Asymptotic Analysis of Modified Erlang-B System with Sensing Time and Stochastic Loss of Secondary Users

Kazuma Abe[1] and Tuan Phung-Duc[2(✉)]

[1] Graduate School of Science and Technology, University of Tsukuba, 1-1-1, Tennoudai, Tsukuba, Ibaraki, Japan
abe.kazuma.su@alumni.tsukuba.ac.jp

[2] Institute of Systems and Information Engineering, University of Tsukuba, Tsukuba, Japan
tuan@sk.tsukuba.ac.jp

Abstract. This paper considers a modified Erlang-B model for cognitive radio networks with both primary users (PUs) and secondary users (SUs). PUs have absolute priority over SUs, and are blocked whenever all channels are used by other PUs upon their arrivals. SUs must sense an idle channel upon their arrivals. If, after sensing, an SU finds an idle channel, it can occupy the channel immediately; otherwise, the SU may either sense again or leave the system forever. Under Poisson arrival assumptions (both PUs and SUs) and exponential sensing and service times, we formulate the system as a three-dimensional Markov chain, and consider an asymptotic regime when the sensing rate is extremely low. We prove that a scaled version of the number of sensing SUs converges to a deterministic process. Furthermore, we prove that the deterministic process has a unique stationary point which we use to approximate the mean number of sensing SUs and the distribution of the states of the channels. Numerical examples reveal that these approximations are highly accurate when the sensing rate is low.

Keywords: Retrial Queue · Cognitive wireless networks (CRN) · Erlang-B model · ODE

1 Introduction

Today's communication network carries very high traffic, making it essential to allocate limited resources efficiently. Therefore, cognitive radio networks (CRN) are expected to play a vital role in developing and expanding next-generation networks that address the bandwidth shortage problem. In CRN, unlicensed devices or users can transmit on the channels of licensed devices or users to satisfy their own spectrum demands as long as they ensure that the interference thus

E. Hyytiä and V. Kavitha (Eds.): VALUETOOLS 2022, LNICST 482, pp. 67–80, 2023.
https://doi.org/10.1007/978-3-031-31234-2_5

caused does not interrupt the activities of the licensed users. These unlicensed users are known as secondary users (SUs), and licensed users are called primary users (PUs).

Three types of spectrum-sharing paradigms for cognitive radio networks are widely discussed [5]: overlay access, underlay access and interweave access. In overlay access, an SU cooperatively relays PU transmissions, permitting the SU to transmit its own data concurrently with a PU. Underlay access allows the SU to transmit simultaneously with a PU, provided that the interference quantity at the PU receiver due to SU transmissions is below some certain predefined threshold. In interweave access, the SUs can use a channel only when it is not occupied by any PU.

In this paper, we focus on interweave access. SUs must sense the available channels before using the frequency bands. If an SU discovers an idle channel after sensing, it can access the channel and start transmission; otherwise, the SU must sense again until it finds an idle channel later or leave the system.

Queueing systems for cognitive radio networks were extensively studied [9]. The SU sensing process is similar to the retrial process in queueing systems. In these systems, arriving customers are blocked when the servers are already fully occupied. These blocked customers instead enter a virtual waiting room called the orbit (the sensing pool in CRN), and try repeatedly after a random waiting time until they obtain a server. For example, [11] is based on a multi-server retrial queue. [13] considered a model with a stochastic choice of channels and a finite number of simultaneously sensing users (i.e., a finite sensing pool). [3] considered the assumption that a newly arriving priority customer is queued in the buffer when all channels are busy with other priority customers. In [10], the size of the sensing pool is infinite.

In this paper, applying the asymptotic method [4,6–8] to our model, we focus on the situation where it takes SUs a long time to sense channel availability. In this case, the number of SUs in the sensing pool diverges, but a scaled version of this number converges to a deterministic process. The limiting results are used to approximate the mean number of sensing SUs and the distribution of the channel states.

It turns out that the main results for the lossless model in [1] can be extended to the current model. In particular, the stability of the deterministic process coincides with that of the system's underlying Markov chain. Furthermore, the uniqueness of the stationary solution of the deterministic process is guaranteed, which is further used to approximate the mean number of sensing SUs in the steady state.

The paper consists of six sections. In Sect. 2, we introduce the mathematical model and preliminary analysis. We consider the first-order approximation of our model in Sect. 3. In Sect. 4, we prove the main results for the stability condition and the uniqueness of the stationary solution of the deterministic process. Section 5 compares simulation results with the approximation obtained in Sect. 3. We conclude with a summary synthesis in Sect. 6.

