On the Treewidth of Planar Minor Free Graphs

Youssou Dieng[1(✉)] and Cyril Gavoille[2]

[1] LI3, Assane Seck University of Ziguinchor, Ziguinchor, Senegal
ydieng@univ-zig.sn
[2] LaBRI, University of Bordeaux, Bordeaux, France
gavoille@labri.fr
https://sites.google.com/a/univ-zig.sn/ydieng/

Abstract. We study in this article, the treewidth of planar graphs excluding as minor a fixed planar graph. We prove that the treewidth of every planar graph excluding a graph having a poly-line $p \times q$-grid drawing is $O(p\sqrt{q})$. As consequences, the treewidth of planar graphs excluding as minor the cylinder $\mathscr{C}_{2,r}$ or its dual $\mathscr{C}^*_{2,r}$ is $O(\sqrt{r})$, where $\mathscr{C}_{2,r}$ denotes the cylinder of height 2 and circumference r. This bound is asymptotically optimal. The treewidth is $O(\sqrt{r \log r})$ if the excluded graph is any outerplanar graph with r vertices.

Keywords: Planar graph · Graph Minor · Treewidth · Graph drawing

1 Introduction

Tree-decomposition is one of the most general and effective technique for designing efficient graph algorithms. Roughly speaking a tree-decomposition of an input graph G is a collection of subgraphs of G, called *bags*, that cover G in a tree-like manner (see Fig. 1 and Sect. 2 for precise definitions). *Treewidth-k* graphs are graphs having a tree-decomposition of *width* k, i.e., into bags of at most $k+1$ vertices. It has been shown (see for instance [Arn85, AP89, Bod96, Cou90]) that many optimization problems on graphs, including NP-hard ones, can be solved by the use of dynamic programming techniques based on tree-decompositions and whose efficiency is directly related to the size of the bags. So, identifying graphs of small treewidth is of great interests.

The problem to decide whether the treewidth of a graph is k is NP-complete [ACP87], but there are linear time algorithms for each fixed k. The best polynomial time approximation algorithm achieves $O(\sqrt{\log k})$ performance ratio where k is the treewidth [FHL08]. For planar graphs, approximation algorithms with performance ratio 1.5 do exist [ST94, GT05].

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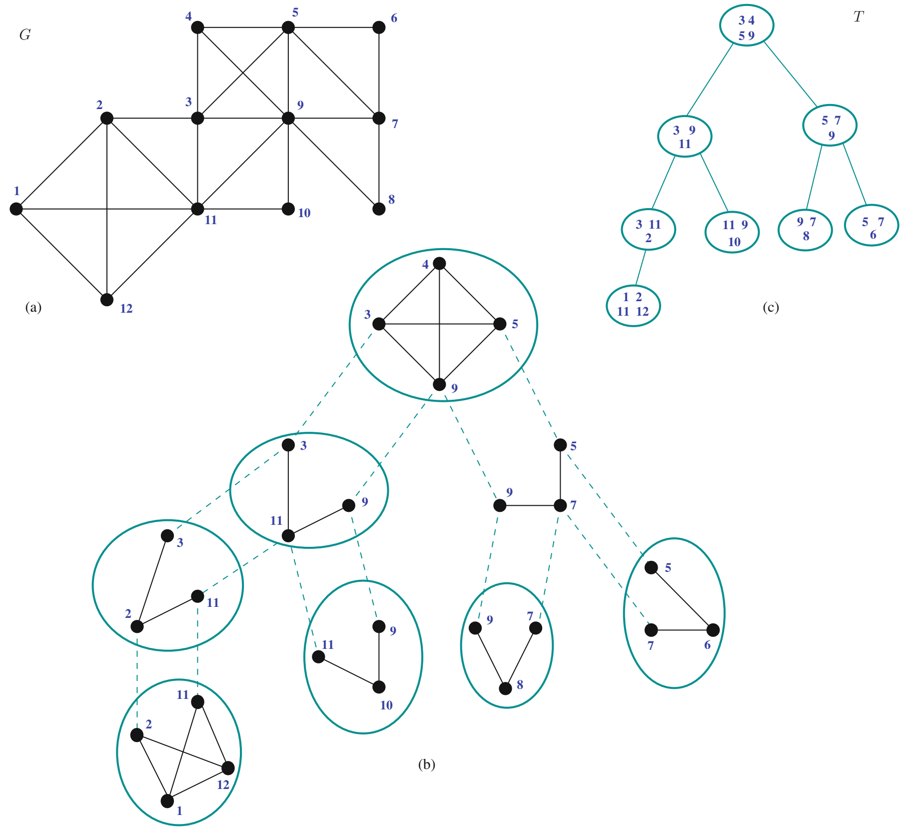

Fig. 1. (a) A graph G; (b) a tree-like representation of G; and (c) a tree-decomposition T of G of width 3.

