Cyclic Retrial Queue for Building Data Transmission Networks

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Abstract. This paper considers a mathematical model of a cyclic multiple access communication network. The model can be used to build specialized "flying" FANET data transmission networks. We consider a single server retrial queue for modeling one node in such network. The input consists of multiple Poisson processes with different arrival rates. Service and retrial rates depend on the origin flow. Thus, each flow has its own orbit for redial. Under the condition when the retrial rate is low, we obtain an asymptotic probability distribution of the number of customers in the orbits.

Keywords: cyclic queueing system · retrial queue · vacations · asymptotic analysis · diffusion approximation

1 Introduction

Special communication networks are destined to provide data transmission between a group of devices. In this paper, we discuss aspects of the organization of special FANET (Flying Ad-Hoc Networks) [1] by stochastic modeling. They are used to organize data transmission in a group of unmanned aerial vehicles (drones).

The actual topology for such communication networks can be a "star", the central node of which performs the functions of controlling groups of drones and is a common network resource. The central node can be a control center for a group of devices, a flight control center, a control room interacting with a group

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of drones. The presence of a network allows each drone to transmit data to the central node.

Resource sharing is a common problem, which can be solved by choosing a protocol for accessing network subscribers to a shared resource. We consider a cyclic protocol, which allocates each drone one time interval, during which it completely transmits data to the control center. Time windows are organized sequentially. Each time window is assigned to one drone. When using the random multiple access protocol, each drone randomly selects a window for transmitting information.

Retrial queues [2–5] are adequate mathematical models of random access protocols and polling systems [6,7] are for cyclic protocols.

In this paper, we intend to investigate a variant of a communication system with a group of drones as a queueing system. When a group of drones is on duty, monitoring the area or delivering cargo, the communication network is in a regular cyclic mode - each drone transmits the collected information in its segment of the cycle to the control center. We consider the cycle as the sum of access intervals to the common resource of each drone. A feature of the proposed model is that the durations of such intervals are random (in particular, they can be deterministic) and independent not only among themselves, but also do not depend on the incoming flows of requests and the duration of their service.

The aim of the paper is to determine the characteristics of the number of messages in orbits of the cyclic retrial queue. The problem is solved using the classical method of "system with server vacations" [8,9]. We have proposed algorithm for assigning access intervals independent of the incoming flow and the time of servicing requests. For our strategy, these times are independent, so the multidimensional probability distribution is factorized and the method of "server vacations" used in the work completely solves the problem.

Polling systems with retrial behavior are considered in papers [11–15]. The authors propose numerical analysis for gated or mixed service disciplines in such systems. We propose the analysis of the system using original method of asymptotic-diffusion analysis under low rate of retrial condition [16]. Our approach allows to build accurate enough approximation for the steady state distribution of the number of calls in the orbit.

The rest of the paper is structured as follows. In Sect. 2, we describe the structure of the model and define its parameters. Section 3 shows the derived Kolmogorov equations for the system state process. In Sects. 4, 5, we describe the main results and prove theorems about approximate diffusion process. In Sect. 6, we present the algorithm of calculation for the approximation of the steady state distribution of the number of customers in the orbit. Section 7 depicts the numerical example of using the derived formulas. Finally, Sect. 8 presents concluding remarks.

2 Mathematical Model

The group of N drones forms N Poisson flows of incoming packages (calls) with intensity λ_n for the n-th drone ($n = 1, \ldots, N$) to the control center (see Fig. 1).