2 Model Description

In this section, we describe an Erlang-B system with two types of users. PUs and SUs arrive at the system of c servers according to Poisson processes with rates λ_1 and λ_2, respectively. The service times are exponentially distributed with parameters μ_1 and μ_2, for PUs and SUs respectively. The arrival processes of both user types and the service times are assumed to be mutually independent. If all of the c channels are occupied by other PUs, a newly arriving PU is blocked. Hence, the model can be treated as Erlang-B from PU's point of view. Arriving SUs must enter the sensing pool and sense for available channels. The sensing times of the SUs follow the exponential distribution with rate σ and do not depend on other SUs. When an SU finishes sensing and cannot find an available channel, there is a probability p that SU will stay in the pool for another sensing; otherwise, it leaves the system with probability $1-p$. When a new PU arrives at the system where all c channels are occupied by other SUs and PUs, the PU terminates the transmission of an SU (if exists) and uses that channel. The interrupted SU must enter the sensing pool to sense again. The sensing pool is assumed infinite.

The model analyzed in [1] is the same as ours with $p = 1$. Note that the service time distribution of an interrupted SU is the same as that of a newly arrived SU because of the memorylessness property of exponential distributions.

Let us denote:

- $n_1(t)$: the number of PUs occupying channels at time t,
- $n_2(t)$: the number of SUs occupying channels at time t,
- $i(t)$: the number of SUs staying at the sensing pool at time t.

Let $P(n_1, n_2, i, t) = P\{n_1(t) = n_1,\, n_2(t) = n_2,\, i(t) = i\}$ be the joint probability distribution of the process $\{(n_1(t),\, n_2(t),\, i(t)) \mid t \geq 0\}$. Under the assumptions of the model, the process is a three-dimensional Markov chain. The transition rates from a state $x = (n_1, n_2, i)$ to another state y $(x \neq y)$ are given as follows.

$$q_{x,y} = \begin{cases} \lambda_1 & \text{if } y = (n_1+1, n_2, i), n_1 + n_2 \leq c-1, \\ \lambda_1 & \text{if } y = (n_1+1, n_2-1, i+1), n_1 + n_2 = c, n_2 \geq 1, \\ \lambda_2 & \text{if } y = (n_1, n_2, i+1), \\ n_1\mu_1 & \text{if } y = (n_1-1, n_2, i), n_1 \geq 1, \\ n_2\mu_2 & \text{if } y = (n_1, n_2-1, i), n_2 \geq 1, \\ i\sigma & \text{if } y = (n_1, n_2+1, i-1), n_1 + n_2 \leq c-1, \\ (1-p)i\sigma & \text{if } y = (n_1, n_2, i-1), n_1 + n_2 = c, 0 \leq p \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The purpose of the paper is to obtain the scaling limits of $(n_1(t), n_2(t))$ and $i(t)$. We solve this problem via the first-order approximation under the asymptotic condition when the sensing time is long: $\sigma \to 0$.

The Kolmogorov forward equations are:

(i) $n_1 + n_2 = 0$

$$\begin{aligned}\frac{dP(0,0,i,t)}{dt} =& -(\lambda_1 + \lambda_2 + i\sigma)P(0,0,i,t) + \lambda_2 P(0,0,i-1,t)\\ &+ \mu_1 P(1,0,i,t) + \mu_2 P(0,1,i,t).\end{aligned}$$

(ii) $1 \le n_1 + n_2 \le c-1$

$$\begin{aligned}\frac{dP(n_1,n_2,i,t)}{dt} =& -(\lambda_1 + \lambda_2 + n_1\mu_1 + n_2\mu_2 + i\sigma)P(n_1,n_2,i,t)\\ &+ \lambda_2 P(n_1,n_2,i-1,t) + \lambda_1 P(n_1-1,n_2,i,t)\\ &+ (n_1+1)\mu_1 P(n_1+1,n_2,i,t)\\ &+ (n_2+1)\mu_2 P(n_1,n_2+1,i,t)\\ &+ (i+1)\sigma P(n_1,n_2-1,i+1,t).\end{aligned}$$

(iii) $n_1 + n_2 = c, n_2 \ge 1$

$$\begin{aligned}\frac{dP(n_1,n_2,i,t)}{dt} =& -(\lambda_1 + \lambda_2 + (1-p)i\sigma + n_1\mu_1 + n_2\mu_2)P(n_1,n_2,i,t)\\ &+ \lambda_1\{P(n_1-1,n_2,i,t) + P(n_1-1,n_2+1,i-1,t)\}\\ &+ \lambda_2 P(n_1,n_2,i-1,t) + (1-p)(i+1)\sigma P(n_1,n_2,i+1,t)\\ &+ (i+1)\sigma P(n_1,n_2-1,i+1,t).\end{aligned}$$

(iv) $n_1 = c$

$$\begin{aligned}\frac{dP(c,0,i,t)}{dt} =& -(\lambda_2 + (1-p)i\sigma + c\mu_1)P(c,0,i,t) + \lambda_2 P(c,0,i-1,t)\\ &+ (1-p)(i+1)\sigma P(n_1,0,i+1,t)\\ &+ \lambda_1\{P(c-1,0,i,t) + P(c-1,1,i-1,t)\}.\end{aligned}$$

We use the convention that $P(n_1,n_2,i,t) = 0$ for $n_1 < 0$ or $n_2 < 0$ or $i < 0$ and let $j = \sqrt{-1}$ for denote imaginary number. We define the partial characteristic function as

$$H(n_1,n_2,s,t) = \sum_{i=0}^{\infty} e^{jsi} P(n_1,n_2,i,t).$$

We obtain the following differential equations:

(i) $n_1 + n_2 = 0$

$$\begin{aligned}\frac{\partial H(0,0,s,t)}{\partial t} =& -(\lambda_1 + \lambda_2)H(0,0,s,t) + e^{js}\lambda_2 H(0,0,s,t)\\ &+ \mu_1 H(1,0,s,t) + \mu_2 H(0,1,s,t) + j\sigma \frac{\partial H(0,0,s,t)}{\partial s}. \qquad (1)\end{aligned}$$

(ii) $1 \leq n_1 + n_2 \leq c - 1$

$$\begin{aligned}\frac{\partial H(n_1, n_2, s, t)}{\partial t} =& -(\lambda_1 + \lambda_2 + n_1\mu_1 + n_2\mu_2)H(n_1, n_2, s, t)\\ &+ e^{js}\lambda_2 H(n_1, n_2, s, t) + \lambda_1 H(n_1 - 1, n_2, s, t)\\ &+ (n_1 + 1)\mu_1 H(n_1 + 1, n_2, s, t)\\ &+ (n_2 + 1)\mu_2 H(n_1, n_2 + 1, s, t)\\ &+ j\sigma\frac{\partial H(n_1, n_2, s, t)}{\partial s} - e^{-js} j\sigma \frac{\partial H(n_1, n_2 - 1, s, t)}{\partial s}. \quad (2)\end{aligned}$$

(iii) $n_1 + n_2 = c, n_2 \geq 1$

$$\begin{aligned}\frac{\partial H(n_1, n_2, s, t)}{\partial t} =& -(\lambda_1 + \lambda_2 + n_1\mu_1 + n_2\mu_2)H(n_1, n_2, s, t)\\ &+ e^{js}\lambda_2 H(n_1, n_2, s, t) + \lambda_1\{H(n_1 - 1, n_2, s, t)\\ &+ e^{js} H(n_1 - 1, n_2 + 1, s, t)\}\\ &+ j(1-p)\sigma(1 - e^{-js})\frac{\partial H(n_1, n_2, s, t)}{\partial s}\\ &- e^{-js} j\sigma \frac{\partial H(n_1, n_2 - 1, s, t)}{\partial s}. \quad (3)\end{aligned}$$

(iv) $n_1 = c$

$$\begin{aligned}\frac{\partial H(c, 0, s, t)}{\partial t} =& -(\lambda_2 + c\mu_1)H(c, 0, s, t) + e^{js}\lambda_2 H(c, 0, s, t)\\ &+ j(1-p)\sigma(1 - e^{-js})\frac{\partial H(n_1, n_2, s, t)}{\partial s}\\ &+ \lambda_1\{H(c - 1, 0, s, t) + e^{js} H(c - 1, 1, s, t)\}. \quad (4)\end{aligned}$$

By adopting linear finite difference operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, we can rewrite (1)–(4) as follows.

$$\begin{aligned}\frac{\partial \mathbf{H}(s, t)}{\partial t} =& \left\{\mathbf{A} + e^{js}(\lambda_1\mathbf{B} + \lambda_2\mathbf{C})\right\}\mathbf{H}(s, t)\\ &+ \left\{\mathbf{I}_0 - e^{-js}\mathbf{I}_1 + (1-p)(1 - e^{-js})\mathbf{I}_2\right\} j\sigma\frac{\partial \mathbf{H}(s, t)}{\partial s}, \quad (5)\end{aligned}$$

where $\mathbf{H}(s, t)$ is a $(c+1) \times (c+1)$ top-left triangle matrix whose (n_1, n_2) element is $H(n_1, n_2, s, t)$ for $n_1 \geq 0$, $n_2 \geq 0$, $n_1 + n_2 \leq c$. Operators in (5) are defined as:

$$\mathbf{AH}(s,t)_{n_1,n_2} = \begin{cases} -(\lambda_1+\lambda_2)H(0,0,s,t) + \mu_1 H(1,0,s,t) + \mu_2 H(0,1,s,t), \\ \quad (n_1+n_2=0), \\ -(\lambda_1+\lambda_2+n_1\mu_1+n_2\mu_2)H(n_1,n_2,s,t) \\ \quad + \lambda_1 H(n_1-1,n_2,s,t) + (n_1+1)\mu_1 H(n_1+1,n_2,s,t) \\ \quad + (n_2+1)\mu_2 H(n_1,n_2+1,s,t), \quad (1 \le n_1+n_2 \le c-1), \\ -(\lambda_1+\lambda_2+n_1\mu_1+n_2\mu_2)H(n_1,n_2,s,t) \\ \quad + \lambda_1 H(n_1-1,n_2,s,t), \quad (n_1+n_2=c), \\ -(\lambda_2+c\mu_1)H(c,0,s,t) + \lambda_1 H(c-1,0,s,t), \quad (n_1=c). \end{cases}$$