A seminal work about tree-decompositions is the one of Robertson and Seymour and their Graph Minor Theorem that has been proved in a more than twenty paper serie spanning over 20 years, from [RS83, RS86, RS03, RS04]. Along their proof, they gave a decomposition theorem [RS03] capturing the structure of graphs excluding a fixed *minor* (see Sect. 2 for precise definitions). Informally, the theorem says that every graph excluding some fixed graph H as minor has a tree-decomposition into bags that can "almost" be embedded on a surface on which H cannot be embedded.

A keystone in the proof of the Graph Minor Theorem is the grid-minor theorem [RS86] which says that every graph of treewidth large enough is guaranteed to have a large grid as minor. In other words, if a graph G excludes a $r \times r$ grid as minor, then the treewidth of G is at most $g(r)$ for some function g. The current best upper bound is $g(r) \leq r^{9+o(1)}$ [CT19], whereas the best lower bound is $g(r) \geq r^{2+o(1)}$ as proved in [RST94]. In this latter paper, it has been conjectured that the best possible bound is in fact $r^{2+o(1)}$, whereas [DHK09] have conjectured $\Theta(r^3)$.

This grid-minor theorem played also a key role for the important Disjoint Paths Problem [KW10], in several other deep meta-theorems and applications as mentioned in [Gro07]. For some applications, efficient solutions can be still computed for graphs having specific tree-decompositions, and not only those of small width. For instance, [DG03,DG07] have studied graphs having a tree-decomposition whose bags contain vertices, possibly many, that are close to each other. Graphs of bounded *tree-length*, i.e., having a tree-decomposition into bags of bounded diameter, admit additive *spanners*, *compact routing schemes*, and *distance labeling schemes* with short labels [BvLTT97,GKK+01,DG04,Dou05, DDGY07,CDE+12], that are important applications for Distributed Computing. Note that many graphs with unbounded treewidth have bounded tree-length. Chordal graphs, interval graphs, split graphs, AT-free graphs, permutation graphs, and many others, are such examples. Further developments on tree-length can be founded in [CDE+08,UY09,Lok10].

In this paper, we study the question about the treewidth of graphs excluding as minor a planar graph. The paper is organized as follows. We start in Sect. 2 with a formal description of minors, tree-decompositions and graph drawings. Section 3 presents our contribution with an overview of the main ingredients for our results. Section 4 gives the proof of the main theorem.

2 Minor, Tree-Decomposition and Drawing

Let G be a simple connected undirected graph with vertex-set $V(G)$ and edge-set $E(G)$. An edge between two vertices u and v of G is denoted by $\{u, v\}$. The *contraction* of $\{u, v\}$ in G is the result of identifying the vertices u and v and removing from G all resulting loops and multiple edges. A *minor* H of G is a subgraph of a graph that can be obtained from G by a sequence of edge contractions (see Fig. 2).

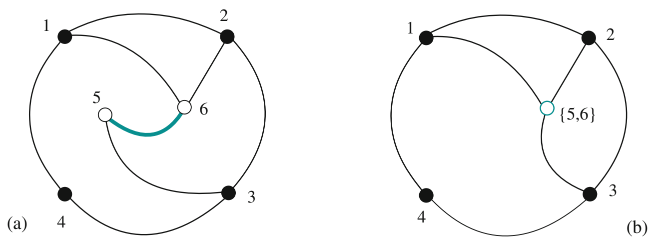

Fig. 2. (a) A graph G; and (b) a minor of G obtained by contracting edge $\{5, 6\}$.

We say that G *excludes* H if H is not a minor of G. By transitivity of the minor relation, if G excludes H, then G excludes every graph having H as minor.

We now present the basic notions of *tree-decomposition* and *treewidth*. See Fig. 1 for an illustration.

