG(3, 3)$ *is one of the following:*

*(1) the sum of a plane and two lines.*
*(2) a minihyper with* $\lambda_2 = 1$*,* $a_{12} = 2$*;*
*(3) a minihyper with* $\lambda_2 = 0$*,* $a_{12} = 1$*.*

**Corollary 7.** *There exist five* $(21, 6)$*-minihypers (up to isomorphism).*

The five minihypers of the three possible types described in Theorem 6 are presented graphically on the pictures below. The doublecircled nodes represent 2-points (Figs. 1, 2 and 3).

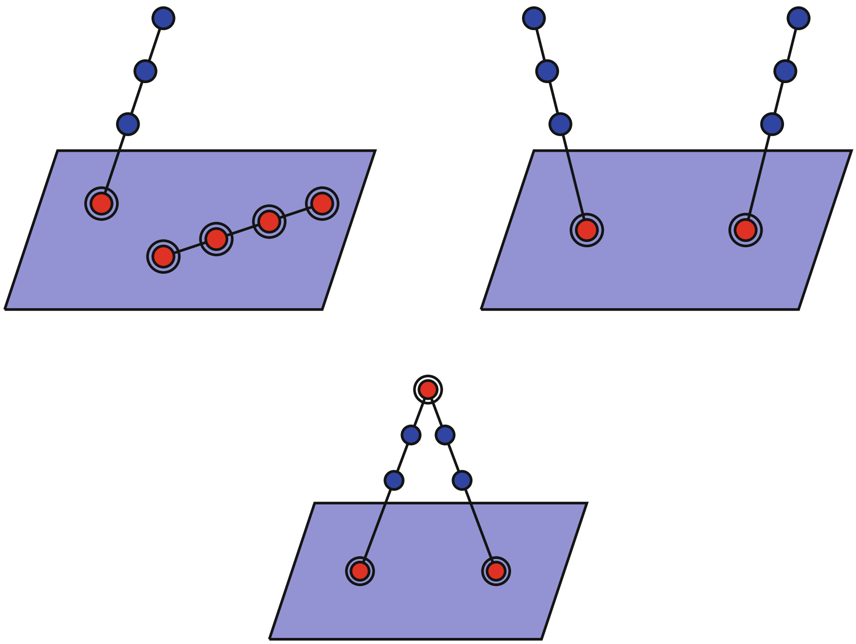

**Fig. 1.** $(21, 6)$-minihypers in PG$(3, 3)$ of type (1)

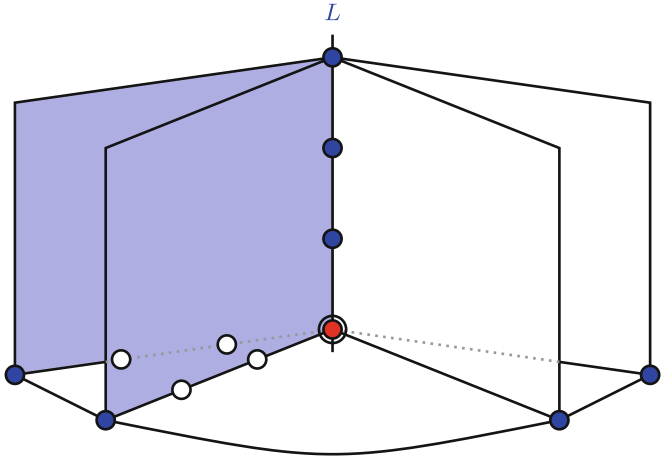


**Fig. 2.** $(21, 6)$-minihypers in $PG(3, 3)$ of type (2)

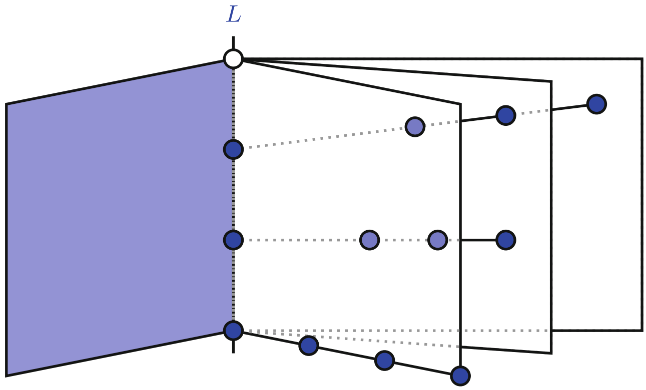


**Fig. 3.** $(21, 6)$-minihypers in $PG(3, 3)$ of type (3)

## 5 $(31, 7)$-minihypers in $PG(3, 4)$

In this section we characterize the $(v_3 + 2v_2, v_2 + 2v_1)$-minihypers in $PG(3, 4)$.

