# Ellipse Fitting Based on a Hybrid $l1l2$-Norm Algorithm

Lingling Luo[1](✉), Wenqing Yang[2], Wei Ren[2], and Xiaojun Yu[3]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025, China

[2] Guangzhou Power Supply Bureau of Guangdong Power Grid Co., Ltd., Guangzhou, China

[3] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025, China

**Abstract.** This paper proposes a new efficient direct method noted the Hybrid $l1l2$-Norm Model for fitting ellipse to resist the Gaussian and Laplacian disturb simultaneously which will be solved by split Bregman iteration and shrink operator. The experimental results reveal that the proposed method not only works well with Gaussian Noise data but Laplacian Noise data and its mixed version. Several experimental examples are reported to demonstrate the robustness of the proposed approach.

**Keywords:** Ellipse fitting · $l1l2$-Norm · Split-Bregman iteration

## 1 Introduction

Ellipse fitting is much more difficult than circle fitting due to the reason that the curvature of the ellipse is not uniform and has more parameters. And Ellipse fitting is widely applied in the fields of computer vision. Thus, a large amount of ellipse fitting algorithms have been developed. Liang et al. [1] adopte the $l(p)$-norm with $p < 2$ in the direct least square fitting method to achieve outlier resistance, and also developed a robust ellipse fitting approach using the alternating direction method of multipliers. A robust and direct algorithm for the least-square fitting of ellipses to scattered data is proposed. Maini [2] propose a new algorithm to improve the robustness of direct least squares fitting under the premise of increasing the amount of computation. Zhang et al. [3] present an ellipse-fitting algorithm based on residual $p$-norm minimum is proposed to deal with the outliers of massive 3D point data. The experiments validate that the p-norm minimum is more robust than the least-squares algorithm, and the application of an adaptive threshold allows the algorithm to clearly distinguish the

Supported by Humanities and Social sciences project of Guizhou Provincial Department of Education, No.: 2019qn032, the National Natural Science Foundation of China, No.: 71761005 and 2017 Academic New Seedling Cultivation and Innovation Exploration Project of Guizhou University of Finance and Economics, No.: [2017]5736-002.


H. Song and D. Jiang (Eds.): SIMUtools 2020, LNICST 370, pp. 766–776, 2021.
https://doi.org/10.1007/978-3-030-72795-6_61

outliers. Zhang et al. [4] note that in order to achieve high measurement accuracy, a random phase retrieval approach based on difference map Gram-Schmidt othonormalization and Lissajous ellipse fitting method (DGS-LEF) is proposed. Shcherbakova et al. [5] demonstrate a feasibility of a novel, ellipse fitting approach, for simultaneous estimation of relaxation times T-1 and T-2 from a single 3D phase-cycled balanced steady-state free precession sequence. Liang et al. and Mulleti et al. [6,7] show a robust ellipse fitting method to alleviate the influence of outliers. The proposed algorithm solves ellipse parameters by linearly combining a subset of data points. Kurt et al. [8] propose an error measure which can be used both to measure the accuracy of any ellipse fitting method and to compare the accuracy of the ellipses fitted with different methods. By computing this measure it is possible to obtain the precision of the ellipse parameters with respect to the orthogonal distance residuals. Fu et al. [9] demonstrate a rapid and precise phase-retrieval method based on Lissajous ellipse fitting and ellipse standardization. After compensating phase-shift errors by ellipse standardization, the phase distribution is extracted with high precision. In fact, ellipse fitting is widely used in many fields. Liao et al. [10] use Ellipse fitting to judge the correct splitting point pair of the automatic segmentation for cell images. Mitchell et al. [11] develope Ellipse fitting to analyse electron diffraction patterns from polycrystalline materials, the result shown that the technique is robust and can be used to determine the pattern centre and the diameter of diffraction rings with a high degree of accuracy. Gabriel Mistelbauer et al. [12] show an elliptical blood vessel model can quantitatively gauge blood vessels in order to better understand their morphology. Augustyn et al. [13] apply the ellipse fitting algorithm in incoherent sampling measurements of complex ratio of AC voltages, the result show that it was possible to perform coherent and incoherent sampling measurements with controlled frequency deviation. Li [14] propose a class of multi ellipse fitting problems for densely connected contours and propose a evaluation framework of various composite data sets in practical application. Yatabe et al. [15] propose a method of avoiding such untrusted pixels within the estimation processes of HEFS, and used an experiment of measuring a sound field in air for real data.

