The Asymmetric Join the Shortest Orbit Queue: Analysis via a Dimensionality Reduction

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Abstract. In this work, we investigate the stationary behaviour of the asymmetric join the shortest queue system with a single server and two infinite capacity orbit queues. Arriving jobs that find the server busy, are forwarded to the least loaded orbit queue, and in case of a tie, they choose an orbit randomly. Orbiting jobs retry to connect with the server at different retrial rates, i.e., heterogeneous orbit queues. Our system is described by a Markov modulated two-dimensional random walk. By exploiting its special structure, its invariant measure is obtained in terms of the invariant measure of a non-modulated two-dimensional random walk by applying the compensation method.

Keywords: Asymmetric Join the Shortest Orbit Queue · Retrials · Compensation Method

1 Introduction

Our primary aim in this work is twofold. First, to investigate the stationary analysis of a certain class of non-homogeneous Markov-modulated random walks in the quarter plane, where modulation allows for a tractable analysis via a dimensionality reduction. In particular, by exploiting its special structure, the problem of deriving its invariant measure is reduced to the problem of solving for the invariant measure of a related non-homogeneous, **non**-modulated two dimensional random walk, which has a *complicated* boundary behaviour. The invariant measure of this standard non-homogeneous, **non**-modulated two dimensional random walk is derived by using the compensation method (CM) [3,5]. To accomplish this task, we consider a retrial system with two infinite capacity orbits and different retrieval rates, operating under the *join the shortest queue* strategy. This is the simplest queueing example, which can be described by such a Markov-modulated two-dimensional random walk. The approach we follow can be adapted to more complicated systems including finite priority lines in front of the server (i.e., priorities), the possibility of joining the priority line directly from the orbits, the case of Erlang arrivals etc.

Second, by application point of view, the simple retrial queueing network that we use has potential applications in the performance modelling of cooperative

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relay assisted wireless networks [15]. Such networks operate as follows: A finite number of source users transmit packets (i.e., the arrival stream) to a common destination node (i.e., the service station). We assume that the destination node can handle at most one job (e.g., due to limited memory capacity). Blocked jobs (i.e., jobs that find the service station occupied) are routed according to a cooperative policy to a finite number of relay nodes (i.e., the orbit queues) that assist the source users by retransmitting their blocked packets to the destination node. The cooperative policy among source users and relays refers to a strategy that dictates in which relay station a blocked job will be forwarded. In this work, we consider the join the shortest queue (JSQ) policy.

Cooperative relaying gives rise to pure wireless self-organizing networks without any need for base stations, and it can be employed in various applications of networked embedded systems, e.g., Cars use it to communicate directly with each other, for instance, to exchange reports on accidents, traffic jams, or bad road conditions. Autonomous robots may use it to build a wireless network in areas without infrastructure, e.g., in deserts and in space. This concept was mostly considered at the physical layer, based on information-theoretic considerations, e.g., [13], and have become a powerful technique to combat fading and attenuation in wireless networks. Recent works [14,16] shown that similar gains are achieved by network-layer cooperation, i.e., when relaying is assumed to take place at the protocol level avoiding physical layer considerations.

Thus, we consider a Markovian network with three nodes: the main service node where blocking is possible, and two delay nodes for repeated attempts. External arrivals are routed initially to the main service node. If the main service node is empty, the arriving job starts service immediately, and leaves the network after the service completion. Otherwise, the arriving blocked job is routed to the least loaded delay node. Each delay station is responsible to retransmit its blocked jobs to the main service station.

The standard (i.e., non-modulated) two-dimensional JSQ problem was initially studied in [11,12], and further investigated by using generating functions in [7,10]. However, their analyses do not lead to an explicit characterization of the equilibrium probabilities. The CM introduced in [3–5] is an elegant and direct method to obtain explicitly the stationary distribution as infinite series of product form terms; see also [6,17]. In [1], the CM was used to analyze the JSQ policy in a two-queue system fed by Erlang arrivals. The queueing system in [1] is described by a multilayer random walk in the quarter plane; see also [18]. To our best knowledge, the works in [8,9] are the only that deal with the stationary analysis of retrial queues under the JSQ rule. In [8], the CM was applied for the symmetric (i.e., identical retrial rates), two orbit queue system with Poisson arrivals, by following the multilayer framework in [1]. Contrary to the work in [8], in this work, we are dealing with the asymmetric case (i.e., different retrial rates), which leads to a random walk in the half-plane. Moreover, our methodology is different since in this work the invariant measure is obtained in terms of the invariant measure of a standard non-modulated random walk, which is constructed by the original one through the dimensionality reduction.

In [9], exact tail asymptotics, and stationary approximations were investigated for the asymmetric two-orbit case, with additional dedicated Poisson streams that upon blocking, they forward their blocked jobs to specific orbits.

2 The System Model and the Equilibrium Equations

Consider a single server retrial system with two infinite capacity orbit queues. Jobs arrive at the system according to a Poisson process with rate $\lambda > 0$. If an arriving job finds the server idle, it starts service immediately. Otherwise, it is routed to the least loaded orbit queue. In case both orbit queues have the same occupancy, the blocked job is routed either to orbit 1 with probability q, or to orbit 2 with probability $1-q$. Orbiting jobs of either type retry independently to occupy the server according to a constant retrial policy. More precisely, the first job in orbit queue 1 (resp. 2) retry after an exponentially distributed time period with rate α_1 (resp. α_2). Without loss of generality, let $\alpha_1 < \alpha_2$. Service times are independent and exponentially distributed with rate μ.

Let $Q_l(t)$ be the number of jobs stored in orbit l, $l = 1, 2$, at time t, and by $C(t)$ the state of the server, i.e., $C(t) = 1$, when it is busy and $C(t) = 0$ when it is idle at time t, respectively. $Y(t) = \{(Q_1(t), Q_2(t), C(t)); t \geq 0\}$ is an irreducible Markov process on $\{0, 1, \ldots\} \times \{0, 1, \ldots\} \times \{0, 1\}$. Denote by $Y = \{(Q_1, Q_2, C)\}$ its stationary version, and define the set of stationary probabilities $p_{i,j}(k) = \mathbb{P}(Q_1 = i, Q_2 = j, C = k)$. The equilibrium equations are as follows. Define the set of stationary probabilities

$$p_{i,j}(k) = \mathbb{P}(Q_1 = i, Q_2 = j, C = k).$$

Let J be a random variable indicating the orbit queue which an arriving blocked job joins. Clearly, J is dependent on the vector (Q_1, Q_2, C). Then, the equilibrium equations are