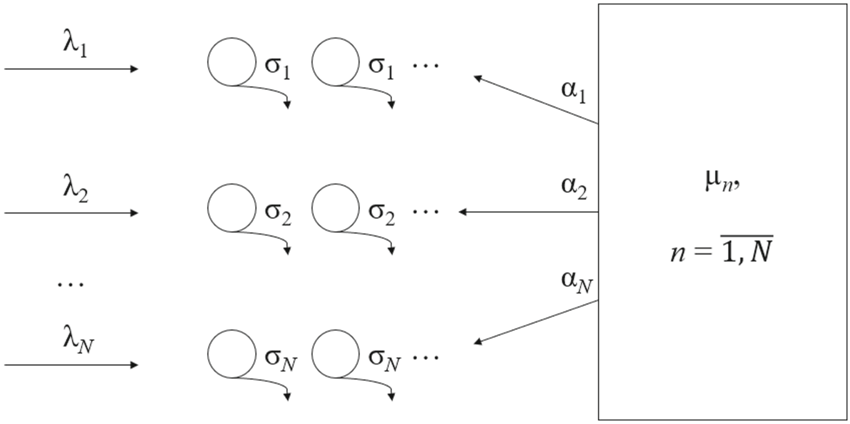

Fig. 1. Cyclic retrial queue

Calls of n-th flow form their own unlimited orbit. The pair of flow and orbit will be called the n-th RQ-system. The control center (server) visits retrial queues in a cyclic order, starting from the first and ending with N-th, then the cycle repeats. The time spent by the server at the n-th retrial queue follows the exponential distribution with mean $1/\alpha_n$, $n = 1, \ldots, N$. During this time, the server transmits packages from the incoming flow and corresponding orbit.

If a call of n-th flow detects the server busy or not connected, it instantly goes to n-th orbit and performs a random delay, otherwise the incoming customer is served. The retrial intervals follow the exponential distribution with mean $1/\sigma_n$, $n = 1, \ldots, N$, after which the call reapplies to the server. The service times of calls are exponentially distributed with mean $1/\mu_n$, $n = 1, \ldots, N$.

If the orbit is empty at the time the server arrives, or it has served all the calls that were in orbit, and no more new calls have arrived from the incoming flow, the server still remains connected to the retrial queue until the connection time expires. We study this cyclic system by decomposing it to N separated systems and analyze them as a retrial queue with vacation.

2.1 Modeling of Server Vacations

To study cyclic retrial queue, we consider the model with server vacations (see Fig. 2), which represents one node in our network.

Let us consider the first system with repeated calls with one server and an orbit of infinite capacity. The system receives the requests from a Poisson process with rate λ. The system operates in a cyclic mode whose cycle consists of two consecutive intervals. During the first interval, the server receives calls that come from the incoming flow during an exponentially distributed time with rate μ.

If an incoming call detects the server busy, it instantly goes into orbit, where it performs a random delay during the exponential time with parameter σ, after which it returns to the server.

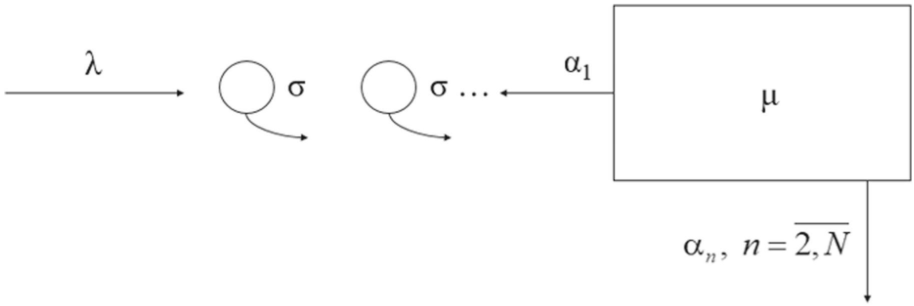

Fig. 2. Retrial queue with server vacations

The duration of the server connection follows the exponential distribution with mean $1/\alpha_1$. Upon the termination of the server connection time, the ongoing call (if any) is pushed to the orbit and retries to enter the server later.

After this interval, the server starts vacation, the duration of which consists of $N-1$ phases. Each phase follows the exponential distribution with parameters α_n, $n = 2, \ldots, N$. During the vacation, the calls that came into the system accumulate in orbit and wait for the server to return.

The process $k(t)$ depicts the state of server at time t: $k(t) = 0$ if the server is free, $k(t) = 1$ if the server is busy servicing a call, $k(t) = n$ if the server is on the n-th phase of the vacation, $n = 2,\ldots,N$.

We also introduce the random process $i(t)$ as the number of calls in the orbit at time t.