$$\mathbf{BH}(s,t)_{n_1,n_2} = \begin{cases} H(n_1-1,n_2+1,s,t), & (n_1+n_2=c, n_2 \ge 1), \\ 0, & (\text{otherwise}). \end{cases}$$

$$\mathbf{CH}(s,t)_{n_1,n_2} = H(n_1,n_2,s,t), \quad (0 \le n_1+n_2 \le c)$$

$$\mathbf{I}_0\mathbf{H}(s,t)_{n_1,n_2} = \begin{cases} H(n_1,n_2,s,t), & (n_1+n_2 \le c-1), \\ 0, & (\text{otherwise}). \end{cases}$$

$$\mathbf{I}_1\mathbf{H}(s,t)_{n_1,n_2} = \begin{cases} H(n_1,n_2-1,s,t), & (1 \le n_2 \le c), \\ 0, & (\text{otherwise}). \end{cases}$$

$$\mathbf{I}_2\mathbf{H}(s,t)_{n_1,n_2} = \begin{cases} H(n_1,n_2,s,t), & (n_1+n_2=c), \\ 0, & (\text{otherwise}). \end{cases}$$

Summing (1)-(4), we obtain

$$\left\{\frac{\partial}{\partial t} \sum_{n_1+n_2 \le c} H(n_1,n_2,s,t)\right\} = (e^{js}-1)\Big\{\lambda_1 \sum_{\substack{n_1+n_2=c \\ n_2 \ge 1}} H(n_1,n_2,s,t) + \lambda_2 \sum_{n_1+n_2 \le c} H(n_1,n_2,s,t) + e^{-js} j\sigma \sum_{n_1+n_2 \le c-1} \frac{\partial H(n_1,n_2,s,t)}{\partial s}\Big\} + (1-p)(1-e^{-js}) j\sigma \sum_{n_1+n_2=c} \frac{\partial H(n_1,n_2,s,t)}{\partial s}. \quad (6)$$

Let $\mathbf{S_1}$ be the summing operator for the elements with $n_1+n_2=c$ and $n_2 \ge 1$, $\mathbf{S_2}$ and $\mathbf{S_3}$ be those for the elements with $n_1+n_2 \le c-1$ and with $n_1+n_2=c$ respectively. Furthermore, let $\mathbf{S}$ be the total summing operator. We hence rewrite (6) in the following form

$$\frac{\partial}{\partial t}[\mathbf{SH}(s,t)] = (e^{js}-1)\Big\{\lambda_1 \mathbf{S_1H}(s,t) + \lambda_2 \mathbf{SH}(s,t) + e^{-js} j\sigma \frac{\partial}{\partial s}[\mathbf{S_2H}(s,t)]\Big\} + (1-p)(1-e^{-js}) j\sigma \frac{\partial}{\partial s}[\mathbf{S_3H}(s,t)]. \quad (7)$$

3 First-Order Approximation

In this section, we solve the system of equations (5) and (7) using the first-order analysis under the asymptotic condition, $\sigma \to 0$. Here, we put the following substitutions for the sensing rate and time:

$$\sigma = \epsilon,\ \tau = \epsilon t,\ s = \epsilon\omega,\ \mathbf{H}(s,t) = \mathbf{F}(\omega,\tau,\epsilon).$$

Then, we have the following equations:

$$\begin{aligned}\epsilon\frac{\partial \mathbf{F}(\omega,\tau,\epsilon)}{\partial \tau} =& \{\mathbf{A} + e^{j\epsilon\omega}(\lambda_1\mathbf{B} + \lambda_2\mathbf{C})\}\mathbf{F}(\omega,\tau,\epsilon) \\ &+ j\{(\mathbf{I}_0 - e^{-j\epsilon\omega}\mathbf{I}_1) + (1-p)(1-e^{-j\epsilon\omega})\mathbf{I_2}\}\frac{\partial \mathbf{F}(\omega,\tau,\epsilon)}{\partial \omega},\end{aligned} \tag{8}$$

$$\begin{aligned}\epsilon\frac{\partial}{\partial \tau}[\mathbf{SF}(\omega,\tau,\epsilon)] =& (e^{j\epsilon\omega} - 1)\{\lambda_1\mathbf{S_1}\mathbf{F}(\omega,\tau,\epsilon) + \lambda_2\mathbf{SF}(\omega,\tau,\epsilon) \\ &+ e^{-j\epsilon\omega}j\mathbf{S_2}\mathbf{F}(\omega,\tau,\epsilon)\} + (1-p)(1-e^{-j\epsilon\omega})j\frac{\partial}{\partial \omega}\mathbf{S_3}\mathbf{F}(\omega,\tau,\epsilon).\end{aligned} \tag{9}$$

Lemma 1. The following equality holds as $\sigma \to 0$.

$$\lim_{\sigma\to 0}\mathbb{E}[e^{j\omega\sigma i(\frac{\tau}{\sigma})}] = e^{j\omega x(\tau)}, \tag{10}$$

where $x(\tau)$ is a solution of

$$x'(\tau) = a(x) = (\lambda_1\mathbf{S_1} + \lambda_2\mathbf{S})\mathbf{R} - x(\mathbf{S_2}\mathbf{R} + (1-p)\mathbf{S_3}\mathbf{R}). \tag{11}$$