$$\begin{array}{l} p_{i,j}(0)(\lambda + \alpha_1 1_{\{i>0\}} + \alpha_2 1_{\{j>0\}})) = \mu p_{i,j}(1), \\ p_{i,j}(1)(\lambda + \mu) = \lambda p_{i,j}(0) + \alpha_1 p_{i+1,j}(0) + \alpha_2 p_{i,j+1}(0) \\ +\lambda[p_{i-1,j}(1)\mathbb{P}(J = 1|Q = (i-1, j, 1))1_{\{i>0\}} \\ +p_{i,j-1}(1)\mathbb{P}(J = 2|Q = (i, j-1, 1))1_{\{j>0\}}], \end{array} \tag{1}$$

where, $1_{\{E\}}$ the indicator function of the event E and

$$\begin{array}{l} \mathbb{P}(J = l|Q = (Q_1, Q_2, 1)) = \frac{\psi_l}{\sum_{k=1}^{2} \psi_k 1_{\{Q_k = Q_l\}}}, \\ \mathbb{P}(J = l|Q = (Q_1, Q_2, 0)) = 0,\ l = 1, 2, \end{array}$$

where $\psi_l = \begin{cases} p,\ l = 1, \\ q,\ l = 2 \end{cases}$.

To apply the compensation method, we have to consider the following transformation: Let $X_1(t) = min\{Q_1(t), Q_2(t)\}$, $X_2(t) = Q_2(t) - Q_1(t)$. The system is described by a 3-dimensional continuous time Markov chain with state space $\{(m, n, k); m \geq 0, n \in \mathbb{Z}, k = 0, 1\}$. Let $q_{m,n}(k)$ be the equilibrium probability of being in state (m, n, k), and denote $\mathbf{q}_{m,n} = (q_{m,n}(0), q_{m,n}(1))^T$, with $\mathbf{x}^T$ the transpose of a vector $\mathbf{x}$. Its matrix transition diagram is given in Fig. 1. The equilibrium equations are as follows:

$$\begin{aligned}
(\mathbf{A}_{0,0} + (\alpha_1+\alpha_2)\mathbf{M})\mathbf{q}_{m,n} + \mathbf{A}_{0,-1}\mathbf{q}_{m,n+1} + \mathbf{A}_{1,-1}\mathbf{q}_{m-1,n+1}& \\
+\mathbf{A}_{-1,1}\mathbf{q}_{m+1,n-1} = \mathbf{0},\ m \geq 1, n \geq 2,& \\
(\mathbf{A}_{0,0} + (\alpha_1+\alpha_2)\mathbf{M})\mathbf{q}_{m,n} + \mathbf{B}_{0,1}\mathbf{q}_{m,n-1} + \mathbf{B}_{1,1}\mathbf{q}_{m-1,n-1}& \\
+\mathbf{B}_{-1,-1}\mathbf{q}_{m+1,n+1} = \mathbf{0},\ m \geq 1, n \leq -2,& \\
(\mathbf{A}_{0,0} + (\alpha_1+\alpha_2)\mathbf{M})\mathbf{q}_{m,0} + \mathbf{A}_{0,-1}\mathbf{q}_{m,1} + \mathbf{A}_{1,-1}\mathbf{q}_{m-1,1}& \\
+\mathbf{B}_{0,1}\mathbf{q}_{m,-1} + \mathbf{B}_{1,1}\mathbf{q}_{m-1,-1} = \mathbf{0},\ m \geq 1,& \\
(\mathbf{A}_{0,0} + (\alpha_1+\alpha_2)\mathbf{M})\mathbf{q}_{m,1} + \mathbf{A}_{0,-1}\mathbf{q}_{m,2} + \mathbf{A}_{-1,1}\mathbf{q}_{m+1,0}& \\
+\mathbf{A}_{1,-1}\mathbf{q}_{m-1,2} + \mathbf{A}_{0,1}\mathbf{q}_{m,0} = \mathbf{0},\ m \geq 1,& \qquad (2)\\
(\mathbf{A}_{0,0} + (\alpha_1+\alpha_2)\mathbf{M})\mathbf{q}_{m,-1} + \mathbf{B}_{0,1}\mathbf{q}_{m,-2} + \mathbf{B}_{-1,-1}\mathbf{q}_{m+1,0}& \\
+\mathbf{B}_{1,1}\mathbf{q}_{m-1,-2} + \mathbf{B}_{0,-1}\mathbf{q}_{m,0} = \mathbf{0},\ m \geq 1,& \\
(\mathbf{A}_{0,0} + \alpha_2\mathbf{M})\mathbf{q}_{0,n} + \mathbf{A}_{0,-1}\mathbf{q}_{0,n+1} + \mathbf{A}_{-1,1}\mathbf{q}_{1,n-1} = \mathbf{0},\ n \geq 2,& \\
(\mathbf{A}_{0,0} + \alpha_1\mathbf{M})\mathbf{q}_{0,n} + \mathbf{B}_{0,1}\mathbf{q}_{0,n-1} + \mathbf{B}_{-1,-1}\mathbf{q}_{1,n+1} = \mathbf{0},\ n \leq -2,& \\
\mathbf{A}_{0,0}\mathbf{q}_{0,0} + \mathbf{A}_{0,-1}\mathbf{q}_{0,1} + \mathbf{B}_{0,1}\mathbf{q}_{0,-1} = \mathbf{0},& \\
(\mathbf{A}_{0,0} + \alpha_2\mathbf{M})\mathbf{q}_{0,1} + \mathbf{A}_{0,-1}\mathbf{q}_{0,2} + \mathbf{A}_{-1,1}\mathbf{q}_{1,0} + \mathbf{A}_{0,1}\mathbf{q}_{0,0} = \mathbf{0},& \\
(\mathbf{A}_{0,0} + \alpha_1\mathbf{M})\mathbf{q}_{0,-1} + \mathbf{B}_{0,1}\mathbf{q}_{0,-2} + \mathbf{B}_{-1,-1}\mathbf{q}_{1,0} + \mathbf{B}_{0,-1}\mathbf{q}_{0,0} = \mathbf{0}.&
\end{aligned}$$

where

$$\mathbf{A}_{1,-1} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \mathbf{A}_{0,-1} = \mathbf{B}_{-1,-1} = \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix}, \mathbf{A}_{0,0} = \begin{pmatrix} -\lambda & \mu \\ \lambda & -(\lambda+\mu) \end{pmatrix},$$
$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{A}_{-1,1} = \mathbf{B}_{0,1} = \begin{pmatrix} 0 & 0 \\ \alpha_1 & 0 \end{pmatrix}, \mathbf{A}_{0,1} = (1-q)\mathbf{A}_{1,-1},\ \mathbf{B}_{0,-1} = q\mathbf{A}_{1,-1}$$

2.1 Preliminary Results

In the following we provide necessary conditions for the system to be stable.

