The Discrete-Time Queueing System with Inversive Service Order and Probabilistic Priority

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ABSTRACT

This paper considers a queueing system $Geo/G/1/\infty$ with inversive service order and probabilistic priority that is determined as follows. Upon arrival into the system of a new customer its length i is compared with the (remaining) length j of the customer in the device and with the probability of $\overline{d}(i,j)$, which depends only on i and j, a new arrival will occupy the server while pushing out a servicing customer to the first place in the queue, and with the supplemental probability $d(i,j) = 1 - \overline{d}(i,j)$, alternatively, a customer that was under service keeps on being served while the newly arrived one occupies the first position in the queue. The remaining queue shifts by one state. The main stationary characteristics of such system's behavior have been found.

General Terms

Queueing theory

Keywords

Queueing system, discrete time, "non-standard" service discipline

1. INTRODUCTION

It is known that application of "nonstandard" service discipline can significantly improve user characteristics of queueing systems (QS). Thus, for the QSs with waiting an absolute champion in terms of the minimal queue length within the conservative service disciplines class (a discipline is called conservative if service procedure does not depend on the customer arrival process and at any time when the customers are available in the system the total service rate is equal to one unit) is the priority service discipline for a customer of the minimal remaining length, or SRPT (Shortest Remaining Processor Time) [19, 18]. However, mathematical relations for calculating the characteristics of functioning (in particular the stationary distribution of customers number), even for QS $M/G/1/\infty$ with SRPT discipline are rather complicated [4, 17, 9, 14] and merely demonstrate the capabilities of up-to-date mathematical constructions and can hardly be used for any practical computations.

In the works [13, 11, 12] there was introduced first for a particular case and then generically an inversive service discipline with probabilistic priority and the main stationary characteristics were defined for the QS $M/G/1/\infty$ with this discipline. By this discipline the authors succeeded to prove an interesting A.D.Solovyev hypothesis: at a given load under stationary operating mode the number of customers in the QS $M/G/1/\infty$ with SRPT discipline is maximal in the matter of ordering "$\prec$" of distribution functions (i.e. such ordering that $\xi \prec \eta$, if $F_\xi(x) \geq F_\eta(x)$ for all x) at the constant length of customers, and also obtain different inequalities for queue length for some QSs. Further there appeared many works generalizing discipline of the inversive service order with probabilistic priority (see, for instance, [16, 20, 21, 22, 23, 24]).

Let us refer now to somewhat different problems of contemporary queueing theory. The use of up-to-date technologies in information and telecommunication networks resulted in new burst of studying discrete-time QSs (see, for instance, [5, 7, 10, 6, 8, 2, 1]). Note that with the seeming simplification of the methods of discrete-time QS study in comparison with continuous-time QSs, in some cases the analysis becomes significantly complicated. This is, first of all, due to the fact that in contrast to continuous time, in discrete time several state changes can occur simultaneously (in particular, completion of servicing the customer on the server and the arrival of a new one to the system).

The present work defines the main stationary characteristics of the system $Geo/G/1/\infty$ with inversive service order and probabilistic priority, which is a discrete analog of the QS $M/G/1/\infty$ with the same discipline.

2. SYSTEM DESCRIPTION

Let's consider a queueing system under discrete time $Geo/G/1/\infty$, a geometrical input flow of customers entering into it with the probability a of customer arrival at one slot (below we'll call a slot both the time interval between the nearest neighbor changes in the system state and the moments when such changes occur). The distribution of customer servicing is arbitrarily discrete having the probability of b_i, $i \geq 0$, that servicing a customer would last i slots (it is assumed that $b_0 = 0$).

Henceforth we'll use the following notation:

$\overline{a} = 1 - a$ — the probability of customer non-arrival into

SMCTools 2008, October 20, 2008, Athens, GREECE.

the slot;

$B_i = \sum_{j=i}^{\infty} b_j, \quad i \geq 0,$ — the probability that customer servicing would last not less than i slots;

$\overline{B}_i = \sum_{j=0}^{i-1} b_j = 1 - B_i, \quad i \geq 1,$ — the probability that customer servicing would last less than i slots;

$\overline{b} = \sum_{i=0}^{\infty} i b_i = \sum_{i=1}^{\infty} B_i$ — mean customer servicing time.

Let's assume that the system load $\rho = a\overline{b}$ is less than one unit. This condition is necessary and sufficient for the existence of stationary regime for the system.