3 Kolmogorov Equations

We are going to obtain the stationary distribution of the number of calls in the orbit. To this end, we study the two-dimensional Markov chain $\{k(t), i(t)\}$. For the probability distribution $P\{k(t) = k, i(t) = i\} = P_k(i,t)$, we compose the Kolmogorov system

$$\frac{\partial P_0(i,t)}{\partial t} = -(\lambda + i\sigma + \alpha_1)P_0(i,t) + \mu P_1(i,t) + \alpha_N P_N(i,t),$$

$$\frac{\partial P_1(i,t)}{\partial t} = -(\lambda+\mu+\alpha_1)P_1(i,t) + \lambda P_0(i,t) + \sigma(i+1)P_0(i+1,t) + \lambda P_1(i-1,t),$$

$$\frac{\partial P_2(i,t)}{\partial t} = -(\lambda + \alpha_2)P_2(i,t) + \alpha_1 P_1(i-1,t) + \alpha_1 P_0(i,t) + \lambda P_2(i-1,t),$$

$$\frac{\partial P_n(i,t)}{\partial t} = -(\lambda+\alpha_n)P_n(i,t)+\lambda P_n(i-1,t)+\alpha_{n-1}P_{n-1}(i,t),\ n = 3,\ldots,N. \quad (1)$$

Because it is difficult to directly solve the above system of differential equations, we introduce partial characteristic functions, where $j = \sqrt{-1}$:

$$H_k(u,t) = \sum_{i=0}^{\infty} e^{jui} P_k(i,t),\ k = 0,\ldots,N,$$

and transform (1) as follows for further research

$$\frac{\partial H_0(u,t)}{\partial t} = -(\lambda+\alpha_1)H_0(u,t) + j\sigma\frac{\partial H_0(u,t)}{\partial u} + \mu H_1(u,t) + \alpha_N H_N(u,t),$$

$$\frac{\partial H_1(u,t)}{\partial t} = (\lambda(e^{ju}-1)-\mu-\alpha_1)H_1(u,t) + \lambda H_0(u,t) - j\sigma e^{-ju}\frac{\partial H_0(u,t)}{\partial u},$$

$$\frac{\partial H_2(u,t)}{\partial t} = (\lambda(e^{ju}-1)-\alpha_2)H_2(u,t) + e^{ju}\alpha_1 H_1(u,t) + \alpha_1 H_0(u,t),$$

$$\frac{\partial H_n(u,t)}{\partial t} = (\lambda(e^{ju}-1)-\alpha_n)H_n(u,t) + \alpha_{n-1}H_{n-1}(u,t), n = 3,\ldots,N. \quad (2)$$

Summing up the equations in (2), we obtain

$$\frac{\partial H(u,t)}{\partial t} = (e^{ju}-1)\left(j\sigma e^{-ju}\frac{\partial H_0(u,t)}{\partial u} + \alpha_1 H_1(u,t) + \lambda\sum_{n=1}^{N} H_n(u,t)\right). \quad (3)$$

Because the solution of (2–3) for arbitrary σ is difficult, we consider asymptotic solution under the condition ($\sigma \to 0$).

4 First Step of Asymptotic-Diffusion Analysis

Introducing $\varepsilon = \sigma$ and performing the following substitution in (2–3)

$$\tau = t\varepsilon, u = \varepsilon w, H_k(u,t) = F_k(w,\tau,\varepsilon),$$

to obtain

$$\varepsilon\frac{\partial F_0(w,\tau,\varepsilon)}{\partial \tau} = -(\lambda+\alpha_1)F_0(w,\tau,\varepsilon) + j\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}$$
$$+\mu F_1(w,\tau,\varepsilon) + \alpha_N F_N(w,\tau,\varepsilon),$$

$$\varepsilon\frac{\partial F_1(w,\tau,\varepsilon)}{\partial \tau} = (\lambda(e^{j\epsilon w}-1)-\mu-\alpha_1)F_1(w,\tau,\varepsilon)$$
$$+\lambda F_0(w,\tau,\varepsilon) - je^{-j\varepsilon w}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w},$$

$$\varepsilon\frac{\partial F_2(w,\tau,\varepsilon)}{\partial \tau} = (\lambda(e^{j\epsilon w}-1)-\alpha_2)F_2(w,\tau,\varepsilon) + e^{j\varepsilon w}\alpha_1 F_1(w,\tau,\varepsilon) + \alpha_1 F_0(w,\tau,\varepsilon),$$