Definition 1. *A* tree-decomposition *of a graph G is a tree T whose nodes, called* bags*, are subsets of $V(G)$, and such that:*

1. $\bigcup_{X \in V(T)} X = V(G)$*;*
2. $\forall \{u,v\} \in E(G)$*,* $\exists X \in V(T)$ *such that* $u, v \in X$*; and*
3. $\forall u \in V(G)$*, the set of bags containing u induces a subtree of T.*

The width *of a tree-decomposition T is* $\max_{X \in V(T)}\{|X| - 1\}$*. The* treewidth *of a graph G is the minimum width over all possible tree-decompositions of G.*

The notions of *path-decomposition* and *pathwidth* are defined similarly, except that in Definition 1, T must be a path. It is not difficult to see that if H is a minor of G, then the treewidth (resp. pathwidth) of H is no more than the treewidth (reps. pathwidth) of G.

Our results rely on plane embeddings of graphs. More precisely, a *drawing* of a graph G maps each vertex of G to a point of the plane and each edge to a simple open Jordan curve between its endpoints. A drawing divides the plane into topologically connected regions, called *faces*; the infinite region is called the *outerface*. A *planar graph* is a graph that can be drawn in the plane without crossing edges. The *dual* of a graph drawn G is the graph denoted by G^* whose the vertices are faces of G, and the edges connect faces having a common edge of G on their borders.

In this paper, we consider *grid-drawings* where vertices of the graph have integer coordinates and edges between adjacent vertices are poly-line whose bends have integer coordinates too. Such drawings were developed by de Fraysseix et al. in [dFPP88,dFPP90] and Schnyder in [Sch90].

More precisely, a graph has a *poly-line* $p \times q$-grid drawing if it has a drawing such that vertices are plotted at the vertices of the $p \times q$ grid, and edges are contiguous sequences of segments, each segment being a straight-line between two vertices of the $p \times q$ grid. The grid-drawing is *orthogonal* if edges can be drawn as path of the grid, i.e., represented as sequences of horizontal or vertical segments only. The drawing is *flat* if every vertex is represented by a horizontal line segment. Finally, the drawing is *straight-line* if each edge consists of one segment only. The $p \times q$ *grid* is the graph whose vertex-set $\{(i,j) : i \in [0,p), j \in [0,q)\}$ and for which two vertices (i,j) and (i',j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. The $p \times q$-cylinder, denoted by $\mathscr{C}_{p,q}$, is the graph with same vertex-set of the $p \times q$-grid and such that (i,j) and (i',j') are adjacent if and only if there are adjacent in the $p \times q$-grid or $i = i'$ and $|j - j'| = q - 1$. In other words, $\mathscr{C}_{p,q}$ is the Cartesian product of a path of p vertices by a cycle with q vertices. We refer to [dBETT99] for a wide overview of grid-drawings, and Fig. 3 for illustrations.

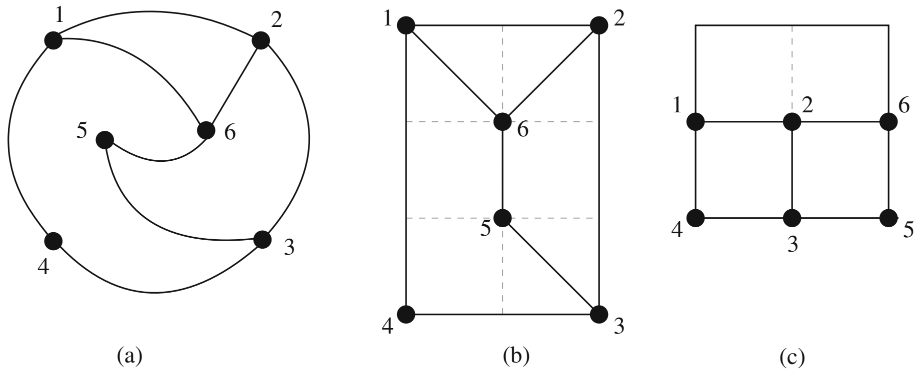

Fig. 3. (a) A planar graph G; (b) a 4×3-grid straight-line drawing of G; and (c) a poly-line orthogonal 3×3-grid drawing of G.