Despite the ellipse fitting is importance and has many methods to solve it, however, there has been until now no computationally efficient ellipse specific fitting algorithm. In this case, this paper introduces a new fitting method called hybrid $l1l2$-Norm algorithm which absorb the advantages of $l1$-norm and $l2$-norm algorithm. By introducing weight $w$ and discussing the ellipse fitting with different noise, we will show its robustness by compare with other convention approaches.

The remaining paper is organized as follows: In Sect. 2, we present a new model called the hybrid $l1l2$-Norm algorithm, and then we introduce the split Bregman iteration method to solve these parameters basing direct least squares fitting of ellipse. In Sect. 3, by analyzing different noises, we conducted some experiments to verity the robustness of the proposed method in ellipse fitting. In Sect. 4, it is the conclusion of the paper.

## 2 The Hybrid *l*1*l*2-Norm Algorithm

### 2.1 Problem Statement

For a two-dimensional plane coordinate system, a general conic can be expressed by an implicit second order polynomial as follows:

$$F(D,\delta) = D \cdot \delta = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where $D = (x^2, xy, y^2, x, y, 1)$ represents data points, $\delta = (a, b, c, d, e, f)^T$ represents parameter of the conic. $(x_k, y_k)$, where $k = 1, 2, ..., n$, represents the coordinates of n data points, $F(D_k, \delta)$ is called the algebraic distance between the point $(x_k, y_k)$ to the conic $F(D, \delta) = 0$. The fitting of a general conic may be approached by minimizing the sum of the squared algebraic distance of points, note

$$\sum_{k=1}^{n} F(D_k, \delta)^2$$

According to change the conic into an ellipse, we must add an ellipse-specific constraint $b^2 - 4ac < 0$ on the conic parameter [1].

The elliptic parameters are solved by minimizing the L2-norm of the algebraic distance, but when there are outliers in the data, the $l2$-norm will expand the influence of outliers, so that the fitting results are biased towards the parameters contained in the outliers. So $l1$-norm presented to estimate parameters can get a more robust. Thus, the parameters of the elliptic equation can be estimated by solving the following L1-norm minimization problems:

$$\begin{cases} \arg\min_\delta \|D\delta\|_1 \\ s.t. \quad b^2 - 4ac < 0 \end{cases} \tag{1}$$

In order to force the conic to be an ellipse, the conic parameter should satisfied the necessary condition $b^2 - 4ac < 0$, for convenience we take its subset $4ac - b^2 = 1$ instead, its matrix form can be presented as $\delta^T C \delta = 1$, where

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By using Lagrange multiplier method, the constrained problem in Eq. (1) can be transformed into the following unconstrained problem:

$$\arg\min_{\delta}\{\|D\delta\|_1 + \lambda(\delta^T C\delta - 1)\} \tag{2}$$

where $\lambda > 0$ is Lagrange parameter.

The advantage of using $l1$-norm is that it can get a satisfied performance on Laplacian noise data, while $l2$-norm fits better for Gaussian noise data than $l1$-norm. In order to improve the accuracy of ellipse fitting and reduce the complexity of ellipse fitting, the hybrid $l1l2$-Norm Model is proposed to achieve data denoising. Unlike the $l1$-norm and $l2$-norm algorithm, the hybrid $l1l2$-Norm Model can impose different constraints on different data points using weight Matrix $w$. Obviously, when the selection of $w$ is reasonable, the hybrid model can inherit the advantages of $l1$-norm and l2-norm algorithm. It has good robustness to Laplacian noise and Gaussian noise data.

### 2.2 Solution of the Hybrid *l*1*l*2-Norm Algorithm

According to the Eq. (2), the hybrid $l1l2$-Norm Algorithm can be shown as:

$$\arg\min_{\delta}\{\|w \circ D\delta\|_1 + \|(1-w)\circ D\delta\|_2 + \lambda(\delta^T C\delta - 1)\} \tag{3}$$

where $\lambda$ is the Lagrange multiplier, $w$ is the weigh. The simulation is a minimization problem with $l1$-norm and $l2$-norm at the same time. In order to get the accurate results, the split Bregman method [16] is adopted to solve the problem. For Eq. (3), it is a very difficult to solve the general $l1$-regularized optimization problem, so we introduce an auxiliary variable $s = D\delta$ in it, then the original problem is converted as

$$\begin{cases} \arg\min_{s}\{\|w \circ s\|_1 + \|(1-w)\circ s\|_2 + \lambda(\delta^T C\delta - 1)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad s.t. \quad s = D\delta \end{cases} \tag{4}$$

By introducing the Bregman iteration variable t, the Eq. 4 can be transformed into the following unconstrained problems:

$$\arg\min_{\delta,s,t}\{\|w \circ s\|_1 + \|(1-w)\circ s\|_2 + \frac{\mu}{2}\|s - D\delta - t\|^2 + \lambda(\delta^T C\delta - 1)\} \tag{5}$$

Now, we solve these subproblems about $\delta, s, t$ one by one.