$$\varepsilon\frac{\partial F_n(w,\tau,\varepsilon)}{\partial \tau} = (\lambda(e^{j\epsilon w}-1)-\alpha_n)F_n(w,\tau,\varepsilon)$$
$$+\alpha_{n-1}F_{n-1}(w,\tau,\varepsilon),\ n = 3,\ldots,N,$$

$$\varepsilon\frac{\partial F(w,\tau,\varepsilon)}{\partial \tau} = (e^{j\varepsilon w}-1)\cdot$$
$$\cdot\left(je^{-j\varepsilon w}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w} + \alpha_1 F_1(w,\tau,\varepsilon) + \lambda\sum_{n=1}^{N} F_n(w,\tau,\varepsilon)\right). \quad (4)$$

Solving (4) for $\varepsilon \to 0$, we prove Theorem 1.

Theorem 1.

$$\lim_{\sigma\to 0}\mathbb{E}e^{jw\sigma i(\frac{\tau}{\sigma})} = e^{jwx(\tau)}, \tag{5}$$

where

$$x'(\tau) = -x(\tau)r_0 + \alpha_1 r_1 + \lambda\sum_{n=1}^{N} r_n.$$

Here, probabilities $r_k = r_k(x), k = 0, \dots, N$ are given by

$$r_0(x) = \left[\frac{\mu+\alpha_1+\lambda+x}{\mu+\alpha_1} + \sum_{n=2}^{N}\frac{\alpha_1(\mu+\alpha_1+\lambda+x)}{\alpha_n(\mu+\alpha_1)}\right]^{-1},$$

$$r_1(x) = \frac{\lambda+x}{\mu+\alpha_1}r_0(x),$$

$$r_2(x) = \frac{\alpha_1(\mu+\alpha_1+\lambda+x)}{\alpha_2(\mu+\alpha_1)}r_0(x),$$

$$r_n(x) = \frac{\alpha_1(\mu+\alpha_1+\lambda+x)}{\alpha_n(\mu+\alpha_1)}r_0(x),\ n = 3, \dots, N.$$

Proof. Taking $\varepsilon \to 0$ in (4), we have

$$-(\lambda+\alpha_1)F_0(w,\tau) + j\frac{\partial F_0(w,\tau)}{\partial w} + \mu F_1(w,\tau) + \alpha_N F_N(w,\tau) = 0,$$

$$-(\mu+\alpha_1)F_1(w,\tau) + \lambda F_0(w,\tau) - j\frac{\partial F_0(w,\tau)}{\partial w} = 0,$$

$$-\alpha_2 F_2(w,\tau) + \alpha_1 F_1(w,\tau) + \alpha_1 F_0(w,\tau) = 0,$$

$$-\alpha_n F_n(w,\tau) + \alpha_{n-1}F_{n-1}(w,\tau) = 0,\ n = 3, \dots, N,$$

$$\frac{\partial F(w,\tau)}{\partial\tau} = jw\left(j\frac{\partial F_0(w,\tau)}{\partial w} + \alpha_1 F_1(w,\tau) + \lambda\sum_{n=1}^{N}F_n(w,\tau)\right), \tag{6}$$

where $F_k^{(2)}(w,\tau) = \lim_{\varepsilon\to 0}F_k^{(2)}(w,\tau,\varepsilon)$.

We will seek the solution of (6) in the form

$$F_k(w,\tau) = r_k(x)e^{jwx(\tau)},\ k = 0, \dots, N, \tag{7}$$

where $x = x(\tau) = \lim_{\sigma\to 0}\sigma i(\tau/\sigma)$.

Substituting (7) into (6), we obtain

$$-(\lambda+\alpha_1+x)r_0(x) + \mu r_1(x) + \alpha_N r_N(x) = 0,$$

$$-(\mu+\alpha_1)r_1(x) + (\lambda+x)r_0(x) = 0,$$

$$-\alpha_2 r_2(x) + \alpha_1 r_1(x) + \alpha_1 r_0(x) = 0,$$

$$-\alpha_n r_n(x) + \alpha_{n-1}r_{n-1}(x) = 0,\ n = 3, \dots, N,$$

$$x'(\tau) = -x(\tau)r_0(x) + \alpha_1 r_1(x) + \lambda \sum_{n=1}^{N} r_n(x). \tag{8}$$