Proposition 1. *Let* $p_{.,0}(0) = \sum_{i=0}^{\infty} p_{i,0}(0)$, $p_{0,.}(0) = \sum_{j=0}^{\infty} p_{0,j}(0)$. *Then,*

$$P(C=1) = \frac{\lambda}{\mu} \ \textit{and}\ \frac{\alpha_1 p_{0,.}(0) + \alpha_2 p_{.,0}(0)}{\alpha_1+\alpha_2} = 1 - \frac{\lambda(\lambda+\alpha_1+\alpha_2)}{\mu(\alpha_1+\alpha_2)} \qquad (3)$$

Proof. For each $i = 0, 1, 2, \ldots$ we consider the cut between the states $\{Q_1 = i, C = 1\}$ and $\{Q_1 = i+1, C = 0\}$. This yields

$$\lambda\mathbb{P}(J=1, Q_1=i, C=1) = \alpha_1 p_{i+1,.}(0).$$

Summing for all $i = 0, 1, \ldots$ results in

$$\lambda\mathbb{P}(J=1, C=1) = \alpha[\mathbb{P}(C=0) - p_{0,.}(0)], \qquad (4)$$

and similarly,

$$\lambda\mathbb{P}(J=2, C=1) = \alpha[\mathbb{P}(C=0) - p_{.,0}(0)]. \qquad (5)$$

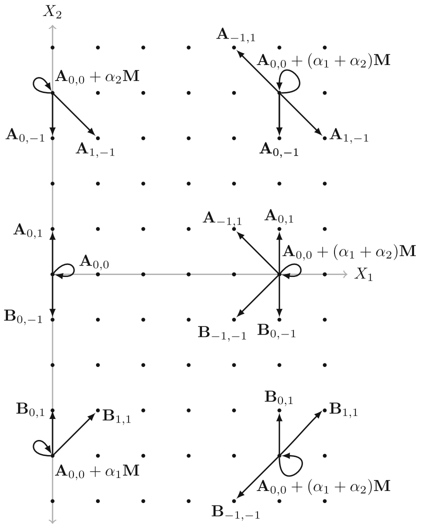

Fig. 1. Transition rate diagram of the transformed process.

Summing (4), (5) yields

$$\lambda\mathbb{P}(C=1)=(\alpha_1+\alpha_2)\mathbb{P}(C=0)-[\alpha_1 p_{0,.}(0)+\alpha_2 p_{0,.}(0)]. \tag{6}$$

Summing the first in (1) for all $i,j\geq 0$ yields

$$(\lambda+\alpha_1+\alpha_2)\mathbb{P}(C=0)-\mu\mathbb{P}(C=1)=\alpha_1 p_{0,.}(0)+\alpha_2 p_{0,.}(0). \tag{7}$$

Using (7), (6) and having in mind that $\mathbb{P}(C=1)+\mathbb{P}(C=0)=1$ we derive $\mathbb{P}(C=1)=\lambda/\mu$. Substituting back in (6) we finally derive

$$\frac{\alpha_1 p_{0,.}(0)+\alpha_2 p_{.,0}(0)}{\alpha_1+\alpha_2}=1-\frac{\lambda(\lambda+\alpha_1+\alpha_2)}{\mu(\alpha_1+\alpha_2)}\geq 0.$$

It can be easily seen that if $\rho:=\frac{\lambda(\lambda+\alpha_1+\alpha_2)}{\mu(\alpha_1+\alpha_2)}=1$, then the system is not stable, i.e., $p_{i,j}(0)=p_{i,j}(1)=0$, $i,j=0,1,\ldots$, and further details are omitted.

Assume hereon that $\rho<1$. In the following sections we will also show that this condition is also sufficient for the system to stable. We now turn our attention to show that the equilibrium probabilities of the transformed process can be written as an infinite series of product forms.

2.2 Dimensionality Reduction

The equilibrium Eq. (2) have a similar structure as Eqs. (1–10) in [3], although in matrix form due to the modulation. Our aim is to exploit the structure of the balance equations, and proceed with a dimensionality reduction, which allows for applying the CM to a transformed two-dimensional random walk constructed by the original one. Such an operation reduces to the half the number of the equilibrium equations that we have to solve through the CM. More precisely:

1. Express the equilibrium probabilities for the idle states in terms of the equilibrium probabilities for the busy states.
2. Substitute the expressions derived in step 1, to the equilibrium equations that refer to the busy states. The resulting equilibrium equations are of the same form as those in [3]. In particular, they describe a standard (i.e., non-retrial) JSQ model of two queues with *appropriate* arrival and service rates.
3. Apply the CM to obtain the invariant measure of the transformed two-dimensional random walk.
4. Use the expressions derived in step 1 and 3 to obtain the equilibrium probabilities of the original random walk.

Simple computations on the balance equations leads to the following relation that relate the idle and the busy states for $n \in \mathbb{Z}$,

$$q_{m,n}(0) = \begin{cases} \frac{\mu}{\lambda+\alpha_1+\alpha_2} q_{m,n}(1), & m > 0, \\ \frac{\mu}{\lambda+\alpha_1 1_{\{n\geq 1\}}+\alpha_2 1_{\{n\leq -1\}}} q_{0,n}(1), & m = 0, \end{cases} \tag{8}$$

Next, we substitute (8) in the rest of equilibrium equations that refer to the busy states. For convenience, set $p_{m,n} := q_{m,n}(1)$, and let $\rho := \frac{\lambda(\lambda+\alpha_1+\alpha_2)}{\mu(\alpha_1+\alpha_2)}$, $\rho_i := \frac{\lambda(\lambda+\alpha_i)}{\mu\alpha_i}$, $i = 1, 2$. Then, after the substitution, the equilibrium equations for the busy states are as follows:

$$\begin{aligned} p_{m,n}(\rho+1) = \rho p_{m-1,n+1} + \frac{\alpha_1}{\alpha_1+\alpha_2} p_{m+1,n-1} \\ + \frac{\alpha_2}{\alpha_1+\alpha_2} p_{m,n+1},\ m > 0, n > 1, \end{aligned} \tag{9}$$

$$\begin{aligned} p_{m,1}(\rho+1) = \rho q p_{m,0} + \rho p_{m-1,2} + \frac{\alpha_1}{\alpha_1+\alpha_2} p_{m+1,0} \\ + \frac{\alpha_2}{\alpha_1+\alpha_2} p_{m,2},\ m > 0, n = 1, \end{aligned} \tag{10}$$