(1) the $\delta$ subproblem is

$$\arg\min_{\delta}\{\frac{\mu}{2}\|s - D\delta - t\|^2 + \lambda(\delta^T C\delta - 1)\} \tag{6}$$

It is a typical least square problem. Thus, the following closed solutions can be obtained as:

$$\delta^{k+1} = \frac{\mu D^T(s^k - t^k)}{\mu D^T D + 2\lambda C} \tag{7}$$

(2) the $s$ subproblem is

$$\arg\min_{s}\{\|w \circ s\|_1 + \|(1-w)\circ s\|_2 + \frac{\mu}{2}\|s - D\delta - t\|^2\} \tag{8}$$

Set $F = \|w \circ s\|_1 + \|(1-w) \circ s\|_2 + \frac{\mu}{2}\|s - D\delta - t\|^2$. For Eq. (8), we first get the partial derivation of variable $s$, and then, we note the necessary conditions, i.e. $\frac{\partial F}{\partial s} = 0$ , at the extreme value of the function. So we can obtain

$$\frac{\partial F}{\partial s} = w\frac{s}{|s|} + 2(1-w)s + \mu(s - D\delta - t)$$

where the symbol $|s|$ means to take absolute value for variable $s$. It can divide the issue into two circumstances.

1) set $s > 0$, get $w + 2(1-w)s + \mu(s - D\delta - t) = 0$
2) set $s < 0$, get $-w + 2(1-w)s + \mu(s - D\delta - t) = 0$

Since $w$ and $1-w$ are the coefficient matrix, each of these elements are between 0 and 1. According to this, the standard shrink operator [17] is used to solve these problems, we can get as:

$$s^{k+1} = \frac{shink(\mu(D\delta^{k+1+t^k}), w)}{2(1-w) + \mu} \tag{9}$$

where $shink(x, r) = \frac{x}{|x|} \max(|x| - r, 0)$

(3) the $t$ subproblem is

$$t^{k+1} = t^k + D\delta^{k+1} - s^{k+1} \tag{10}$$

where $t$ is Bregman variable, $\mu$ is the penalty parameter.

The complete algorithm of the minimization problem Eq. (3) with Bregman iteration is shown in Algorithm 1.

Algorithm 1. the Hybrid $l1l2$-Norm Algorithm based direct least squares fitting of ellipse with Bregman iteration

(1) set $w, \mu, \lambda, \delta^0, s^0, t^0, k = 0, \varepsilon = 10^{-6}.n_{maxiter} = 300$;
(2) For $k < n_{maxiter}$ and $\frac{\|D\delta^{k+1} - D\delta^k\|_2}{\|D\delta^{k+1}\|_2} > \epsilon$;
(3) Solve $\delta^{k+1}$ using Eq. (7);
(4) Solve $s^{k+1}$ using Eq. (9);
(5) Solve $t^{k+1}$ using Eq. (10);
(6) set $k = k + 1$;
(7) end.

when the ellipse parameter $\delta = (a, b, c, d, e, f)'$ is solved, according [18] and [19], we can normalize $f = 1$, so it gets $\delta = (a, b, c, d, e, 1)'$. Thus, ellipse's center $(x_c, y_c)$, major semiaxis $r_c$, minor semiaxis $r_c$ and angle $\theta$ have the following coordinates:

$$\begin{cases} x_c = \frac{2cd-be}{b^2-4ac} \\ y_c = \frac{2ae-bd}{b^2-4ac} \\ r_x = \sqrt{\frac{2(ax_c^2+cy_c^2+bx_cy_c-1)}{a+c-\sqrt{(a-c)^2+b^2}}} \\ r_y = \sqrt{\frac{2(ax_c^2+cy_c^2+bx_cy_c-1)}{a+c+\sqrt{(a-c)^2+b^2}}} \\ \theta = \frac{1}{2}\tan^{-1}\frac{b}{a-c} \end{cases} \tag{11}$$