System (8) together with normalization condition $\sum\limits_{k=0}^{N} r_k(x) = 1$ determine $r_k(x)$

$$r_0(x) = \left[\frac{\mu + \alpha_1 + \lambda + x}{\mu + \alpha_1} + \sum_{n=2}^{N} \frac{\alpha_1(\mu + \alpha_1 + \lambda + x)}{\alpha_n(\mu + \alpha_1)}\right]^{-1},$$

$$r_1(x) = \frac{\lambda + x}{\mu + \alpha_1} r_0(x),$$

$$r_2(x) = \frac{\alpha_1(\mu + \alpha_1 + \lambda + x)}{\alpha_2(\mu + \alpha_1)} r_0(x),$$

$$r_n(x) = \frac{\alpha_1(\mu + \alpha_1 + \lambda + x)}{\alpha_n(\mu + \alpha_1)} r_0(x),\ n = 3, \ldots, N. \tag{9}$$

Let us denote

$$a(x) = x'(\tau) = \lambda - \left(x - \frac{\alpha_1(\lambda + x)}{\mu + \alpha_1} + \lambda\right) r_0(x). \tag{10}$$

As we will see, $a(x)$ represents the drift coefficient of a certain diffusion process related to the scaled number of calls in the orbit. Thus, Theorem 1 is proven.

Corollary 1. *Stability condition in considered queue is given by*

$$\lim_{x \to \infty} a(x) < 0,$$

which is equivalent to

$$\lambda \left(\sum_{n=1}^{N} \frac{\alpha_1}{\alpha_n}\right) < \mu.$$

Proof. From explicit expressions of $r_k(x), k = 0, 1, \ldots, N$, we have

$$\lim_{x \to \infty} r_0(x) = \frac{\mu + \alpha_1}{1 + \sum_{k=2}^{N} \alpha_1/\alpha_n},$$

and

$$\lim_{x \to \infty} r_n(x) = \frac{\alpha_1}{\alpha_n(1 + \sum_{k=2}^{N} \alpha_1/\alpha_n)}, n = 1, 2 \ldots, N.$$

Plugging these limits into $\lim_{x \to \infty} a(x)$ and rearranging the result yields corollary 1.

It should be remarked that if $\alpha_n = \infty$ for $n = 2, 3, \ldots, N$, i.e., the server only serves queue 1, the stability condition is reduced to $\lambda < \mu$ which is consistent with the stability of the M/M/1 retrial queue.

5 Second Step of Asymptotic-Diffusion Analysis

In (2) and (3), we make the following substitutions:

$$H_k(u,t) = H_k^{(2)}(u,t)e^{j\frac{u}{\sigma}x(\sigma t)},\ k = 0,\dots,N.$$

and obtain the system of equations

$$\frac{\partial H_0^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H_0^{(2)}(u,t) = -(\lambda + \alpha_1 + x(\sigma t))H_0^{(2)}(u,t)$$

$$+j\sigma\frac{\partial H_0^{(2)}(u,t)}{\partial u} + \mu H_1^{(2)}(u,t) + \alpha_N H_N^{(2)}(u,t),$$

$$\frac{\partial H_1^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H_1^{(2)}(u,t) = (\lambda(e^{ju}-1) - \mu - \alpha_1)H_1^{(2)}(u,t)$$

$$+(\lambda + x(\sigma t)e^{-ju})H_0^{(2)}(u,t) - j\sigma e^{-ju}\frac{\partial H_0^{(2)}(u,t)}{\partial u},$$

$$\frac{\partial H_2^{(0)}(u,t)}{\partial t} + jux'(\sigma t)H_2^{(2)}(u,t) = (\lambda(e^{ju}-1) - \alpha_2)H_2^{(2)}(u,t)$$

$$+e^{ju}\alpha_1 H_1^{(2)}(u,t) + \alpha_1 H_0^{(2)}(u,t),$$

$$\frac{\partial H_n^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H_n^{(2)}(u,t) = (\lambda(e^{ju}-1) - \alpha_n)H_n^{(2)}(u,t)$$

$$+\alpha_{n-1}H_{n-1}^{(2)}(u,t), n = 3,\dots,N.$$

$$\frac{\partial H^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H^{(2)}(u,t) = (e^{ju}-1)\left(-x(\sigma t)e^{-ju}H_0^{(2)}(u,t)\right.$$

$$\left.+j\sigma e^{-ju}\frac{\partial H_0^{(2)}(u,t)}{\partial u} + \alpha_1 H_1^{(2)}(u,t) + \lambda\sum_{n=1}^{N} H_n^{(2)}(u,t)\right). \tag{11}$$

The characteristic function of $i(t) - \frac{1}{\sigma}x(\sigma t)$ is given by $H^{(2)}(u,t)$. We make the following substitutions.

