Modeling with Words: Steps Towards a Fuzzy Quantum Logic

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Abstract. On 1936, Birkhoff and von Newmann proposed the introduction of a "quantum logic", as the lattice of quantum mechanical proposition which is not distributive and also not a Boolean. Seven years later, Mackey tried to provide a set of axioms for the propositional system to predict of the outcome set of experiments. He indicated that the system is an orthocomplemented partially ordered set. Physical complex systems can be modeled by using linguistic variables which are variables whose values may be expressed in terms of a specific natural or artificial language, for example $\mathbb{L}= \{$*very less young; less young; young; more young; very young; very very young* $\ldots\}$. In language of hedge algebra ($\mathbb{HA}$), $\mathbb{L}$ set which is generated from $\mathbb{HA}$ is the POSET (partial order set). In this paper, we introduce a quantum logic ℓ to assert that, let $\perp$ be the orthocomplementation map $\perp : \ell \rightarrow \ell$, all $\clubsuit, \spadesuit \in \perp$ must satisfy the following conditions:

- $(\clubsuit^{\perp})^{\perp} = \clubsuit$
- If $\spadesuit \leq \clubsuit$ then $\clubsuit^{\perp} \leq \spadesuit^{\perp}$
- The greatest lower bound $\clubsuit \vee \clubsuit^{\perp} \in \ell$ and the least upper bound $\clubsuit \wedge \clubsuit^{\perp} \in \ell$

Keywords: Linguiatic variable · Fuzzy quantum logic · Quantum bit

1 Introduction

Fuzzy set, linguistic fuzzy logic and fuzzy dynamic system have been studied and applied in artificial intelligence such as neural network as well as machine learning. Fuzzy set or "computing with words" (CWW) was introduced by Lotfi A. Zadeh in 1965 as an extension of the classical notion of set [15,22,23] and was just a tool to knowledge represent and reasoning in intelligent system [16]. As Zadeh indicated in [16], human acknowledgment is nothing different from words. In daily activity, we see the real world through words. Many smart devices established based on CWW such as fuzzy neural network, fuzzy querying, fuzzy data mining, and so on have been studied [2,6,7,13,14,20] In Quantum information

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science [1,5,17], fuzzy logic holds important roles in modeling and verifying quantum computing and has been developed for decades [21]. Together with fuzzy logic, fuzzy set and fuzzy Z-number have also widely been applied in quantum fuzzy set [18] and quantum Z-number [3]. The rest of the paper is organized as follows: Sect. 2 recalls some of the main foundation concepts of quantum fuzzy logic and linguistic variables that are set up on hedge algebra $\mathbb{HA}$ [8–10,12–14]. Section 3 proposes a quantum linguistic fuzzy logic. Section 4 outlines summaries and forthcoming work.

2 Preliminary: Linguistic Variables and Quantum Fuzzy Logic

This section revises relational knowledgement in both fuzzy quantum logic ($\mathbb{FQL}$) and linguistic fuzzy quantum logic ($\mathbb{LQL}$) that associate with our study paper.

2.1 Quantum Fuzzy Logic

Element of quantum logic ($\mathbb{QL}$) is orthomodular partial order set (poset), which is defined in [21].

Definition 1. *In [21], a non-empty set L together with a binary relation $\leq$ on it is said to be a poset if for $\forall a, b, c \in L$, we have:*

1. *$a \leq a$ (reflexive property)*
2. *if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetric property).*
3. *if $a \leq b$ and $b \leq c$, then $a = c$ (transitive property).*

For estimating $\mathbb{QL}$, it is based on distributive and modular lattices

Definition 2. *In [21], for any a, b, $c \in L$, a distributive lattice is a lattice, which satisfies the following two conditions:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad (1)$$
$$a \vee (b \vee c) = (a \vee b) \wedge (a \vee c), \quad (2)$$

In which, the meet $\wedge$ operation has a higher priority than the join $\vee$ operation.

Definition 3. *In [21], a mapping $\perp : L \to L$ is said to be an orthocomplementation on a poset L with 1 and 0 if for all a, $b \in L$:*

1. *$(a^{\perp})^{\perp}) = a$*
2. *if $a \leq b$ then $b^{\perp} \leq a^{\perp}$*
3. *$a \vee a^{\perp} = 1$*

Definition 4. [21] *A lattice L is called modular if for any $a, b, c \in L$, the following condition holds:*

$$a \vee (b \wedge c) = (a \vee b) \wedge c \quad \textit{if} \quad a \leq c. \quad (3)$$

Let $\perp$ be the orthocomplementation on poset L, two elements $a, b \in L$ are called orthogonal, denote $a \perp b$ if $a \leq b^{\perp}$

Definition 5. *An orthocomplemented poset L is said to be an orthomodular poset (OMP for short) if $a, b \in L$ then:*

1. *if $a \perp b$ then $a \vee b \in L$.*
2. *if $a \leq b$ then*

$$b = a \vee (b \wedge a^{\perp}) \tag{4}$$

In general, for $a_i \perp a_j$, $i \neq j$ and $a_i, a_j \in L$, an OMP L meets fully expression:

$$\bigvee_{i=1}^{\infty} a_i \in L \tag{5}$$

L is called a quantum logic.

Example 1. In [4], an OMP $A = \{0 = min,\ x,\ y,\ z,\ x^{\perp},\ y^{\perp},\ z^{\perp},\ 1 = max\}$ whose Hasse diagram is presented in Fig. 1 is a $\mathbb{QL}$.