## 3 Experimental Results

In this section, we conduct simulations and experiments to evaluate the performance of the proposed $l1l2$-Norm approach for ellipse fitting with simulated data. We main analysis our algorithm to compare with the $l2$-Norm based direct least-squares ellipse fitting (DLS). We note the scale normalization parameter $f_0 = 600$ when we calculate the center position of the circle and note the max iteration for convergence in hyper-renormalization 30. At the same time, we test the proposed algorithm with idea data from a set of ellipses points with parameters $x_c = 128, y_c = 256,\ r_x = 80, r_y = 64, \theta$ and the weight $w = 0.25$. Unless otherwise explained, all parameters are the same. In this paper, three kinds of experiments are discussed as follows.

Experiment 1: Containing Gaussian Noise in Ellipse Fitting
In this experiment, a serial of ellipse points polluted with gaussian noise with mean 0 and standard deviation 6 is simulated for averting the scatter matrix $D^T D$ to be a singular matrix. And the simulated ellipse parameters are set as $x_c = 128, y_c = 256, r_x = 80, r_y = 64, \theta$. If we keep the center and axis unchanged and rotation angle $\theta$, in this paper, we will discuss the angle $\theta = 25, 45, 65, 85$, thus, 5 ellipses are acquired. For the DLS method and the proposed method, we can compute the parameters $x_c, y_c, r_x, r_y, \theta$ , and then the result is represented in Table 1. We can get that no matter how you change the angle, all methods get a very precise results. But on the whole, the proposed method get higher accuracy than DLS method.

Experiment 2: Containing Laplacian Noise in Ellipse Fitting
In this experiment, a serial of ellipse points polluted with Laplacian noise with mean 9 and standard deviation 16 is simulated. The simulated ellipse parameters are set as $x_c = 128, y_c = 256, r_x = 80, r_y = 64, \theta = 45^0$. In this paper, we apply the DLS method and the proposed method to analyze different noise point, i.e. 5 noisy points, 15 noisy points, 35 noisy points and 65 noisy points. Comparison of their results (see Fig. 1), we can see that our approach gives a more precise than the DLS. While under the condition of laplacian noise, as the number of noise increases, our approach achieves a very competitive performance of the ellipses parameters estimate issue. The more detail are shown in Table 2.

Experiment 3: Containing Laplacian Noise and Gaussian Noise in Ellipse Fitting
In this experiment, we set a serial of ellipse points polluted with gaussian noise with zero-mean and standard deviation 2 and at the same time with Laplacian noise with mean 16 and standard deviation 32. The simulated ellipse parameters are set as $x_c = 128, y_c = 256,\ r_x = 80, r_y = 64, \theta = 45^0$. In this paper, when we set the weight to a fixed value $w = 0.25$, $l1l2$-Norm approach obtain the highest accuracy and far superior to $l2$-norm method (see Fig. 2). It also show that when the input data is polluted with a low density Laplacian Noise and Gaussian Noise, with the increasing of the mixed noise, the hybrid $l1l2$-Norm algorithm produces satisfactory fitting result. The ellipse fitting results of $x_c, y_c, r_x,\ r_y, \theta$ of the two methods are shown in Table 3. Now, let's talk about different weights $w$. In this case, the simulated ellipse parameter is set as $x_c = 128, y_c = 256$,

$r_x = 80, r_y = 64, \theta = 45°$ and we choose 25 noise points. It should be noted that other parameter are not change. When we change the weight $w$, we can estimate the ellipse parameters $x_c, y_c, r_x, r_y, \theta$. According to the Eq. (3), we can see that when our weight $w$ is greater than 0.5, the proportion of $l1$-norm method is larger than that of $l2$-norm method. Otherwise, it's the opposite. The more detail are shown in Table 4. We obtain that it is not the bigger of the weight $w$ the better and at the same it is not the smaller the weight $w$ the better. $w = 0.25$ is more accurate than other data for estimating ellipse parameters.