Denoting $\sigma = \varepsilon^2$ in (11) and substituting

$$\tau = \varepsilon^2 t, u = \varepsilon w, H_k^{(2)}(u,t) = F_k^{(2)}(w,\tau,\varepsilon),\ k = 0,\dots,N,$$

we obtain

$$\varepsilon^2\frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x)F_0^{(2)}(w,\tau,\varepsilon) = -(\lambda + \alpha_1 + x)F_0^{(2)}(w,\tau,\varepsilon)$$

$$+j\varepsilon\frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} + \mu F_1^{(2)}(w,\tau,\varepsilon) + \alpha_N F_N^{(2)}(w,\tau,\varepsilon),$$

$$\varepsilon^2 \frac{\partial F_1^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_1^{(2)}(w,\tau,\varepsilon) = (\lambda(e^{j\epsilon w}-1) - \mu - \alpha_1) F_1^{(2)}(w,\tau,\varepsilon)$$

$$+(\lambda + x e^{-j\varepsilon w}) F_0^{(2)}(w,\tau,\varepsilon) - j\varepsilon e^{-j\varepsilon w} \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w},$$

$$\varepsilon^2 \frac{\partial F_2^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_2^{(2)}(w,\tau,\varepsilon) = (\lambda(e^{j\epsilon w}-1) - \alpha_2) F_2^{(2)}(w,\tau,\varepsilon)$$

$$+e^{j\varepsilon w}\alpha_1 F_1^{(2)}(w,\tau,\varepsilon) + \alpha_1 F_0^{(2)}(w,\tau,\varepsilon),$$

$$\varepsilon^2 \frac{\partial F_n^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_n^{(2)}(w,\tau,\varepsilon) = (\lambda(e^{j\epsilon w}-1) - \alpha_n) F_n^{(2)}(w,\tau,\varepsilon)$$

$$+\alpha_{n-1} F_{n-1}^{(2)}(w,\tau,\varepsilon),\ n = 3,\ldots,N,$$

$$\varepsilon^2 \frac{\partial F^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F^{(2)}(w,\tau,\varepsilon) = (e^{j\varepsilon w}-1)\Big(-x e^{-j\varepsilon w} F_0^{(2)}(w,\tau,\varepsilon)$$

$$+j\varepsilon e^{-j\varepsilon w} \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} + \alpha_1 F_1^{(2)}(w,\tau,\varepsilon) + \lambda \sum_{n=1}^{N} F_n^{(2)}(w,\tau,\varepsilon)\Big). \tag{12}$$

Solving this system, we obtain Theorem 2.

Theorem 2.

$$F_k^{(2)}(w,\tau) = \lim_{\varepsilon\to 0} F_k^{(2)}(w,\tau,\varepsilon),\ k = 0,\ldots,N$$

are given by

$$F_k^{(2)}(w,\tau) = \Phi(w,\tau) r_k(x),\ k = 0,\ldots,N$$

where $r_k(x)$ is given before and $\Phi(w,\tau)$ satisfies

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = a'(x) w \frac{\partial \Phi(w,\tau)}{\partial w} + \frac{(jw)^2}{2} b(x) \Phi(w,\tau). \tag{13}$$