Here, $\mathbf{R} = \mathbf{R}(x)$ is a left-top triangle matrix which is a solution of the following system

$$\{\mathbf{A} + \lambda_1\mathbf{B} + \lambda_2\mathbf{C} - x(\tau)(\mathbf{I}_0 - \mathbf{I}_1)\}\mathbf{R} = 0, \tag{12}$$

and satisfies the normalization condition of a probability distribution

$$\mathbf{SR} = \sum_{n_1+n_2\le c} R(n_1,n_2,x) = 1. \tag{13}$$

From (12) and (13), $R(n_1,n_2,x)$ is the steady-state probability of a Markov chain at state (n_1,n_2). The associated transition diagram is illustrated in Fig. 1 (for simplicity, we show the case of $c = 4$). It follows from (12) that this Markov chain represents the corresponding loss system, where the arrival rates of PUs and SUs are λ_1 and x, respectively. It is noted that the Markov chain is independent of p.

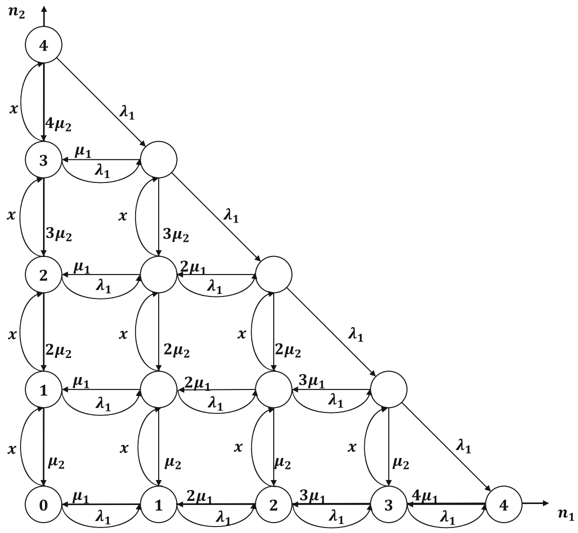

Fig. 1. Transitions among states of the Markov chain.

Proof. Denoting $\lim_{\epsilon \to 0} \mathbf{F}(\omega, \tau, \epsilon) = \mathbf{F}(\omega, \tau)$ and taking the limit $\epsilon \to 0$ in (8), we have

$$(\mathbf{A} + \lambda_1 \mathbf{B} + \lambda_2 \mathbf{C})\mathbf{F}(\omega, \tau) + (\mathbf{I}_0 - \mathbf{I}_1) j \frac{\partial \mathbf{F}(\omega, \tau)}{\partial \omega} = 0. \tag{14}$$

Due to the structure of (14), similar to the scalar case, we find the solution of (14) in the form

$$\mathbf{F}(\omega, \tau) = \mathbf{R} e^{j\omega x(\tau)}, \tag{15}$$

where $\mathbf{R}$ is a left-top triangle matrix and $x(\tau)$ is a scalar function which represents the asymptotic value of the normalized number of SUs in the sensing pool, i.e., $\sigma i(\frac{\tau}{\sigma})$. Substituting (15) into (14), we obtain (12). Because $\mathbf{R}$ encodes the stationary distribution of a Markov chain, it satisfies (13). Taking the limit $\epsilon \to 0$ in (9) yields

$$\mathbf{S} \frac{\partial \mathbf{F}(\omega, \tau)}{\partial \tau} = j\omega \left\{ \lambda_1 \mathbf{S_1} \mathbf{F}(\omega, \tau) + \lambda_2 \mathbf{S} \mathbf{F}(\omega, \tau) + j \frac{\partial}{\partial \omega} [\mathbf{S_2} \mathbf{F}(\omega, \tau)] \right\}$$
$$+ (1-p) j\omega \cdot j \frac{\partial}{\partial \omega} [\mathbf{S_3} \mathbf{F}(\omega, \tau)]. \tag{16}$$

Substituting (15) into (16), we obtain (11). Since the scalar function $x(\tau)$ is the asymptotic value of the normalized number of SUs in the sensing pool $\sigma i(\frac{\tau}{\sigma})$, (10) holds.

For clarity, we can rewrite the explicit form of (11) as follows.

$$
\begin{aligned}
a(x) =&\lambda_1 \sum_{\substack{n_1+n_2=c \\ n_2 \geq 1}} R(n_1,n_2,x) + \lambda_2 \\
&- x\left\{ \sum_{n_1+n_2 \leq c-1} R(n_1,n_2,x) + (1-p) \sum_{n_1+n_2=c} R(n_1,n_2,x) \right\}.
\end{aligned}
$$

Using the first-order analysis, we construct an approximation to the mean of the number of sensing SUs and compare the approximation with simulation results in Sect. 5.