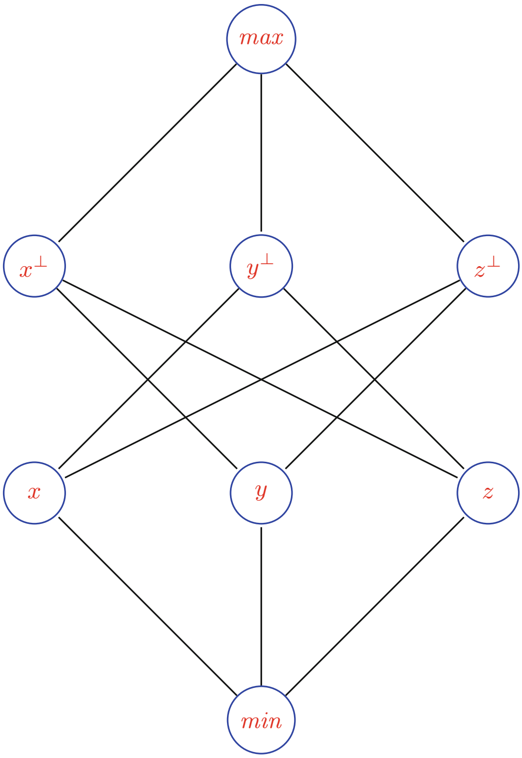

Fig. 1. Hasse diagram for OMP $A = \{min,\ x,\ y,\ z,\ x^{\perp},\ y^{\perp},\ z^{\perp},\ max\}$

2.2 Hedge Algebra and Linguistic Variables

Linguistic variables used in the paper are based on 3-Tuple $\langle X, G, H, \leq\rangle$, see in [13,14]:

- X is the name of linguistic variable
- G is the generating elements
- H is the linguistic hedges.

Example 2. In [11], consider an $\mathbb{HA} = \langle Temperature, \{high,\ low\}, \{0, W, 1\}, \{less,\ more,\ very\} \leq\rangle$ Fuzzy set X *is Temperature*, $G = \{c^+ = high;\ c^- = low\}$, $H = \{moreorless; more; very\}$ so term-set which is generated by the linguistic variable *Temperature* X is $\mathfrak{L}(X)$ or $\mathfrak{L}$ for short:

$\mathfrak{L}$ = {*very low, very less low; more or less low; low; more or less high; more high; very young; very very high* ...}.

3 Steps Towards Linguistic Fuzzy Quantum Logic

3.1 Hedge Algebra and Orthomodular Lattice (OML)

Properties of the poset $\langle \mathfrak{L},\ \leq\rangle$ are depended on generating element G which is totally or partially ordered set.

Theorem 1. *In* [19]*, let* $\mathbb{HA} = \langle X, G, C, H, \leq\rangle$ *be a refined hedge algebra* (RHA) *where* G *is a totally ordered set then* $\mathbb{HA}$ *is a distributive lattice.*

3.2 Fuzzy Quantum Logic Based on Hedge Algebra

In [19], a special case of RHA that called symmetrical refined $\mathbb{HA}$ ($\mathcal{SH}$ for short). On $\mathcal{SH}$, let $\ell \subset \mathfrak{L}$ and $\perp \subseteq \ell \times \ell$ be a map on ℓ set. We have the following important property:

Property 1. Let $\perp : \ell \rightarrow \ell$ such that $\perp(x \in \ell)$ is a contradictory element in ℓ

1. ℓ is the OML (1)
2. ℓ is a $\mathbb{QL}$ and is called linguistic quantum logic ($\mathbb{LQL}$) (2)

Proof. 1. property (1) is immediately inferred from theorem 5.1 in [19] by setting $\perp \leftarrow$ which is the contradictory mapping.
2. property (2) is a consequence from the property (1)

Example 3 and Fig. 2 illustrate the Property 1.

Example 3. Consider a $\mathcal{SH}$ with $G = \{\mathsf{false},\ \mathsf{true}\}$, $H = \{\mathcal{V}\}$. Subset OML $\ell^q = \{0,\ \mathcal{V}\mathsf{false},\ \mathsf{false},\ \mathsf{true},\ \mathcal{V}\mathsf{true},\ 1\}$ is a $\mathbb{LQL}$

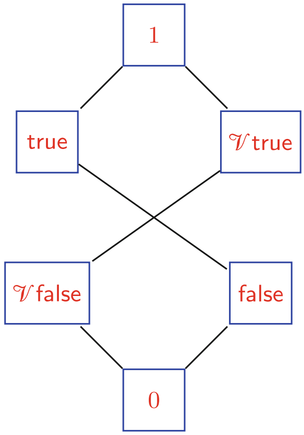

Fig. 2. Hasse diagram for OML $\ell^q = \{0,\ \mathscr{V}\mathsf{false},\ \mathsf{false},\ \mathsf{true},\ \mathscr{V}\mathsf{true},\ 1\}$

4 Conclusions and Forthcoming Study

The paper recommends a method to represent the fuzzy $\mathbb{FQL}$ using linguistic variables which is developed from $\mathbb{HA}$. $\mathbb{HA}$ is a particular abstract algebra whose properties $fm(c^+) + fm(c^-) = 1$ [13], a quantum bit can be written as

$$|qu\rangle = fm(x)^{\frac{1}{2}}|0\rangle + (1 - fm(x))^{\frac{1}{2}}|1\rangle \tag{6}$$

In the future, two forthcoming studies will be:

– Calculating a quantum state $|\Psi\rangle$ from N quantum $|qu\rangle$ bits is based on equation (6)

$$|\Psi\rangle = \bigotimes_{i=1}^{N}[fm(x_i)^{\frac{1}{2}}|0\rangle + (1 - fm(x_i))^{\frac{1}{2}}|1\rangle] \tag{7}$$

– Designing quantum gates and tensor product for fuzzy quantum bits that use linguistic variables.

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