$$p_{0,n}(\rho_2+1) = p_{0,n+1} + \frac{\alpha_1}{\alpha_1+\alpha_2} \frac{\rho_2}{\rho} p_{1,n-1},\ n > 1, \tag{11}$$

$$p_{0,1}(\rho_2+1)=\rho_2 q p_{0,0}+p_{0,2}+\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\rho_2}{\rho}p_{1,0}, \tag{12}$$

$$\begin{aligned}p_{m,0}(\rho+1)=\rho(p_{m-1,1}+p_{m-1,-1})+\frac{\alpha_1}{\alpha_1+\alpha_2}p_{m+1,-1}\\+\frac{\alpha_2}{\alpha_1+\alpha_2}p_{m,1},\ m>0,\end{aligned} \tag{13}$$

$$p_{0,0}=\frac{1}{\rho_1}p_{0,-1}+\frac{1}{\rho_2}p_{1,0}, \tag{14}$$

$$\begin{aligned}p_{m,n}(\rho+1)=\rho p_{m-1,n+1}+\frac{\alpha_1}{\alpha_1+\alpha_2}p_{m,n-1}\\+\frac{\alpha_2}{\alpha_1+\alpha_2}p_{m+1,n+1},\ m>0,n<-1,\end{aligned} \tag{15}$$

$$\begin{aligned}p_{m,-1}(\rho+1)=\rho(1-q)p_{m,0}+\rho p_{m-1,-2}+\frac{\alpha_1}{\alpha_1+\alpha_2}p_{m,-2}\\+\frac{\alpha_2}{\alpha_1+\alpha_2}p_{m+1,0},\ m>0,n=-1,\end{aligned} \tag{16}$$

$$p_{0,n}(\rho_1+1)=p_{0,n-1}+\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\rho_1}{\rho}p_{1,n+1},\ n<-1, \tag{17}$$

$$p_{0,-1}(\rho_1+1)=\rho_1(1-q)p_{0,0}+p_{0,-2}+\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\rho_1}{\rho}p_{1,0}, \tag{18}$$

Note that Eqs. (9)–(18) has the same form as Eqs. (1–10) in [3], a fact that allows for applying the CM and derive a formal solution (i.e., meaning that for the moment we do not pay attention to its convergence) to the equilibrium Eqs. (9)–(18). Then, using the derived formal solution, and (8), we can obtain the stationary distribution of the original system.

3 The Compensation Method

CM attempts to solve the balance equations by a linear combination of product-form terms. First, it characterizes a sufficiently rich basis of product-form solutions satisfying the balance equations in the interior of the state space. Subsequently this basis is used to construct a linear combination that also satisfies the equations for the boundary states. Note that the basis contains uncountably many elements. The procedure to select the appropriate elements is based on a compensation argument: after introducing the first term, countably many terms may subsequently be added so as to alternatingly compensate for the error on one of the two boundaries.

CM starts with the solution describing the asymptotic behaviour of $p_{m,n}$ as $m\to\infty$. In Theorem 4.1, [9] we shown (for a more general model with additional dedicated arrival streams) that $p_{m,n}:=q_{m,n}(1)\sim\gamma_0^m x_n(\gamma_0)$, as $m\to\infty$, for $\gamma_0:=\rho^2$, where ρ equals the decay rate of the corresponding single orbit system with retrial rate equal to $\alpha_1+\alpha_2$. The form of the asymptotic results in [9] helps us to assume that $p_{m,n}=\gamma^m\delta^n$ satisfies (9), and $p_{m,n}=$

$\gamma^m\delta^{-n}$ (15), respectively, i.e., the equilibrium equations at the interior of the transformed state space. The next lemma characterizes a continuum of product-forms satisfying the inner equations.

Lemma 1. *i) The product $\gamma^m\delta^n$ is a solution of (9) iff*

$$\gamma\delta(\rho+1)=\rho\delta^2+\tfrac{\alpha_1}{\alpha_1+\alpha_2}\gamma^2+\tfrac{\alpha_2}{\alpha_1+\alpha_2}\gamma\delta^2. \tag{19}$$

i) The product $\gamma^m\delta^{-n}$ is a solution of (15) iff

$$\gamma\delta(\rho+1)=\rho\delta^2+\tfrac{\alpha_2}{\alpha_1+\alpha_2}\gamma^2+\tfrac{\alpha_1}{\alpha_1+\alpha_2}\gamma\delta^2. \tag{20}$$

Proof. The proof is direct by substituting $p_{m,n}=\gamma^m\delta^n$ in (9), and $p_{m,n}=\gamma^m\delta^{-n}$ in (15).

For fixed γ, (19) is solved by $X_{\pm}(\gamma)=\gamma\frac{\rho+1\pm\sqrt{(\rho+1)^2-\frac{4\alpha_1(\rho+\frac{\alpha_2\gamma}{\alpha_1+\alpha_2})}{\alpha_1+\alpha_2}}}{2(\rho+\frac{\alpha_1\gamma}{\alpha_1+\alpha_2})}$. Let also by $Y_{\pm}(\gamma)$ the roots of (19) for fixed δ. Similarly $x_{\pm}(\gamma)$, $y_{\pm}(\delta)$ are the roots of (20) for fixed γ and δ, respectively.

Lemma 1 characterizes basic solutions satisfying (9), (15). The next lemma specifies the initial solution satisfying (9), (10), (13), (15), (16).

Lemma 2. *(Initial solution) For $\gamma_0=\rho^2$, let $\delta_1:=X_-(\rho^2)=\frac{\alpha_1\rho^2}{\alpha_1+\alpha_2(1+\rho)}$, $\delta_2:=x_-(\rho^2)=\frac{\alpha_2\rho^2}{\alpha_2+\alpha_1(1+\rho)}$, such that $|\delta_1|,|\delta_2|\in(0,\rho^2)$. Then, the solution*

$$z_{m,n}=\begin{cases} d_1\gamma_0^m\delta_1^n, & m\geq 0, n\geq 1,\\ \gamma_0^m, & m\geq 0, n=0,\\ d_2\gamma_0^m\delta_2^{-n}, & m\geq 0, n\leq -1,\end{cases} \tag{21}$$

satisfies (9), (10), (13), (15), (16), iff $d_1=\frac{(\alpha_1+\alpha_2)\rho q+\alpha_1\gamma}{\alpha_1\gamma_0}$, $d_2=\frac{(\alpha_1+\alpha_2)\rho(1-q)+\alpha_2\gamma_0}{\alpha_2\gamma}$.