4 Main Results

In this section, we consider the necessary stability condition. The system can be shown always stable when probability $p < 1$ via the theory of Lyapunov functions [2,12]. In this paper, we show that the limit condition $\lim_{x\to\infty} a(x) < 0$ is consistent with the stability condition by the Lyapunov function approach.

Theorem 1 (Stability condition). When $p < 1$, the system is always stable for all parameters, which coincides with $\lim_{x\to\infty} a(x) < 0$.

Proof. From (11), we rewrite the function $a(x)$ as below:

$$
\begin{aligned}
a(x) =&(\lambda_1 \mathbf{S_1} + \lambda_2 \mathbf{S})\mathbf{R}(x) - x(\mathbf{S_2}\mathbf{R}(x) + (1-p)\mathbf{S_3}\mathbf{R}(x)) \\
=&\lambda_1\{1 - \mathbf{S_2}\mathbf{R}(x) - R(c,0,x)\} + \lambda_2 - x(\mathbf{S_2}\mathbf{R}(x) + (1-p)\mathbf{S_3}\mathbf{R}(x)) \\
=&\lambda_1(1-\pi_c) + \lambda_2 - (\lambda_1 + x)\mathbf{S_2}\mathbf{R}(x) - x(1-p)\mathbf{S_3}\mathbf{R}(x).
\end{aligned} \tag{17}
$$

Here, π_c is Erlang-B formula for PUs and is given as follows:

$$
\pi_c = \frac{\frac{\lambda_1^c}{c!\mu_1^c}}{\sum_{k=0}^{c} \frac{\lambda_1^k}{k!\mu_1^k}}.
$$

We note that π_c does not depend on x. The third term of (17), $(\lambda_1 + x)\mathbf{S_2}\mathbf{R}(x)$ indicates the incoming flow of the two types of users successfully entering the system without blocking and interrupting. In other words, the third term expresses the arrival rate that increases the number of users in the channels (the throughput of all users).

Moreover, the fourth term $x(1-p)\mathbf{S_3R}(x)$ diverges to infinity as $x \to \infty$ since $\mathbf{S_3R}(x)$ means the blocking probability and increases to 1 as $x \to \infty$. Hence, $\lim_{x\to\infty} a(x) < 0$ holds for all parameters with $p < 1$. Therefore, $a(x)$ decreases with the increase in x, and thus the theorem is proved.

Next, we study the uniqueness of the fixed point κ such that $a(\kappa) = 0$, to derive the stationary solution of $x(\tau)$ for our model with $p < 1$. This point also plays a pivotal role in determining the asymptotic number of SUs in the orbit.

Theorem 2 (Uniqueness).
When $p < 1$, the solution of $a(x) = 0$ is unique.

Proof. It is obvious that $a(0) > 0$ and $\lim_{x\to\infty} a(x) < 0$ from (11) and Theorem 1. From the intermediate value theorem, there is at least one value κ in $(0, \infty)$ for which $a(\kappa) = 0$. We prove that this equation has a unique solution κ.

In Eq. (17), the latter two terms represent the throughput of all users and the rate of SUs being blocked and leaving the orbit. As a consequence, $a(x)$ monotonically decreases as x increases.

5 Numerical Experiment

In this section, we carry out some numerical experiments to evaluate the model's performance. We set the service rates to $\mu_1 = 4$, $\mu_2 = 20$ and $c = \{1, 5\}$ and investigate the impact of λ_1 and λ_2. Under these same service rates, we also evaluate performance, comparing results from simulations with those from the asymptotic method. All of the experimental simulations here are run for 10^6 time steps, which is sufficient to converge to their corresponding numerical solutions. The simulation results in all the figures are denoted by "Sim."

Figures 2 and 3 show the average number of SUs in the orbit against the arrival rate of PUs for $c = \{1, 5\}$. Here, the mean number of SUs in the orbit is approximated by

$$\mathbb{E}[N_{\text{orbit}}] \approx \frac{\kappa}{\sigma}.$$

We can see that the larger the probability p is, the more SUs there are in the orbit. This trend is due to the higher chances that an SU continues to stay in the orbit when it gets blocked. In Fig. 3, there are small differences between each p when λ_1 is small. In contrast, the bigger λ_1 is, the bigger the differences are. Because the number of SUs in the orbit is fewer when p is lower, and more channels are available to allocate, they have more chances to occupy the channel on first arriving. The greater the probability p is, the bigger the differences become because the number of sensing SUs increases and an SU is more likely to be blocked.