3 Our Contributions

Bounding the treewidth of a graph by a function of a minor it excludes is one of the most surprising property of the Graph Minor Theory. As previously discussed, if a graph G excludes some planar graph H, then the treewidth of G is at most some constant depending on H. The best current upper bound on the treewidth is $r^{9+o(1)}$ where $r = |V(H)|$. In fact, the bound holds even if H is the $r \times r$ grid. The treewidth bound can be reduced to $O(r)$ if the excluded minor is a $K_{2,r}$ [BvLTT97], a forest [BRST91] or a cycle [FL89] of r vertices. It is also known that the treewidth of graphs excluding a 3×3 grid (resp. a 4×4 grid) is at most respectively 7 [BBR09] (resp. 7 262 [BBR07]).

We investigate the question of the treewidth of a graph G that excludes a planar graph H whenever G is itself planar. It is known [RST94] that the treewidth of a planar graph excluding an r-vertex planar graph, or an $r \times r$ grid, is only $O(r)$. The most accurate bound on the term $O(r)$ can be derived from [GT12] and [RST94], as explained later in the Lemma 4.

Our main result is (see Sect. 4 for the proof):

Theorem 1. *The treewidth of every planar graph excluding as minor a graph having a poly-line $p \times q$-grid drawing is $O(p\sqrt{q})$.*

Because $\mathscr{C}_{2,r}$ and its dual $\mathscr{C}^*_{2,r}$ have $4 \times r$-grid drawings (see Fig. 4), and that $K_{2,r}$ is a minor of $\mathscr{C}^*_{2,r}$, we derive directly from Theorem 1 that:

Corollary 1. *Let $H \in \{\mathscr{C}_{2,r}, \mathscr{C}^*_{2,r}, K_{2,r}\}$. The treewidth of every planar graph excluding H as minor is $O(\sqrt{r})$.*

This bound significantly improves upon the $r+2$ upper bound of [Thi99]. As we will see later, the bound of $O(\sqrt{r})$ is actually asymptotically optimal.

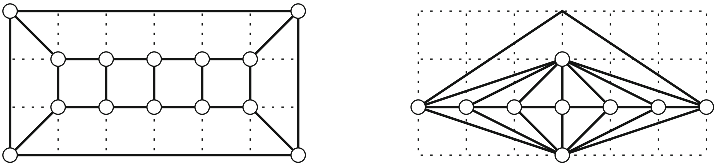

Fig. 4. A straight-line $4 \times r$-grid drawing of the $2 \times r$-cylinder $\mathscr{C}_{2,r}$ and a poly-line drawing of its dual $\mathscr{C}^*_{2,r}$, here for $r = 7$.

We now derive a similar bound if the excluded graph is outerplanar. Theorem 1 allows us the plug results from literature of Graph Drawing Theory. For instance, using the result of [Bie14][Th. 9], every outerplanar graph H with r vertices and pathwidth k has straight-line $O(k) \times O(r)$-grid drawing. From Theorem 1, the treewidth of every planar graph excluding H is $O(k\sqrt{r})$.

However, this latter bound can be slightly improved as follows:

Proposition 1. *Every planar graph excluding as minor an outerplanar graph with r vertices and pathwidth k has treewidth $O(\sqrt{kr})$, which is at most $O(\sqrt{r}\log r)$.*

Proof. Let H be the excluded minor. We observe that the straight-line $O(k) \times O(r)$-grid drawing of the minor H as built in [Bie14][Th. 9] is actually based on a flat orthogonal drawing due to [Bie12][Th. 1]. The property of this drawing is that, if H is 2-connected, then it has a flat orthogonal $(4k-3) \times \lceil 3(r-2)/2 \rceil$-grid drawing.

The connectivity condition on H can be overcome, because as proved in [BBCR14], any outerplanar graph can be made 2-connected while increasing its pathwidth by a constant factor. We also remark that if H has a flat orthogonal $p \times q$-grid drawing, then H is a minor of the $p \times q$-grid (simply contract the horizontal segments representing the vertices). By Lemma 1 H is also a minor of the $O(\sqrt{pq}) \times O(\sqrt{pq})$-grid. And, by Lemma 4(ii), the treewidth of a planar graph excluding such a grid, and thus H, is $O(\sqrt{pq})$. Overall, plugging $p = O(k)$ and $q = O(r)$, we get that the treewidth of a planar graph excluding H as minor is $O(\sqrt{kr})$.

We concluding by noting that $k = O(\log r)$, because the treewidth of H is at most two, and the pathwidth of any graph with r vertices is at most its treewidth plus one times $O(\log r)$ [KS93][Theorem 6]. □

The end of this section is devoted to a discussion about the optimality of the treewidth bounds we have obtained.