The initial solution (21) does not satisfy the positive and negative vertical boundary equations (11), (17). To compensate for the error produced by $d_1\gamma_0^m\delta_1^n$, we seek for c_1, γ, δ such that $d_1\gamma_0^m\delta_1^n+d_1c_1\gamma^m\delta_1^n$ satisfies (11). We need to take $\delta=\delta_1$, $\gamma=\gamma_1:=Y_-(\delta_1)$. Insert $d_1\gamma_0^m\delta_1^n+d_1c_1\gamma_1^m\delta_1^n$ in (11), we obtain

$$c_1=-\frac{(1+\rho_2)\delta_1-\delta_1^2-\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\rho_2}{\rho}\gamma_0}{(1+\rho_2)\delta_1-\delta_1^2-\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\rho_2}{\rho}\gamma_1}. \tag{22}$$

Similarly, we can compensate for the error produced in the negative vertical boundary by $d_2\gamma_0^m\delta_2^{-n}$, i.e., by adding a term $d_2c_2\gamma_2^m\delta_2^n$ with $\gamma_2:=y_-(\delta_2)$, and a similar expression as in (22) is derived. In general, the vertical compensation step is summarized in the following lemma.

Lemma 3. *(Vertical compensation (VC)) i) Let $z_{m,n}=Y_+^m(\delta)\delta^n+cY_-^m(\delta)\delta^n$, $m\geq 0$, $n>0$. Then, $z_{m,n}$ satisfies (9) and (11) if c satisfies*

$$c=-\frac{(1+\rho_2)\delta-\delta^2-\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\rho_2}{\rho}Y_-(\delta)}{(1+\rho_2)\delta-\delta^2-\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\rho_2}{\rho}Y_+(\delta)}.$$

ii) Let $z_{m,n} = y_+^m(\delta)\delta^{-n} + cy_-^m(\delta)\delta^{-n}$, $m \geq 0$, $n < 0$. *Then,* $z_{m,n}$ *satisfies* (15) *and* (17) *if* c *satisfies*

$$c = -\frac{(1+\rho_1)\delta-\delta^2-\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\rho_1}{\rho}y_-(\delta)}{(1+\rho_1)\delta-\delta^2-\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\rho_1}{\rho}y_+(\delta)}.$$

Proof. The proof is similar to the one in [3, Lemma 2] and further details are omitted.

The updated solution after the VC step, does not satisfy the horizontal boundary Eqs. (10), (13), (16), so we need to compensate for the error produced by the VC step. This is done separately for each of the terms used to compensate the error in the vertical boundary, i.e., for $d_1c_1\gamma_1^m\delta_1^n$, $d_2c_2\gamma_2^m\delta_2^{-n}$. Specifically, for the term $d_1c_1\gamma_1^m\delta_1^n$, we seek for d_3, d_4, f_1, δ_3, δ_4 such that

$$\begin{cases} d_1c_1\gamma_1^m\delta_1^n + d_3c_1\gamma_1^m\delta_3^n, & m \geq 0, n > 0,\\ d_4c_1\gamma_1^m\delta_4^{-n}, & m \geq 0, n < 0,\\ c_1f_1\gamma_1^m, & m \geq 0, n = 0,\end{cases}$$

with $\delta_3 := X_-(\gamma_1)$, $\delta_4 := x_-(\gamma_1)$, such that $\delta_1 > \gamma_1 > \delta_3 > 0$, and the constants d_3, d_4, f_1 are derived by substituting in (10), (13), (16). The formal compensation step on the horizontal boundary is outlined in the following lemma.

Lemma 4. *(Horizontal compensation (HC)) i) Let*

$$z_{m,n} = \begin{cases} k_1\gamma^m X_+(\gamma)^n + k_2\gamma^m X_-(\gamma)^n, & m \geq 0, n > 0,\\ k_3\gamma^m x_-(\gamma)^{-n}, & m \geq 0, n < 0,\\ k_4\gamma^m, & m \geq 0.\end{cases}$$

Then, $z_{m,n}$ *satisfies* (9), (15), (10), (13), (16) *if*

$$\begin{aligned} k_2 &= -k_1\frac{\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_-(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_+(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)}{\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_+(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_+(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)},\\ k_3 &= -k_1\frac{\alpha_1(\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2))(\frac{1}{X_-(\gamma)}-\frac{1}{X_+(\gamma)})}{\alpha_2[\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_+(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_+(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)]},\\ k_4 &= k_3\frac{\alpha_2\gamma}{\rho(1-q)(\alpha_1+\alpha_2)+\alpha_2\gamma} = (k_1+k_2)\frac{\alpha_1\gamma}{\rho q(\alpha_1+\alpha_2)+\alpha_1\gamma}.\end{aligned}$$

ii) Let

$$z_{m,n} = \begin{cases} k_1\gamma^m x_+(\gamma)^{-n} + k_2\gamma^m x_-(\gamma)^n, & m \geq 0, n < 0,\\ k_3\gamma^m X_-(\gamma)^n, & m \geq 0, n > 0,\\ k_4\gamma^m, & m \geq 0.\end{cases}$$

Then, $z_{m,n}$ *satisfies* (9), (15), (10), (13), (16) *if*

$$\begin{aligned} k_2 &= -k_1\frac{\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_+(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_-(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)}{\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_+(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_+(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)},\\ k_3 &= -k_1\frac{\alpha_2(\alpha_1\gamma+q\rho(\alpha_1+\alpha_2))(\frac{1}{x_-(\gamma)}-\frac{1}{x_+(\gamma)})}{\alpha_1[\frac{\alpha_1\gamma+q\rho(\alpha_1+\alpha_2)}{X_+(\gamma)}+\frac{\alpha_2\gamma+(1-q)\rho(\alpha_1+\alpha_2)}{x_+(\gamma)}-(\alpha_1+\alpha_2)(1+\rho)]},\\ k_4 &= (k_1+k_2)\frac{\alpha_2\gamma}{\rho(1-q)(\alpha_1+\alpha_2)+\alpha_2\gamma} = k_3\frac{\alpha_1\gamma}{\rho q(\alpha_1+\alpha_2)+\alpha_1\gamma}.\end{aligned}$$

Proof. The proof is similar to the one in [3, Lemma 3] and further details are omitted.