$a(x)$ is obtained in (10) and

$$b(x) = a(x) + 2\Big(- x g_0(x) + \alpha_1 g_1(x) + \lambda \sum_{k=1}^{N} g_k(x) + x r_0(x)\Big), \tag{14}$$

where the functions $g_k(x),\ k = 0,\ldots,N$ are defined by an heterogeneous system:

$$-(\lambda + \alpha_1 + x) g_0 + \mu g_1 + \alpha_N g_N = a(x) r_0,$$

$$-(\mu + \alpha_1) g_1 + (\lambda + x) g_0 = a(x) r_1 - \lambda r_1 + x r_0,$$

$$-\alpha_2 g_2 + \alpha_1 g_1 + \alpha_1 g_0 = a(x) r_2 - \lambda r_2 - \alpha_1 r_1,$$

$$-\alpha_n g_n + \alpha_{n-1} g_{n-1} = a(x) r_n - \lambda r_n,\ n = 3,\ldots,N,$$

$$\sum_{k=0}^{N} g_k = 0. \tag{15}$$

Proof. Let us use the following decomposition in (12)

$$F_k^{(2)}(w,\tau,\varepsilon) = \Phi(w,\tau)(r_k + j\varepsilon w f_k) + O(\varepsilon^2),\ k = 0,\ldots,N. \tag{16}$$

Substituting (16) into the first four equations of system (12), we obtain the following system in the limit by $\varepsilon \to 0$:

$$\begin{aligned}
-(\lambda+\alpha_1+x)f_0+\mu f_1+\alpha_N f_N &= a(x)r_0-\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}r_0,\\
-(\mu+\alpha_1)f_1+(\lambda+x)f_0 &= a(x)r_1-\lambda r_1+xr_0+\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}r_0,\\
-\alpha_2 f_2+\alpha_1 f_1+\alpha_1 f_0 &= a(x)r_2-\lambda r_2-\alpha_1 r_1,\\
-\alpha_n f_n+\alpha_{n-1}f_{n-1} &= a(x)r_n-\lambda r_n,\ n=3,\ldots,N.
\end{aligned} \tag{17}$$

We propose finding the solution as

$$f_k = Cr_k + g_k - \phi_k\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}, \tag{18}$$

and substitute in (17) to obtain two systems

$$\begin{aligned}
-(\lambda+\alpha_1+x)\phi_0+\mu\phi_1+\alpha_N\phi_N &= r_0,\\
-(\mu+\alpha_1)\phi_1+(\lambda+x)\phi_0 &= -r_0,\\
-\alpha_2\phi_2+\alpha_1\phi_1+\alpha_1\phi_0 &= 0,\\
-\alpha_n\phi_n+\alpha_{n-1}\phi_{n-1} &= a(x)\phi_n-\lambda\phi_n,\ n=3,\ldots,N.
\end{aligned} \tag{19}$$

$$\begin{aligned}
-(\lambda+\alpha_1+x)g_0+\mu g_1+\alpha_N g_N &= a(x)r_0,\\
-(\mu+\alpha_1)g_1+(\lambda+x)g_0 &= a(x)r_1-\lambda r_1+xr_0,\\
-\alpha_2 g_2+\alpha_1 g_1+\alpha_1 g_0 &= a(x)r_2-\lambda r_2-\alpha_1 r_1,\\
-\alpha_n g_n+\alpha_{n-1}g_{n-1} &= a(x)r_n-\lambda r_n,\ n=3,\ldots,N.
\end{aligned} \tag{20}$$

We take into account the last equation of (12) with the substitution (16) up to $O(\varepsilon^3)$. Taking the limit by $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\frac{\partial\Phi(w,\tau)}{\partial\tau} &= (jw)^2\Big(-a(x)\sum_{k=0}^{N}f_k+xr_0-xf_0+\alpha_1 f_1+\lambda\sum_{n=1}^{N}f_n\Big)\Phi(w,\tau)\\
&+j^2w\frac{\partial\Phi(w,\tau)}{\partial w}r_0+\frac{(jw)^2}{2}\Big(-xr_0+\alpha_1 r_1+\lambda\sum_{n=1}^{N}r_n\Big)\Phi(w,\tau).
\end{aligned} \tag{21}$$

We will make a substitution (18) into (21) to obtain

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} = (jw)^2\Big(xr_0-xg_0+\alpha_1 g_1+\lambda\sum_{n=1}^{N}g_n\Big)\Phi(w,\tau)$$

$$-(jw)^2(-x\phi_0+\alpha_1\phi_1+\lambda\sum_{n=1}^{N}\phi_n-r_0)\frac{\partial\Phi(w,\tau)/\partial w}{w}$$
$$+\frac{(jw)^2}{2}(-xr_0+\alpha_1 r_1+\lambda\sum_{n=1}^{N}r_n)\Phi(w,\tau). \quad (22)$$