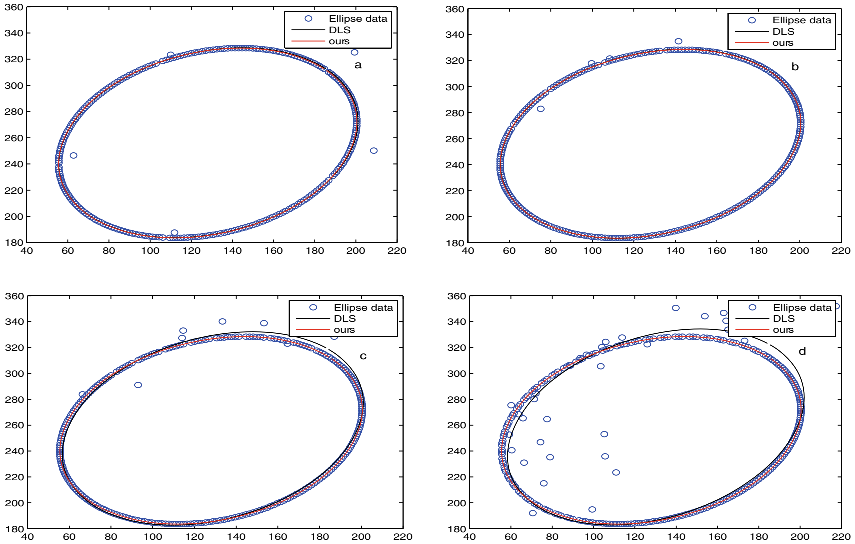


**Fig. 1.** Ellipse Fitting with Laplacian noise.(a) 5 noisy points.(b) 15 noisy points.(c) 35 noisy points.(d) 65 noisy points.

**Table 1.** Ellipse parameters fitting results with Gaussian noise.

| Angle | Method | $r_x$ | $r_y$ | $x_c$ | $y_c$ | $\theta$ |
|---|---|---|---|---|---|---|
| 25 | DLS | 79.7815 | 64.5943 | 127.6936 | 256.3401 | 23.6879 |
| 25 | Ours | 79.8369 | 64.6328 | 127.6527 | 256.3573 | 24.3919 |
| 45 | DLS | 80.6144 | 63.8754 | 127.8803 | 256.1238 | 45.1474 |
| 45 | Ours | 80.3639 | 63.9584 | 127.9081 | 256.0573 | 44.8488 |
| 65 | DLS | 80.2682 | 64.3844 | 128.0290 | 256.3719 | 65.9796 |
| 65 | Ours | 79.7223 | 64.8028 | 128.1065 | 256.5538 | 66.0449 |
| 85 | DLS | 80.6070 | 63.9293 | 128.2050 | 256.4321 | 84.4931 |
| 85 | Ours | 80.0320 | 64.1070 | 128.2758 | 256.3606 | 84.7569 |

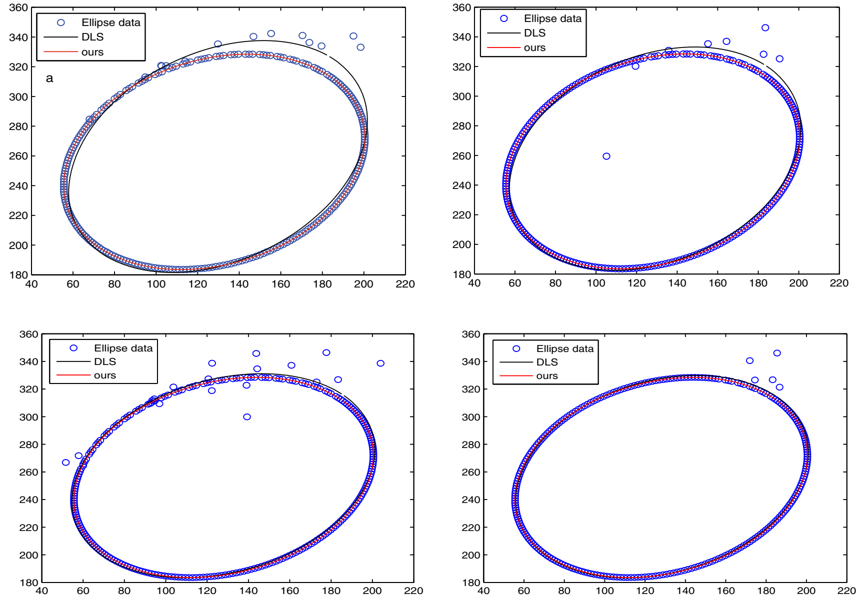


**Fig. 2.** Ellipse Fitting with mixed noise.(a) 5 noisy points.(b) 15 noisy points.(c) 35 noisy points.(d) 65 noisy points.