The CM leads to an expressions for the $q_{m,n}(1)$ (which is equal to the equilibrium probability for the busy states of the model at hand), written as an infinite sum of terms in the upper and lower quadrant. These γs and δs are represented by a binary tree as given in [3, Fig. 2, p. 14]. The γs and δs are numbered from the root and from left to right. We use the same notation as in [3]: $\delta_{l(i)}$ = the left descendant of γ, $\delta_{r(i)}$ = the right descendant of γ, $\gamma_{p(i)}$ = the $\gamma-$parent of δ_i. Let $L = \{l(i), i = 0, 1, \ldots\}$, $R = \{r(i), i = 0, 1, \ldots\}$, with $l(i) = 2i + 1$, $r(i) = 2i + 2$, $p(i) = \lfloor \frac{i-1}{2} \rfloor$, $i = 0, 1, \ldots$. Starting from $\gamma_0 = \rho^2$, $\rho = \frac{\lambda(\lambda+\alpha_1+\alpha_2)}{\mu(\alpha_1+\alpha_2)}$. For $i \geq 0$ the left descendant $\delta_{l(i)}$ of γ_i is defined as the smaller root of Eq. (19) with $\gamma = \gamma_i$; and the right descendant $\delta_{r(i)}$ of γ_i is defined as the smaller root of Eq. (20) with $\gamma = \gamma_i$. The descendant $\gamma_{l(i)}$ of $\delta_l(i)$ is defined as the smaller root of (19) with $\delta = \delta_l(i)$, and descendant $\gamma_{r(i)}$ of $\delta_r(i)$ is defined as the smaller root of (20) with $\delta = \delta_r(i)$ (Note also that $0 < \gamma < 1$). Using induction, we show that $\delta_{l(i)} = X_-(\gamma_i)$, $\delta_{r(i)} = x_-(\gamma_i)$, $\gamma_l(i) = Y_-(\delta_{l(i)})$, $\gamma_{r(i)} = y_-(\delta_{r(i)})$, and $\gamma_i > \delta_{l(i)} > \gamma_{l(i)} > 0$, $\gamma_i > \delta_{r(i)} > \gamma_{r(i)} > 0$, so that $\{\gamma_i\}$, $\{\delta_i\}$ form a decreasing positive tree. Then, define the infinite sums

$$z_{m,n} = \sum_{i \in L} d_i (c_{p(i)} \gamma_{p(i)}^m + c_i \gamma_i^m) \delta_i^n, \; m \geq 0, n > 0, \tag{23}$$

$$z_{m,n} = \sum_{i \in R} d_i (c_{p(i)} \gamma_{p(i)}^m + c_i \gamma_i^m) \delta_i^{-n}, \; m \geq 0, n < 0. \tag{24}$$

By linearity the sum $p_{m,n}$ satisfies the conditions (9), (15), and we have to define coefficients d_i, c_i to satisfy also the boundaries (10), (11), (13), (16), (17). The coefficients c_i are such that $(c_{p(i)}\gamma_{p(i)}^m + c_i \gamma_i^m)\delta_i^n$ satisfies (10) for all $i \in L$ and such that $(c_{p(i)}\gamma_{p(i)}^m + c_i \gamma_i^m)\delta_i^{-n}$ satisfies (16) for all $i \in R$; see Lemma 3. So c_i can be obtained by $c_{p(i)}$ with initial condition $c_0 = 1$. We now rewrite (23), (24) as

$$z_{m,n} = \begin{cases} d_1 \gamma_0^m \delta_1^n + \sum_{i \in L} c_i \gamma_i^m (d_i \delta_i^n + d_{l(i)} \delta_{l(i)}^n) + \sum_{i \in R} c_i d_{l(i)} \gamma_i^m \delta_{l(i)}^n), & m \geq 0, n > 0, \\ d_2 \gamma_0^m \delta_2^{-n} + \sum_{i \in R} c_i \gamma_i^m (d_i \delta_i^{-n} + d_{r(i)} \delta_{r(i)}^{-n}) + \sum_{i \in L} c_i d_{r(i)} \gamma_i^m \delta_{r(i)}^{-n}), & m \geq 0, n < 0, \end{cases} \tag{25}$$

and define

$$z_{m,0} = \gamma_0^m + \textstyle\sum_{i \in L} c_i f_i \gamma_i^m + \sum_{i \in R} c_i f_i \gamma_i^m, \; m \geq 0. \tag{26}$$

The coefficients d_i and f_i are such that for $i \in L$ the terms $\gamma_i^m(d_i\delta_i^n + d_{l(i)}\delta_{l(i)}^n)$, $d_{r(i)}\gamma_i^m\delta_{r(i)}^{-n}$, $f_i\gamma_i^m$, satisfies the horizontal boundary equations (i.e., by using Lemma 4), while for $i \in R$, the same conditions are satisfied by $\gamma_i^m d_{l(i)}\delta_{l(i)}^n$, $\gamma_i^m(d_i\delta_i^{-n} + d_{r(i)}\delta_{r(i)}^{-n})$, and $f_i\gamma_i^m$, starting with d_i, $f_0 = 1$ as in Lemma 2. Application of Lemma 4 yields that the coefficients $d_{l(i)}$, $d_{r(i)}$ can be expressed in terms of d_i (e.g., see [3, pp. 16–17]).

We may conclude that (25) satisfies the equilibrium Eqs. (9)–(11), (13), (15)–(17), except (12), (14), (18). Substituting (23) in (12) and make use of how we derive c_i through Lemma 3, we can show that (25) satisfies (12). Similarly, we can show that (25) is a formal solution to all equilibrium Eqs. (9)(18).

The next step is to formally prove that the series as formulated in (23), (24), (26), are absolutely convergent. However, these series may diverge for small values of m, n. Using similar arguments as in [3] we can show the following theorem:

Theorem 1. *There is an integer N such that:*

1. *the series (23) converges absolutely for all $m \geq 0$, $n \geq 0$ with $m+n > N$,*
2. *the series (24) converges absolutely for all $m \geq 0$, $n \leq 0$ with $m-n > N$,*
3. *the series (26) converges absolutely for all $m \geq N$,*
4. $\sum_{m \geq 0,\, m+|n|>N} |z_{m,n}| < \infty$.

The proof of Theorem 1 needs the proof of a series of results. First, we need to show that γ_i, δ_i decrease exponentially fast. So we need the following lemma:

Lemma 5. *For all $\gamma \in (0, \rho^2]$,*

1. *the ration $X_+(\gamma)/\gamma$ is decreasing and $X_-(\gamma)/\gamma$ is increasing.*
2. $X_+(\gamma) > \gamma > X_-(\gamma) > 0$.

The same properties hold also for $Y_\pm(\delta)$, $x_\pm(\gamma)$ and $y_\pm(\delta)$.