**Theorem 8.** *A* $(31, 7)$*-minihyper in* $PG(3, 4)$ *is one of the following:*

*(1) the sum of a plane and two lines.*
*(2) a minihyper with* $\lambda_3 = 1$*; it is a cone with vertex the 3-point and base curve – a Baer subplane.*

*Proof.* Let $\mathcal{B}$ be a $(31, 7)$-minihyper in $\mathrm{PG}(3, 4)$. The restriction of $\mathcal{B}$ to a plane is either the sum of a line and two points or a Baer subplane.

By a result of Ward [12] which is a refinement of Theorem 1 all planes have odd multiplicity. Since each line has to be blocked at least once, a plane with 0-points has at most 15 points. This rules out planes of multiplicity 17 and 19.

Furthermore 21-planes are impossible since an affine blocking set has at least 13 points which gives $|\mathcal{B}| \geq 21 + 13 = 34$, a contradiction. The existence of a plane with at least 23 points gives a minihyper of the type described in (1).

Hence we have to characterize just the minihypers with planes of multiplicity 7, 11, 15. For the spectrum of $\mathcal{B}$ we have

$$\begin{aligned} a_7 + \quad a_{11} + \quad a_{15} &= 85 \\ 7a_7 + 11a_{11} + \ 15a_{15} &= 651 \\ 21a_7 + 55a_{11} + 105a_{15} &= 2325 + 16\lambda_2 + 48\lambda_3 \end{aligned}$$

whence

$$a_7 = 75 + \lambda_2 + 3\lambda_3, \ a_{11} = 6 - 2\lambda_2 - 6\lambda_3, \ a_{15} = 4 + \lambda_2 + 3\lambda_3.$$

A 15-plane is (i) the sum of three concurrent lines; (ii) the sum of three non-concurrent lines (iii) the complement of an oval.

In case (iii) the planes through a 3 line must be all 7-planes and hence the minihyper should be projective, i.e. $\lambda_2 = \lambda_3 = 0$. On the other side through a 5-line there is one further 15-plane whence $a_{15} = 7$, a contradiction.

In case (ii) we get easily $\lambda_2 = 3$ since all planes through a 3-line without 0-points should be 7-planes without 0-points. On the other hand a 7-line is incident with two further 15-planes, and each of them has one additional 2-point, whence $\lambda_2 \geq 5$, a contradiction.

In case (i), it is easily seen that $\lambda_3 = 1, \lambda_2 = 0$. Hence $a_7 = 78, a_{11} = 0, a_{15} = 7$. In addition all 7-planes should be Baer planes. This implies that $\mathcal{B}$ is a cone with vertex the 3-point and base curve – a Baer subplane.

## 6 (22, 6)-minihypers in PG(3, 3)

The goal here is to prove that every $(22, 6)$-minihyper in $\mathrm{PG}(3, 3)$ is reducible to a $(21, 6)$-minihyper.

Denote by $\mathcal{B}$ a minihyper in $\mathrm{PG}(3, 3)$ with parameters $(22, 6)$. Counting arguments give that for every plane $\pi$, it holds $\mathcal{B}(\pi) \in \{6, 7, \ldots, 22\}$. Morover we have the following lemma.

**Lemma 9.** *Let $\mathcal{B}$ be a $(22, 6)$-minihyper in $PG(3, 3)$. If $\pi$ is a hyperplane of multiplicity $\mathcal{B}(\pi) = 22 - 3i - j$, $i \in \{0, \ldots, 5\}$, $j \in \{0, 1, 2\}$ then $\mathcal{B}|_\pi$ is a $(22 - 3i - j, 6 - i)$ minihyper*

**Corollary 10.** *A $(22, 6)$-minihyper in $PG(3, 3)$ does not have 11-plane. An 8-plane in such a minihyper is the sum of two lines.*

As in the previous subsection, a 14-plane does not have 0-points. If $\mathcal{B}(\pi) \geq 15$ for some plane $\pi$ then it is the sum of a plane and a $(9,2)$-minihyper in $\mathrm{PG}(3,3)$. The latter is the sum of two lines and point [1]. This implies that in this case $\mathcal{B}$ is reducible.