**Table 2.** Ellipse parameters fitting results with Laplacian noise.

| Number | Method | $r_x$ | $r_y$ | $x_c$ | $y_c$ | $\theta$ |
|---|---|---|---|---|---|---|
| 5 noisy points | DLS | 80.3658 | 63.9807 | 128.0540 | 256.2458 | 44.1923 |
| 5 noisy points | Ours | 79.9992 | 63.9995 | 127.9986 | 256.0000 | 45.0039 |
| 15 noisy points | DLS | 81.2586 | 63.6584 | 128.5470 | 256.6195 | 47.0657 |
| 15 noisy points | Ours | 79.9932 | 64.0006 | 127.9956 | 255.9967 | 45.0030 |
| 35 noisy points | DLS | 84.1981 | 62.4486 | 127.5874 | 256.0333 | 45.5419 |
| 35 noisy points | Ours | 80.0087 | 64.0040 | 128.0051 | 256.0035 | 45.0075 |
| 65 noisy points | DLS | 88.7864 | 60.7384 | 131.8915 | 259.9820 | 46.0048 |
| 65 noisy points | Ours | 79.8599 | 64.0552 | 128.0141 | 256.0038 | 44.9933 |

**Table 3.** Ellipse parameters fitting results with mixed noise.

| Number | Method | $r_x$ | $r_y$ | $x_c$ | $y_c$ | $\theta$ |
|---|---|---|---|---|---|---|
| 5 noisy points | DLS | 80.2828 | 63.9424 | 128.3090 | 256.1941 | 44.66127 |
| 5 noisy points | Ours | 80.0002 | 64.0003 | 128.0005 | 255.9998 | 44.9983 |
| 15 noisy points | DLS | 80.1221 | 63.9887 | 128.0631 | 256.1300 | 45.5244 |
| 15 noisy points | Ours | 80.0494 | 64.0185 | 128.0227 | 256.0621 | 45.1249 |
| 35 noisy points | DLS | 80.2954 | 63.9292 | 128.1387 | 256.2292 | 45.7377 |
| 35 noisy points | Ours | 80.0697 | 64.0161 | 128.0311 | 256.0652 | 45.0858 |
| 65 noisy points | DLS | 81.0642 | 63.6442 | 128.3892 | 256.6061 | 46.2952 |
| 65 noisy points | Ours | 80.0013 | 63.9998 | 128.0008 | 256.0004 | 44.9979 |

**Table 4.** Ellipse parameters fitting results with different weight $w$.

| $w$ | $r_x$ | $r_y$ | $x_c$ | $y_c$ | $\theta$ |
|---|---|---|---|---|---|
| $w = 0.05$ | 80.6784 | 63.9100 | 128.3385 | 256.5320 | 45.4967 |
| $w = 0.15$ | 80.0115 | 64.0032 | 128.0038 | 256.0118 | 45.0186 |
| $w = 0.25$ | 80.0000 | 63.9999 | 128.0001 | 255.9999 | 44.9987 |
| $w = 0.5$ | 80.0019 | 63.9993 | 128.0008 | 256.0009 | 44.9996 |
| $w = 0.65$ | 80.2000 | 64.0419 | 128.1200 | 256.0378 | 45.1254 |
| $w = 0.75$ | 79.9596 | 64.0534 | 127.9523 | 256.0484 | 45.2212 |
| $w = 0.85$ | 80.0193 | 63.9942 | 128.0033 | 256.0007 | 44.9756 |
| $w = 0.95$ | 80.5697 | 63.7796 | 128.0124 | 256.1112 | 44.8990 |

## 4 Conclusions

This paper has presented a new method for direct $l1l2$-Norm algorithm based ellipse fitting. In this study, we describe three experiments: ellipse parameters fitting results with Gaussian noise, ellipse parameters fitting results with Laplacian noise and ellipse parameters fitting results with mixed noise.

For the experiment 1, under the condition that other parameters remain unchanged, the ellipse parameters with DLS model and the hybrid $l1l2$-Norm model are predicted by the rotation angle. At 45°, the prediction result of our proposed model is the most accurate. For the experiment 2, two models are used to estimate the ellipse parameters by introducing different amount of noise. No matter how the amount of noise changes, we can see that the prediction result of our proposed model is better than that of DLS model. For the experiment 3, we mainly discuss the influence of the change of noise quantity and weight on the prediction results of ellipse parameters for DLS model and the hybrid $l1l2$-Norm model. It is also concluded that the prediction of our proposed is better than that of DLS model no matter how the amount of noise changes. As for the change of weight $w$, $w = 0.25$ is the optimal value. In fact, we can also discuss that the weight is a matrix form, which will be discussed in the next topic.