Proof. We only focus on $X_\pm(\gamma)$. Since $0 < \gamma \leq \gamma_0 = \rho^2$ the discriminant of (19) for fixed γ is strictly positive since

$$(\rho+1)^2 - \frac{4\alpha_1(\rho+\frac{\alpha_2\gamma}{\alpha_1+\alpha_2})}{\alpha_1+\alpha_2} \geq (\rho+1)^2 - \frac{4\alpha_1(\rho+\frac{\alpha_2\rho^2}{\alpha_1+\alpha_2})}{\alpha_1+\alpha_2} = (\tfrac{\rho}{\alpha_1+\alpha_2})^2(\alpha_1+\alpha_2+(\alpha_2-\alpha_1)\rho)^2 > 0,$$

since $\alpha_1 < \alpha_2$. Thus, $X_+(\gamma)$, $X_-(\gamma)$ are distinct positive zeros and $X_+(\gamma)/\gamma$ is decreasing. Moreover,

$$\frac{X_-(\gamma)}{\gamma} = \frac{\alpha_1}{\rho(\alpha_1+\alpha_2)}\frac{\gamma}{X_+(\gamma)} = \frac{2\alpha_1}{(\alpha_1+\alpha_2)(1+\rho)+\sqrt{((\alpha_1+\alpha_2)(1+\rho))^2-4\alpha_1(\rho(\alpha_1+\alpha_2)+\alpha_2\gamma)}},$$

which is increasing in γ. Since $\gamma \leq \gamma_0$,

$$\frac{X_+(\gamma)}{\gamma} \geq \frac{X_+(\gamma_0)}{\gamma_0} = \frac{1}{\rho} > 1,\ \frac{X_-(\gamma)}{\gamma} \geq \frac{X_-(\gamma_0)}{\gamma_0} = \frac{\alpha_1}{\alpha_1+\alpha_2(1+\rho)} < 1.$$

An immediate corollary, which can be proved by induction states that

Corollary 1. *For all $i = 0, 1, \ldots$, $\gamma_0 \geq \gamma_i > \delta_{l(i)} > \gamma_{l(i)} > 0$, $\gamma_0 \geq \gamma_i > \delta_{r(i)} > \gamma_{r(i)} > 0$, where*

$$\begin{aligned}
\delta_{l(i)} &\leq \frac{\alpha_1}{\alpha_1+\alpha_2(1+\rho)}\gamma_i,\ \ \gamma_{l(i)} \leq \frac{2\rho(\alpha_1+\alpha_2)\delta_{l(i)}}{(\rho+1)(\alpha_1+\alpha_2)-\alpha_2\delta_1+\sqrt{((\rho+1)(\alpha_1+\alpha_2)-\alpha_2\delta_1)^2-4\alpha_1(\alpha_1+\alpha_2)\rho}},\\
\delta_{r(i)} &\leq \frac{\alpha_2}{\alpha_1+\alpha_2(1+\rho)}\gamma_i,\ \gamma_{r(i)} \leq \frac{2\rho(\alpha_1+\alpha_2)\delta_{r(i)}}{(\rho+1)(\alpha_1+\alpha_2)-\alpha_1\tilde{\delta}_1+\sqrt{((\rho+1)(\alpha_1+\alpha_2)-\alpha_1\tilde{\delta}_1)^2-4\alpha_1(\alpha_2+\alpha_2)\rho}},
\end{aligned}$$

for $\delta_1 := X_-(\gamma_0)$, $\tilde{\delta}_1 := x_-(\gamma_0)$.

Proof. Based on Lemma 5, $0 < \delta_{l(i)} = X_-(\gamma_i) \leq \frac{X_-(\gamma_0)}{\gamma_0}\gamma_i = \frac{\alpha_1}{\alpha_1+\alpha_2(1+\rho)}\gamma_i$. Moreover, $0 < \delta_{l(i)} = X_-(\gamma_i) \leq X_-(\gamma_0) = \delta_1$, and thus, $\gamma_{l(i)} = Y_-(\delta_{l(i)}) \leq \frac{Y_-(\delta_1)}{\delta_1}\delta_{l(i)}$, which leads after some algebra in the above expression. Similarly we can handle the other assertion; see also [3, p. 20].

Then, the asymptotic behaviour of γ_i, δ_i is stated in the next lemma.

Lemma 6. *If the depth of γ_i in the parameter tree tends to infinity, i.e., $\gamma_i \to 0$ as $i \to \infty$, then*

$$\frac{\delta_{l(i)}}{\gamma_i} \to \frac{1}{A_2}, \ \frac{\delta_{r(i)}}{\gamma_i} \to \frac{1}{a_2}$$

and if $\delta_{l(i)} \to 0$, $\delta_{r(i)} \to 0$, then,

$$\frac{\gamma_{l(i)}}{\delta_{l(i)}} \to A_1, \ \frac{\gamma_{r(i)}}{\delta_{r(i)}} \to a_1,$$

where

$$\begin{aligned}
A_1 &= \frac{(\alpha_1+\alpha_2)(\rho+1)-\sqrt{((\alpha_1+\alpha_2)(\rho+1))^2-4\alpha_1\rho(\alpha_1+\alpha_2))}}{2\alpha_1},\\
a_1 &= \frac{(\alpha_1+\alpha_2)(\rho+1)-\sqrt{((\alpha_1+\alpha_2)(\rho+1))^2-4\alpha_2\rho(\alpha_1+\alpha_2))}}{2\alpha_2},\\
A_2 &= \frac{(\alpha_1+\alpha_2)(\rho+1)+\sqrt{((\alpha_1+\alpha_2)(\rho+1))^2-4\alpha_1\rho(\alpha_1+\alpha_2))}}{2\alpha_1},\\
a_2 &= \frac{(\alpha_1+\alpha_2)(\rho+1)+\sqrt{((\alpha_1+\alpha_2)(\rho+1))^2-4\alpha_2\rho(\alpha_1+\alpha_2))}}{2\alpha_2}.
\end{aligned}$$

We further need to study the asymptotic behaviour of the coefficients c_i, d_i, following the lines in [3, pp. 21–24]. Then, these results ensure the proof of Theorem 1. The next theorem summarizes the main result.

Theorem 2. *For all m, n, such that $m \geq 0$, $m + |n| > N$, and for $m = N$, $n = 0$,*

$$\begin{aligned}
q_{m,n}(1) &= C^{-1} z_{m,n},\\
q_{m,n}(0) &= C^{-1} \times \begin{cases} \frac{\mu}{\lambda+\alpha_1+\alpha_2} z_{m,n}, & m > 0,\\ \frac{\mu}{\lambda+\alpha_1 1_{\{n\geq 1\}}+\alpha_2 1_{\{n\leq -1\}}} z_{0,n}, & m = 0, \end{cases}
\end{aligned} \tag{27}$$

where $z_{m,n}$ as given in (25), (26) and C^{-1} a normalization constant. The value of constant N depends on the system parameters. In the symmetric case, i.e., $\alpha_1 = \alpha_2$, $N = 0$. In case $N > 0$, the equilibrium equations for $m + |n| \leq N$ have to be solved numerically; see [3, Sect. 18].