We note that

$$-x\phi_0+\alpha_1\phi_1+\lambda\sum_{n=1}^{N}\phi_n-r_0=a'(x).$$

We denote

$$b(x)=a(x)+2\Big(-xg_0(x)+\alpha_1 g_1(x)+\lambda\sum_{k=1}^{N}g_k(x)+xr_0(x)\Big). \quad (23)$$

We have

$$\frac{\partial\Phi(w,\tau)}{\partial\tau}=a'(x)w\frac{\partial\Phi(w,\tau)}{\partial w}+\frac{(jw)^2}{2}b(x)\Phi(w,\tau).$$

Theorem is proved.

Lemma 1. *The stationary probability density of the normalized and centered number of calls in the orbit is given by*

$$s(z)=\frac{C}{b(z)}exp\Big\{\frac{2}{\sigma}\int_0^z\frac{a(x)}{b(x)}\,dx\Big\}, \quad (24)$$

where C is subject to the normalization condition.

The proof of the Lemma 1 is described in paper [17].

We define a non-negative function $G(i)$ of discrete argument i in the form

$$G(i)=\frac{C}{b(\sigma i)}exp\Big\{\frac{2}{\sigma}\int_0^{\sigma i}\frac{a(x)}{b(x)}\,dx\Big\}. \quad (25)$$

After that, we construct an approximation $P_{dif}(i)$ of the probability distribution $P(i)=P\{i(t)=i\}$ of i calls in orbit for RQ-systems using the formula (24):

$$P_{dif}(i)=\frac{G(i)}{\sum_{i=0}^{\infty}G(i)}. \quad (26)$$

6 Algorithm of Calculation Probability Distribution $P_{dif}(i)$

1. Let us define the parameters of the system λ, σ, μ, and α_n, $n=1,...,N$.

2. We calculate $r_0(x)$ and determine probabilities $r_n(x)$, $n = 1, ..., N$ as functions of x using (9).
3. We substitute $r_n(x)$ into (10) and obtain drift coefficient $a(x)$.
4. We determine additional functions $g_n(x)$, $n = 0, ..., N$ as the solution of system (15) for each x.
5. Substituting $r_n(x)$ and $g_n(x)$ into (23), we obtain diffusion coefficient $b(x)$.
6. Having $a(x)$ and $b(x)$, we can calculate values $G(i)$.
7. Finally, we apply formula (26) to determine stationary probability distribution $P_{dif}(i)$ of the number of customers in the orbit.

7 Domain of Applicability of the Asymptotic-Diffusion Analysis

This section is devoted to determining the area of applicability of the obtained asymptotic-diffusion results. Comparing the asymptotic results with the pre-limit distribution obtained earlier in [10], we can determine the parameters, for which approximate probability distribution is close to the pre-limit one. Let us define the parameters of the system $\lambda = 1$, $\mu = 3$, $N = 5$, $\alpha_n = 0.2 \times n$, $n = 1, ..., N$.

Table 1 shows the values of Kolmogorov distance for various σ.

Table 1. Kolmogorov distance Δ.

	$\sigma = 0.5$	$\sigma = 0.1$	$\sigma = 0.05$	$\sigma = 0.01$	$\sigma = 0{,}005$
Δ	0.078	0.073	0.056	**0.049**	**0.037**

Analyzing the data of Table 1, we can say that the approximation accuracy increases with a decrease in σ. The given approximations are applicable for the Kolmogorov distance not exceeding 0.05. Values of accuracy, which we consider as satisfactory, are marked bold in Table 1. From the obtained values, we can conclude that the approximation of the asymptotic-diffusion analysis is applicable when $\sigma \leq 0.01$.

8 Conclusion

This paper presents a study of a cyclic retrial queue for a special data transmission network FANET. The analysis of the presented model is carried out by the method of asymptotic-diffusion analysis. Based on the results obtained, an approximation of the probability distribution of the number of calls in orbit was constructed. In the chapter on numerical analysis, the accuracy of the constructed approximation of the asymptotic-diffusion analysis is shown.

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