It can see that the proposed model is good for Laplacian and all kinds of mixed noise without degradation of Gaussian performance. Comparison experiments suggest that the proposed method can bring great challenge for artificial intelligence regardless of the noise type.

## References

1. Liang, J.L., Li, P.L., Zhou, D.Y.: Robust ellipse fitting via alternating direction method of multipliers. Signal Process. **164**, 30–40 (2019)
2. Maini, E.S.: Enhanced direct least square fitting of ellipses. Int. J. Pattern Recogn. Artif. Intell. **20**, 939–953 (2006)
3. Zhang, L., Cheng, X., Wang, L.Y.: Ellipse-fitting algorithm and adaptive threshold to eliminate outliers. Survey Rev. **51**, 250–256 (2019)
4. Zhang, L., Cheng, X., Wang, L.Y.: Random three-step phase retrieval approach based on difference map Gram-Schmidt orthonormalization and Lissajous ellipse fitting method. Optics and Lasers in Eng. **121**, 11–17 (2019)
5. Shcherbakova, Y., Van den Berg, C.A.T., Moonen, C.T.W., Bartels, L.M.: PLANET: An ellipse fitting approach for simultaneous T-1 and T-2 mapping using phase-cycled balanced steady-state free precession. Magnetic Resonnace Med. **79**, 711–722 (2018)
6. Liang, J.L., Zhang, M.H., Liu, D., Zeng, X.J.: Robust ellipse fitting based on sparse combination of data points. IEEE Trans. Image Process. **22**, 2207–2218 (2013)
7. Mulleti, S., Seelamantula, C.S.: Ellipse fitting using the finite rate of innovation sampling principle. IEEE Trans. Image Process. **25**, 1451–1464 (2015)
8. Kurt, O., Arslan, O.: A general accuracy measure for quality of elliptic sections fitting. Measurement **145**, 640–647 (2019)
9. Fu, Y., Wu, Q., Yao, Y., Gan, Y., Liu, C., Yang, Y., Tian, J., Xu, K.: Rapid and precise phase retrieval from two-frame tilt-shift based on Lissajous ellipse fitting and ellipse standardization. Optics Express **28**(3), 3952–3964 (2020)
10. Liao, M., Zhao, Y.Q., Li, X.H.: Automatic segmentation for cell images based on bottleneck detection and ellipse fitting. Neurocomputing **173**, 615–622 (2016)
11. Mitchell, D.R.G., Van den Berg, J.A.: Development of an ellipse fitting method with which to analyse selected area electron diffraction patterns. Ultramicroscopy **160**, 140–145 (2016)
12. Mistelbauer, G., Zettwitz, M., Schernthaner, R.: Visual assessment of vascular torsion using ellipse fitting. Proc. Eurographics Workshop on Visual Comput. Biol. Med. **12**, 129–133 (2018)
13. Augustyn, J., Kampik, M.: Application of ellipse fitting algorithm in incoherent sampling measurements of complex ratio of AC voltages. IEEE Trans. Instrument. Measurement **66**, 1117–1123 (2017)
14. Li, H.: Multiple ellipse fitting of densely connected contours. Inf. Sci. **502**, 330–345 (2019)
15. Yatabe, K., Ishikawa, K., Oikawa, Y.: Hyper ellipse fitting in subspace method for phase-shifting interferometry: practical implementation with automatic pixel selection. Optics Express **25**(23), 29401–29416 (2017)
16. Miao, C., Yu, H.Y.: A general-thresholding solution for l(p)(0<p<1) regularized CT reconstruction. IEEE Thans. Image Process. **24**, 5455–5468 (2015)

17. Goldstein, T., Osher, S.: A new wavelet threshold function and denoising application. SIAM J. Imag. Sci. **2**, 323–343 (2009)
18. Zhou, G., Zhong, K., Li, Z., Shi, Y.: Direct least absolute deviation fitting of ellipses. Math. Prob. Eng. **6**, 1–11 (2020)
19. Rosin, P.L.: Further five point fit ellipse fitting. Graph. Models Image Process. **61**(5), 245–259 (1999)