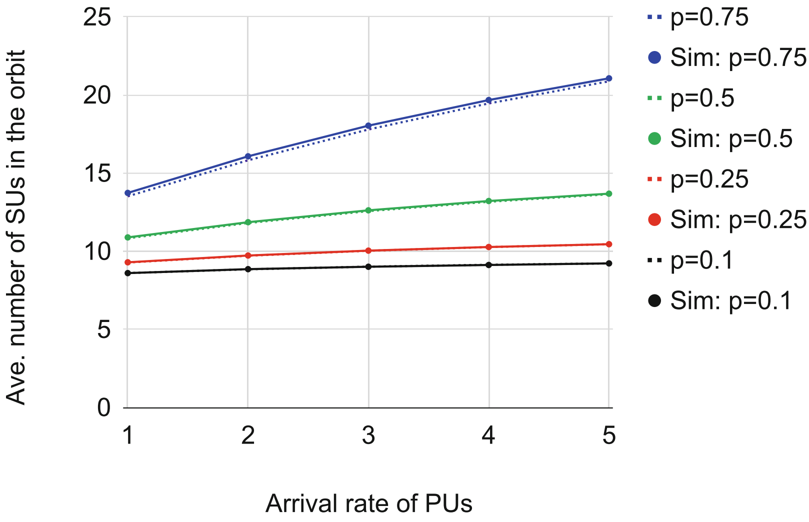

Fig. 2. Mean number SUs in the orbit versus λ_1, for $\sigma = 1.0$, $c = 1$.

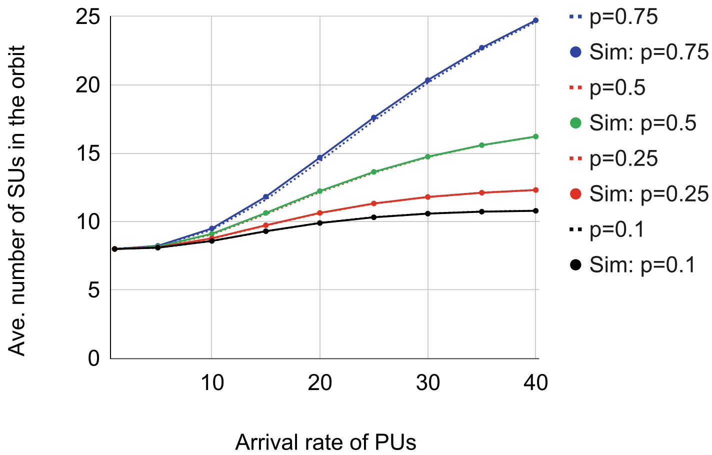

Fig. 3. Mean number SUs in the orbit versus λ_1, for $\sigma = 1.0$, $c = 5$.

Figures 4 and 5 show the average number of SUs in the orbit against their arrival rate. When λ_2 is small, the difference between the values for each p also becomes smaller, and the difference is more significant with larger λ_2. Furthermore, we can interpret that the crowding of the orbit increases the demand upon available channels when the probability is high.

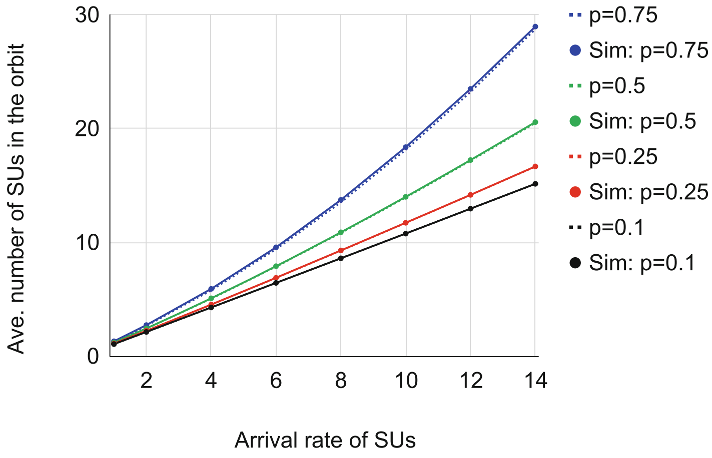

Fig. 4. Mean number SUs in the orbit versus λ_2, for $c = 1$.

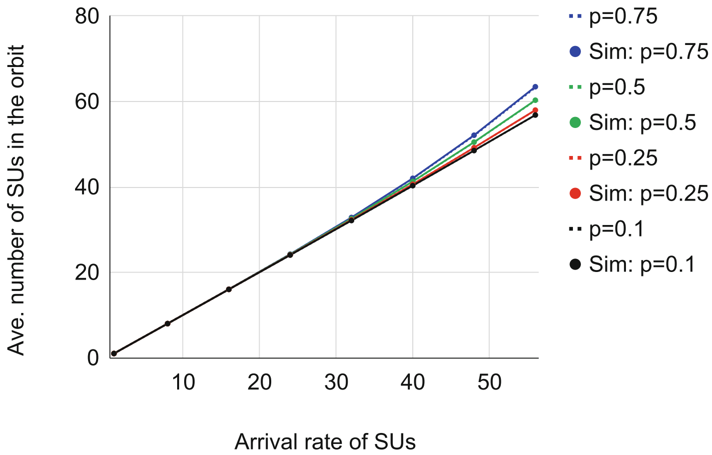

Fig. 5. Mean number SUs in the orbit versus λ_2, for $c = 5$.

Finally, we compare the transition of the behavior between simulations and $x(\frac{\tau}{\sigma})$ in Fig. 6. We can see that the simulation results evolve around $x(\frac{\tau}{\sigma})$ over time. We can see that $x(\cdot)$ is appropriate enough to approximate the normalized number of SUs in the orbit.