4 Discussion on the Case of a Priority Line of a Finite Capacity

We briefly describe how we can handle the case where the main service station can handle at most $K > 1$ jobs, i.e., there is a finite priority line of capacity $K - 1$ in front of the service station. In such a scenario, arriving jobs are routed to the orbits if upon arrival they find the priority line fully occupied. Orbiting jobs can access the server only if upon a retrial they find the server idle, i.e., we do not allow orbiting jobs to access the priority line.

The matrix form of the equilibrium equations is as those in (2), where now the matrices are of order $(K+1)\times(K+1)$. We are still able to consider the dimensionality reduction, by expressing all the equilibrium probabilities $q_{m,n}(k)$, $k=0,1,\ldots,K-1$, in terms of $q_{m,n}(K)$. However, upon doing this operation, we will see that the resulting equations, i.e., the equations that result after the dimensionality reduction and are expresses in terms of $q_{m,n}(K)$, are more general than those in (9)–(18). In particular, the resulting two-dimensional random walk in the half plane has now fatter boundaries, and transitions to non-neighbour states, although the required assumptions to apply the compensation method are satisfied; to our best knowledge, the most related available literature refers to [2,17] that dealt with large jumps (i.e., to non-neighbour states) but only in the positive quadrant. In case $K>1$, the transformed equations after the dimensionality reduction contain transitions to non-neighbour states both in the positive and in the negative quadrant of the half plane. For example, for $K=3$, tedious computations yield

$$\begin{aligned}
&q_{m,n}(w)=s_{3-w}q_{m,n}(3)-s_{2-w}\left[q_{m-1,n+1}(3)1_{\{n\geq 1\}}+q_{m-1,n-1}(3)1_{\{n\leq -1\}}+\right.\\
&+qq_{m,0}(3)1_{\{n=-1\}}+(1-q)q_{m,0}(3)1_{\{n=1\}}\\
&\left.+(q_{m-1,1}(3)+q_{m-1,-1}(3))1_{\{n=0\}}\right],\ m>0,n\neq 0,w=1,2,\\
&q_{0,n}(w)=s_{3-w}q_{0,n}(3),n\neq 0,w=1,2,\\
&q_{m,n}(0)=\tfrac{\mu}{\lambda+\alpha_1 1_{\{n<0\}}+\alpha_2 1_{\{n>0\}}}q_{m,n}(1),\ m\geq 0,n\in\mathbb{Z}.
\end{aligned}$$

where

$$s_{3-w}=\tfrac{\sum_{m=0}^{3-w}\lambda^{3-w-m}\mu^m}{\lambda^{3-w}},\ w=1,2,3.$$

After the dimensionality reduction, the resulting inner equilibrium equations in the positive and negative quadrant, respectively, reads for $p_{m,n}:=q_{m,n}(3)$:

$$\begin{aligned}
p_{m,n}[s_2(\rho+1)-\tfrac{s_1(\lambda+\alpha_2)}{\alpha_1+\alpha_2}]+p_{m-1,n+2}\tfrac{\alpha_2 s_1}{\alpha_1+\alpha_2}&=p_{m-1,n+1}[s_1(\rho+1)-\tfrac{\lambda+\alpha_1+\alpha_2}{\alpha_1+\alpha_2}]\\
&+p_{m+1,n-1}\tfrac{s_2\alpha_1}{\alpha_1+\alpha_2}+p_{m,n+1}\tfrac{s_2\alpha_2}{\alpha_1+\alpha_2},\ m>0,n>2,\\
p_{m,n}[s_2(\rho+1)-\tfrac{s_1(\lambda+\alpha_1)}{\alpha_1+\alpha_2}]+p_{m-1,n-2}\tfrac{\alpha_1 s_1}{\alpha_1+\alpha_2}&=p_{m-1,n-1}[s_1(\rho+1)-\tfrac{\lambda+\alpha_1+\alpha_2}{\alpha_1+\alpha_2}]\\
&+p_{m,n-1}\tfrac{s_2\alpha_1}{\alpha_1+\alpha_2}+p_{m+1,n+1}\tfrac{s_2\alpha_2}{\alpha_1+\alpha_2},\ m>0,n<-2,
\end{aligned}$$

It can be easily seen from the above that for any state (m,n), $m>0$, $n>2$ (resp. $n<-2$) (of the transformed process) we can have transitions to $(m+1,n-2)$, $(m+1,n-1)$, $(m-1,n+1)$, $(m,n-1)$ (resp. to $(m+1,n+2)$, $(m+1,n+1)$, $(m-1,n-1)$, $(m,n+1)$).

Our intuition indicates that following similar arguments as in [2,17] we are able to apply the CM in the half plane. At that point, a crucial point is the choice of the initial candidate solution for the inner equilibrium equations, i.e., the initial γ_0, and δ_0. Note that by substituting the product $\gamma^m\delta^n$ in the above equations yields a cubic equation with respect to δ and a quadratic with respect to γ. Thus, it is first important to study the asymptotic behaviour of $q_{m,n}(K)$ as $m\to\infty$, and investigate whether it is of geometric form.

5 Conclusion and Future Work

In this work, by using a Markovian queueing network with retrials under the JSQ policy, we investigated the stationary behaviour of a certain non-homogeneous Markov modulated two-dimensional random walk. By exploiting its special structure, its invariant measure is obtained by applying the CM to a standard, i.e., non-modulated random walk in the half plane with a complicated vertical boundary behaviour. In particular, the resulting two-dimensional random walk describes the JSQ problem in two standard (i.e., non-retrial) queues with appropriate arrival/service rates. With such an operation, we reduce to the half the number of equilibrium equations to be solved through the CM.

The analysis we follow seems to be fully applicable for even more general cases, including: multiserver service station, Erlang (or PH) arrival streams, several stations that can accommodate a finite number of customers, as well as to incorporate the possibility of the bi-level JSQ policy, i.e., the possibility of retransmitting from orbit queues to the service node with the fewest number of jobs. The only limitation (due to the use of CM) relies on the number of orbit queues, which is restricted to two. Even with the above setting, we can ensure a dimensionality reduction, which allows to obtain the invariant measure through the CM by solving fewer (based on the model at hand) equilibrium equations.

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