Next we tackle the case when $a_{14} > 0$.

**Lemma 11.** *There exists no* $(22,6)$*-minihyper in* $PG(3,3)$ *with a 14-plane.*

*Proof.* Since a 14-plane does not have 0-points, it has one 2-point and twelve 1-points, and spectrum $a'_4 = 9$, $a'_5 = 4$. Morover all planes different from the 14-plane have multiplicity at most 10. We have the following equations for the spectrum of $\mathcal{B}$:

$$\sum a'_i = 40 \tag{5}$$

$$\sum i a'_i = 286 \tag{6}$$

$$\sum \binom{i}{2} a'_i = 924 + 9\lambda'_2 + 27\lambda'_3 \tag{7}$$

which, taking into account that $a'_{14} = 1$ implies

$$a_8 + 3a_9 + 6a_{10} = 20 + 9\lambda'_2 + 27\lambda'_3. \tag{8}$$

Now we count the contributions to the LHS of (8) of the planes through the different lines in the 14-plane. For a 4-line this contribution is 0 since a 8-plane does not have a 4-line (it is the sum of two lines and has just 2-, 5- or 8-lines). For a 5-line this contribution is at most 6. Hence (8) implies

$$20 + 9\lambda'_2 + 27\lambda'_3 \leq 9 \cdot 0 + 4 \cdot 6 = 24.$$

This implies $\lambda'_2 = \lambda'_3 = 0$, which is a contradiction because we have at least one 2-point (the one in the 14-plane).

Finally, we are going to rule out the existence of 8-planes in $\mathcal{B}$. Then $\mathcal{B}$ will be reducible by the lemma of Hill and Lizak. Let us note that if $\pi$ is an 8-plane then $\mathcal{B}|_\pi$ is the sum of two lines and has one of the two spectra:

$$(A)\ a'_8 = 1, a'_2 = 12, \lambda'_2 = 4; \quad (B)\ a'_5 = 2, a'_2 = 11, \lambda'_2 = 1.$$

Again from (5), we have

$$a_8 + 3a_9 + 6a_{10} + 15a'_{12} + 21a'_{13} = 48 + 9\lambda'_2 + 27\lambda'_3. \tag{9}$$

**Lemma 12.** *There exists no* $(22,6)$*-minihyper in* $PG(3,3)$ *with a 8-plane.*

*Proof.* 1) Assume there exists an 8-plane with spectrum $(A)$, and let us count the contribution of the planes through the lines in the 8-plane to the left-hand side of (9). The maximal contribution through the 8-line is 57, while the comaximal contribution through a 2-line is 1. Hence we have (since $\lambda_2' \geq 4$)

$$70 = 1 + 1 \cdot 57 + 12 \cdot 1 \geq 48 + 9\lambda_2' + 27\lambda_3' \geq 48 + 9 \cdot 4 = 84,$$

a contradiction.

2) The same argument for an 8-plane with spectrum $(B)$ gives

$$66 = 1 + 2 \cdot 27 + 11 \cdot 1 \geq 48 + 9\lambda_2' + 27\lambda_3', \tag{10}$$

whence $\lambda_2' = 1$ or 2, $\lambda_3' = 0$.

Assume $\lambda_2' = 2$. Then in (10) we have equality and the spectrum of $\mathcal{B}$ is

$$a_{13}' = 2, a_{10}' = 2, a_8' = 12, a_6' = 24.$$

Since the largest cap in PG$(3,3)$ has 10 points, there exist three 8-planes sharing a common line which is forced to be a 2-line. This is clearly impossible since then we have:

$$|\mathcal{B}| \geq 3 \cdot 8 + 6 - 3 \cdot 2 = 24 > 22.$$

Now assume $\lambda_2' = 1$. We get from (10) that $a_{13}' \geq 1$. But $a_{13}' \geq 2$ is impossible since two 13-planes should meet in a line of multiplicity at least 6. This is a contradiction since we have just one 2-point. Hence $a_{13}' = 1$. But there must be a plane of multiplicity (otherwise, we have again a contradiction to (10)).