If H is a general planar graph, then the $O(r)$ bound of [RST94] is optimal (asymptotically). This is because there are planar graphs H with $O(r)$ vertices that are not minor of the $r \times r$ grid. In other words, if G is the $r \times r$ grid, then G is planar and has treewidth[1] r, whereas it excludes H, an $O(r)$-vertex planar

[1] It is well-known the $p \times q$ grid has treewidth $\min\{p, q\}$.

graph. For concreteness, consider $H = \mathscr{C}_{3,r+2}$, the $3 \times (r+2)$-cylinder. It is easy to show that: (1) any drawing of H contains at least $\delta \geq r/2+1$ disjoint nested cycles[2]; and (2) H cannot be a minor of a graph having a drawing with less than δ disjoint nested cycles, since edge contraction and taking subgraph cannot increase the number of disjoint nested cycles. Unfortunately, the $r \times r$ grid has only $r/2 < \delta$ disjoint nested cycles. So, H that has $3r+6 = O(r)$ vertices is not a minor of a planar graph G of treewidth r (the $r \times r$ grid).

However, as demonstrated by Corollary 1, the $O(r)$ bound can be reduced if we restrict furthermore the family of excluded minors. To formalize this idea, let us consider an infinite graph family $\mathscr{H}$ closed under taking subgraphs. And, let $\theta_{\mathscr{H}}(r)$ be the function defined as the smallest t such that every planar graph excluding any graph $H \in \mathscr{H}$ with at most r vertices has treewidth at most t. The main question we have addressed in this paper is to find a large family $\mathscr{H}$ of planar graphs such that $\theta_{\mathscr{H}}(r) = o(r)$.

Observe that Lemma 4(ii) implies that $\theta_{\mathscr{H}}(r) \leq 9r$ for each family $\mathscr{H}$ of planar graphs. Function $\theta_{\mathscr{H}}$ must be linear in general, since from the discussion above, if $\mathscr{C}_{3,r+2} \in \mathscr{H}$, then there are planar graphs excluding $\mathscr{C}_{3,r+2}$ and of treewidth at least r. Thus $\theta_{\mathscr{H}}(|V(\mathscr{C}_{3,r+2})|) = \theta_{\mathscr{H}}(3r+6) \geq r$.

On the other hand, it is easy to see that $\theta_{\mathscr{H}}(r) = \Omega(\sqrt{r})$ for every graph family $\mathscr{H}$. Indeed, for any $H \in \mathscr{H}$, by denoting $r = |V(H)|$, the $\lceil \sqrt{r} - 1 \rceil \times \lceil \sqrt{r} - 1 \rceil$ grid has $< r$ vertices, so it excludes H. However, this grid has treewidth $\lceil \sqrt{r} - 1 \rceil$.

From the above discussions, we have therefore:

Proposition 2. *For every family $\mathscr{H}$ of planar graphs, $\sqrt{r} - 1 \leq \theta_{\mathscr{H}}(r) \leq 9r$. Furthermore, if $\mathscr{C}_{3,\lfloor r/3 \rfloor} \in \mathscr{H}$, then $\theta_{\mathscr{H}}(r) \geq r/3$.*

So, from Proposition 2, the $O(\sqrt{r})$ bound of Corollary 1 is optimal. And more generally, the family $\mathscr{H}_p$ composed of all r-vertex graphs having a poly-line $p \times O(r/p^2)$-grid drawing have $\theta_{\mathscr{H}_p}(r) = O(\sqrt{r})$ which is optimal.

4 Proof of the Main Theorem

The goal of the section is to prove Theorem 1 that we recall the statement:

Theorem 1. *The treewidth of every planar graph excluding as minor a graph having a poly-line $p \times q$-grid drawing is $O(p\sqrt{q})$.*

We start with a simple lemma:

Lemma 1. *The $p \times q$ grid is a minor of the $\lfloor (2+\sqrt{2})\sqrt{pq} \rfloor \times \lfloor (2+\sqrt{2})\sqrt{pq} \rfloor$ grid.*

[2] Note that if a triangle is chosen as outerface of $\mathscr{C}_{3,r+2}$, then the resulting drawing has $r+2$ nested triangles. However, a drawing with the minimal number of nested disjoint cycles can be obtained by choosing a quadrangle of $\mathscr{C}_{3,r+2}$ as outerface.