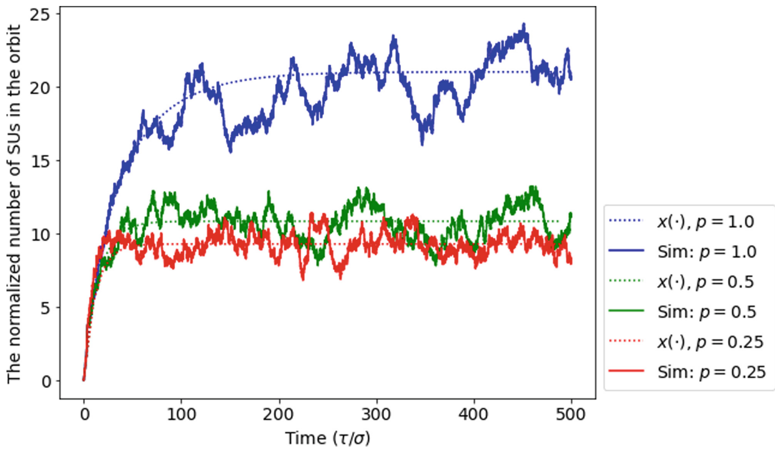

Fig. 6. The transition of the normalized number of SUs in the orbit for $\lambda_1 = 1$, $\lambda_2 = 8$, $\mu_1 = 4$, $\mu_2 = 20$, $\sigma = 0.1$, $c = 1$.

6 Concluding Remark

We analyzed the behavior of secondary users in a modified Erlang-B system using the asymptotic approach under $\sigma \to 0$. We derived the probability distribution of the states of occupied servers under first-order approximation. Using the method of asymptotic analysis, we obtained the stability condition, which is consistent with previous work. Besides, we showed the uniqueness of the fixed point κ such that $a(\kappa) = 0$ to derive the stationary solution of $x(\tau)$ for our model with $p < 1$. We also performed some numerical experiments to investigate the distribution of the states of servers and the number of SUs in the orbit.

Acknowledgement. The research of Tuan Phung-Duc was supported in part by JSPS KAKENHI Grant Number 21K11765.

References

1. Abe, K., Phung-Duc, T.: Diffusion limit of a modified Erlang-B system with sensing time of secondary users. Annals Oper. Res. 1–22 (2022)
2. Falin, G., Templeton, J.G.: Retrial Queues, vol. 75. CRC Press, Boca Raton (1997)
3. Gómez-Corral, A., Krishnamoorthy, A., Narayanan, V.C.: The impact of self-generation of priorities on multi-server queues with finite capacity. Stoch. Model. **21**(2–3), 427–447 (2005)
4. Moiseev, A., Nazarov, A., Paul, S.: Asymptotic diffusion analysis of multi-server retrial queue with hyper-exponential service. Mathematics **8**(4), 531 (2020)
5. Nasser, A., Al Haj Hassan, H., Abou Chaaya, J., Mansour, A., Yao, K.: Spectrum Sensing for a cognitive radio: recent advances and future challenge. Sensors **21**(7), 2408 (2021)

6. Nazarov, A., Moiseev, A., Phung-Duc, T., Paul, S.: Diffusion limit of multi-server retrial queue with setup time. Mathematics **8**(12), 2232 (2020)
7. Nazarov, A., Phung-Duc, T., Paul, S., Lizura, O.: Asymptotic-diffusion analysis for retrial queue with batch Poisson input and multiple types of outgoing calls. In: Vishnevskiy, V.M., Samouylov, K.E., Kozyrev, D.V. (eds.) DCCN 2019. LNCS, vol. 11965, pp. 207–222. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-36614-8_16
8. Nazarov, A., Phung-Duc, T., Paul, S., Lizyura, O.: Diffusion approximation for multiserver retrial queue with two-way communication. In: Vishnevskiy, V.M., Samouylov, K.E., Kozyrev, D.V. (eds.) DCCN 2020. LNCS, vol. 12563, pp. 567–578. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-66471-8_43
9. PalunčIć, F., Alfa, A.S., Maharaj, B.T., Tsimba, H.M.: Queueing models for cognitive radio networks: a survey. IEEE Access **6**, 50801–50823 (2018)
10. Phung-Duc, T., Akutsu, K., Kawanishi, K., Salameh, O., Wittevrongel, S.: Queueing models for cognitive wireless networks with sensing time of secondary users. Ann. Oper. Res. **310**(2), 641–660 (2022)
11. Phung-Duc, T., Kawanishi, K.: Multiserver retrial queue with setup time and its application to data centers. J. Ind. Manage. Optim. **15**(1), 15–35 (2019)
12. Phung-Duc, T.: Retrial queueing models: a survey on theory and applications. arXiv preprint arXiv:1906.09560 (2019)
13. Salameh, O., De Turck, K., Bruneel, H., Blondia, C., Wittevrongel, S.: Analysis of secondary user performance in cognitive radio networks with reactive spectrum handoff. Telecommun. Syst. **65**(3), 539–550 (2017)