Now denote by $\pi_o$ the 13-plane, by $\pi_1$ – the 12-plane. These two should meet in a 5-line which contains the 2-point $P$. The two remaining planes through $L$, $\pi_2$ and $\pi_3$ say, are 6-planes. Denote by $R_1$ (resp. $R_2$) the 1-point in $\pi_2$ (resp. $\pi_3$) outside $L$. Furthermore, let $Q_0$ be the 0-point in $\pi_0$, and $Q_1$, $Q_2$ – the two 0-points in $\pi_1$. As before $Q_1Q_2$ is incident with $P$ and so the plane $\langle Q_0, Q_1, Q_2 \rangle$ is incident with the point $P$.

Now assume there exists a 10-plane, $\delta$ say. This plane should meet $\pi_0$ and $\pi_1$ in 5-lines (through $P$), and it should contain also $R_1$ and $R_2$. The line $Q_0Q_1$ contains one of the points $R_1$, $R_2$, $R_1$ say, since it should be blocked at least once. By a similar argument, $Q_0Q_2$ contains $R_2$. But now we have that $\langle Q_0, Q_1, Q_2 \rangle \cap \delta$ contains $P$, $R_1$, and $R_2$, whence $\langle Q_0, Q_1, Q_2 \rangle \equiv \delta$, which is clearly impossible. So far, we have shown that $\mathcal{B}$ contains no 10-plane. It remains to use (10) once more to get

$$57 + 48 + 9\lambda_2' \leq 1 + 1 \cdot 24 + 1 \cdot 19 + 11 \cdot 1 = 55,$$

a contradiction.

**Theorem 13.** *A* $(22,6)$*-minihyper in* $PG(3,3)$ *is reducible.*

Using similar arguments it can proved that the same result holds for arbitrary 3-dimensional geometries.

**Theorem 14.** *Every* $(v_3 + 2v_2 + v_1, v_2 + 2v_1)$*-minihyper in* $PG(3, q)$*,* $q \geq 3$*, is reducible.*

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## References

1. Hamada, N.: A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry. Discrete Math. **116**, 229–268 (1993)
2. Hill, R.: Optimal linear codes. In: Mitchell, C. (ed.) Cryptography and Coding II. Oxford Univ, pp. 75–104. Press, Oxford (1992)
3. Hill, R.: An extension theorem for linear codes. Des. Codes Cryptogr. **17**, 151–157.11111 (1999)
4. Hill, R., Lizak, P.: Extensions of linear codes. In: Proceedings IEEE International Symposium on Information Theory, Whistler, Canada, p. 345 (1995)
5. Kanda, H.: A new extension theorem for ternary linear codes and its application. Finite Fields Appl. **67**, 1017111 (2020). https://doi.org/10.1016/j.ffa.2020.101711
6. Landjev, I., Rogachev, E., Rousseva, A.: On ternary codes and blocking sets in $\mathrm{PG}(\mathrm{r}, \mathrm{q})$, manuscript
7. Landjev, I., Vandendriesche, P.: A study of $(xv_t, xv_{t-1})$-minihypers in $\mathrm{PG}(\mathrm{t}, \mathrm{q})$. J. Combin. Theory Ser. A **119**, 1123–1131 (2012)
8. Maruta, T., Oya, Y.: On optimal ternary linear codes of dimension 6. Adv. Math. Comm. **5**(3), 505–520 (2011)
9. Sawashima, T., Maruta, T.: Nonexistence of some ternary linear codes with minimum weight$-2$ modulo 9, Adv. Math. Comm. (2022). (to appear)
10. Takenaka, M., Okamoto, K., Maruta, T.: On optimalnon-projective termary linear codes. Discrete Math. **308**, 842–854 (2008)
11. Yoshida, Y., Maruta, T.: Ternary linear codes and quadrics. Electronic J. Combin. **16**, #R9 (2009)
12. Ward, H.N.: Divisibility of codes meeting the Griesmer bound. J. Comb. Theory Ser. A **83**(1), 79–93 (1998)