Proof. W.l.o.g., assume that $p \leq q$. The construction is illustrated on Fig. 5. The $p \times q$ grid H is first split into squares, each one being a $p \times p$ grid. There are $s = \lceil q/p \rceil$ such squares, the last one may be completed by some columns if p does not divide q.

Let $k = \lceil \sqrt{s+1} \rceil + 1$. Observe that k is the smallest integer such that $k \cdot (k-2) \geq s$, since $k \cdot (k-2) \geq (\sqrt{s+1}+1) \cdot (\sqrt{s+1}-1) = s$. In the illustration on Fig. 5, $s = 8$ and $k = 4$.

Now, the s squares are organized into k strips, each containing $k-2$ squares. Extra squares may be added to complete the last strip. Then, each such strip is surrounded by two extra squares: one at the beginning and one at the end (cf. the red squares on Fig. 5). The final grid M is composed of the k strips each of k squares. Therefore M is a $pk \times pk$ grid.

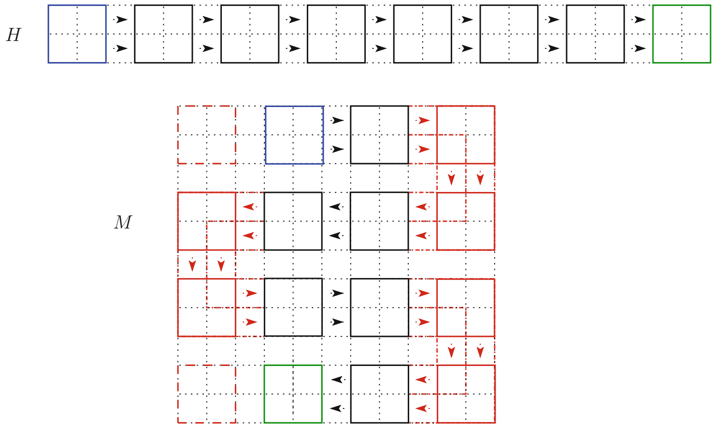

Fig. 5. From a 3×24 grid H to a 12×12 grid M. (Color figure online)

Grid M contains H as a minor, since the edges between two adjacent squares in H exist also in M either horizontally, if the squares belong to the same strip, or vertically and then horizontally to make the turn between consecutive strips.

We have $pk = p \cdot (\lceil \sqrt{s+1} \rceil + 1)$. Using the fact that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$, we have $\lceil \sqrt{s+1} \rceil = \lceil \sqrt{\lceil q/p \rceil + 1} \rceil = \lceil \sqrt{\lceil q/p + 1 \rceil} \rceil = \lceil \sqrt{q/p+1} \rceil$. It follows that $pk = p \cdot \lceil \sqrt{q/p+1} \rceil + p < p\sqrt{q/p+1} + 2p = \sqrt{pq+p^2} + 2p$. Since $p \leq q$, $p = \sqrt{p^2} \leq \sqrt{pq}$, and it follows that $pk < \sqrt{pq+pq} + 2\sqrt{pq} = (\sqrt{2}+2)\sqrt{pq}$. Thus, M is a $\lfloor (2+\sqrt{2})\sqrt{pq} \rfloor \times \lfloor (2+\sqrt{2})\sqrt{pq} \rfloor$ grid. This completes the proof. □

The proof of our second lemma, relies on a special drawing transformation preserving height due to [Bie14].

Lemma 2. **([Bie14], Theorem 5).** *Any poly-line $p \times q$-grid drawing can be transformed into a flat orthogonal $p \times w$-grid drawing with $w \leq \max\{n, m\} + b$,* *where n and m are respectively the number of vertices and edges of the graph, and b is the maximum number of local minima and maxima of polygonal curves in the poly-line drawing.*

We are now ready to prove:

Lemma 3. *Every graph having a poly-line $p \times q$-grid drawing is minor of the $p \times (3pq)$ grid.*

Proof. Let G be a graph having a poly-line $p \times q$-grid drawing Γ. Transform Γ into a flat orthogonal $p \times w$-grid drawing Γ' thanks to Lemma 2, where $w \leq \max\{n, m\} + b$ with $n = |V(G)|$ and $m = |E(G)|$. The number b can be upper bounded by the number of bends of any edge drawn as poly-line in Γ. Each bend occupies a point of the grid in Γ. Therefore, the number of vertices of G is $n \leq pq - b$. From the planarity of G, $m < 3n = 3(pq - b)$. It follows that $w \leq \max\{n, m\} + b \leq 3(pq - b) + b \leq 3pq$, and thus Γ' is a flat orthogonal $p \times (3pq)$-grid drawing of G.

We conclude the proof by observing that the grid supporting Γ', say the graph M, contains G as minor. Indeed, in M, original vertices and edges of G are represented has sequences of horizontal or vertical segments. These sequences of segments are connected subgraphs of M, and actually internally disjoint paths as they can meet only at vertices of G. Thus, contracting in M vertices of G into single vertices, and edges of G into single edges, provides a graph G' containing G as subgraph. Therefore, G is a minor of M minor, completing the proof. □

To conclude the proof of Theorem 1, we will use an accurate bound of the excluded grid-minor theorem of [RST94].

Lemma 4. **([GT12,RST94]).** *Every planar graph G excluding a planar graph H as minor has treewidth at most:*

i. $9r/2 - 4$, if H is the $r \times r$ grid with $r \geq 2$; and
ii. $9r - 22$, if H has $r \geq 3$ vertices.

Proof. It has been proved in [GT12, Theorem 1.4, pp. 419] that every planar graph either contains a $h \times k$-cylinder $\mathscr{C}_{h,k}$ as minor or has branchwidth at most $k + 2h - 2$ (for $k \geq 3$ and $h \geq 1$). In particular, if a planar graph G excludes a $r \times r$ grid as minor, then its branchwidth is at most $r + 2r - 2 = 3r - 2$ (since, if G excludes an $r \times r$ grid, then it excludes an $r \times r$-cylinder as well). It is also well known that the treewidth of G is at most $\max\{3b/2 - 1, 1\}$, where b is the branchwidth of G. It follows that the treewidth of G is at most $3(3r-2)/2-1 = 9r/2 - 4$ if G is planar and excludes an $r \times r$ grid as minor, for $r \geq 3$. Observe that if H is a 2×2 grid (i.e., a cycle C_4) then G must be an outerplanar graph (since it cannot contains neither a K_4 nor a $K_{2,3}$ as minor that both contains a C_4). Thus G has treewidth $2 \leq 9 \cdot 2/2 - 4 = 5$. Therefore, the bound holds also for $r = 2$, proving the first point.

It is known that every Hamiltonian planar graph with r vertices is contained as minor in a $r \times r$ grid [RST94, Theorem (1.3)]. Moreover, every r-vertex planar graph is contained as minor in a Hamiltonian planar graph with $2r-4$ vertices for $r \geq 4$. The graph is obtained by replacing one edge per separating triangle by a degree-4 vertex (cf. [RST94, Theorem (1.4)]). It follows that every planar graph H with r vertices is contained as minor in a $(2r-4) \times (2r-4)$ grid.

Therefore, if a planar graph G excludes an r-vertex planar graph H as minor with $r \geq 4$, then G excludes an $(2r-4) \times (2r-4)$ grid as minor, and thus has treewidth at most $9 \cdot (2r-4)/2 - 4 = 9r - 22$. Observe that if H has $r = 3$ vertices, then G must be a forest and thus has treewidth $1 \leq 9 \cdot 3 - 22 = 5$. Therefore, the bound holds also for $r = 3$, proving the second point. □

By combining Lemma 3 and Lemma 1, we get that a graph H having a poly-line $p \times q$-grid drawing is the minor of the $O(p\sqrt{q}) \times O(p\sqrt{q})$ grid. By Lemma 4(i), the treewidth of a planar graph excluding such a square grid, and thus H, has treewidth $O(p\sqrt{q})$, which completes the proof of Theorem 1.

Using the constants in Lemma 3, 1, and 4(i), we can obtained a more accurate upper bound on the treewidth for Theorem 1, namely of $9\left\lfloor (2+\sqrt{2})\sqrt{p \cdot 3pq} \right\rfloor/2 - 4 \approx 27p\sqrt{q}$.

5 Conclusion

In this paper, we establish a connection between the treewidth of a planar graph G excluding as minor a graph H and the ability of poly-line grid-drawing of H with small height. One of the consequences of our main result is that the treewidth of every planar excluding such graph H is $O(\sqrt{r})$ where $r = |V(H)|$, which is optimal. We also show that if H is outerplanar, then the bound slightly increases to $O(\sqrt{r \log r})$, leaving open the question of the optimality of this bound.

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