The inversive service order with probabilistic priority consists in the following. It is assumed that at any instant of time the remaining length (henceforth we'll call it simply the length) of each customer in the system is known, i.e. a number of slots needed to finalize servicing of a given customer. At the moment of new customer arrival its length i is compared with the (remaining) length j of a customer on the server and with the probability $\overline{d}(i,j)$, dependent only on i and j the newly entered customer occupies the server and pushes out a previously served customer to the first place in the queue while alternatively with the supplemental probability of $d(i,j) = 1 - \overline{d}(i,j)$ a customer that had been under service continues to be served while the newly arrived one occupies the first position in the queue. The remaining queue shifts by one position. For the sake of definiteness we'll consider that if at a certain moment the servicing of one customer was finalized while another one arrived into the system, then independently on the lengths of all customers residing in the queue a newly arrived one would be entered into the server. Customers with interrupted servicing continue their servicing.

3. STATIONARY DISTRIBUTION OF THE QUEUE

Let's find the stationary distribution of the number of customers in the system.

Let's enter the following notions:

p_0 — stationary probability that right after the next slot the system would become idle;

$p_n(i), \quad i \geq 1,$ — stationary probability that right after the next slot there would be n customers in the system and prior to completion of servicing a customer on the server there will remain i slots.

We set

$$\overline{P}_n(j) = \sum_{i=j+1}^{\infty} p_n(i), \quad n \geq 1, \quad j \geq 0,$$

$$P_n(j) = \sum_{i=1}^{j} p_n(i) = \overline{P}_n(0) - \overline{P}_n(j), \quad n \geq 1, \quad j \geq 1,$$

$$p_n = \sum_{i=1}^{\infty} p_n(i) = \overline{P}_n(0), \quad n \geq 1.$$

Let's put down a system of steady-state equations (SSE), that is satisfied by the introduced functions.

First of all, the local balance usage gives:

$$a p_0 = \overline{a} p_1(1), \quad n \geq 0, \tag{1}$$

$$a(p_n - p_n(1)) = \overline{a} p_{n+1}(1), \quad n \geq 1. \tag{2}$$

Next, it is easy to proceed as follows [13]. Let's introduce a new QS $Geo/G/1/n$ with a finite buffer of volume n, which is different from the original one only by the fact that if n customers present in the queue, a customer of i length is being served on the server and a new customer of j length arrives, then with the probability $d(i,j)$ the newly arrived customer stays in the system (on the server) while the one that had been previously served abandons it, and vice versa, with the probability $\overline{d}(i,j)$ the newly arrived customer abandons the system while the one that had previously been on the server continues to be served.

By using the method introduced in [13] and presented in detail in [3], pages 22–29, one can easily demonstrate that stationary probabilities of the states in the original and new QSs differ from each other only by the constant multiplier. This allows to write down the following SSE:

$$p_1(i) = p_0 a b_i + p_1(i+1)\overline{a} + p_1(1) a b_i +$$
$$+ a b_i \sum_{j=1}^{\infty} p_1(j+1) d(i,j) + a p_1(i+1) \sum_{j=1}^{\infty} b_j \overline{d}(j,i), \quad i \geq 1, \tag{3}$$

$$p_n(i) = a b_i \sum_{j=1}^{\infty} p_{n-1}(j+1)\overline{d}(i,j) +$$
$$+ a p_{n-1}(i+1) \sum_{j=1}^{\infty} b_j d(j,i) + p_n(i+1)\overline{a} +$$
$$+ p_n(1) a b_i + a b_i \sum_{j=1}^{\infty} p_n(j+1) d(i,j) +$$
$$+ a p_n(i+1) \sum_{j=1}^{\infty} b_j \overline{d}(j,i), \quad n \geq 2, \quad i \geq 1, \tag{4}$$

to which it is necessary to add the normalization requirement

$$p_0 + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} p_n(i) = p_0 + \sum_{n=1}^{\infty} \overline{P}_n(0) = p_0 + \sum_{n=1}^{\infty} p_n = 1. \tag{5}$$

Possible way of solving a system of equations (1)–(5) consists in the following. In the first instance from equation (1) $p_1(1)$ is expressed via p_0. Then from equation (3) one can determine $p_1(i), \quad i \geq 2$, where it is convenient to use the iterative method for making calculations. Next, with the help of equation (2) $p_2(1)$ is expressed via p_0, from the equation (4) we find $p_2(i), \quad i \geq 2$, etc. Finally, p_0 is withdrawn from the normalization requirement (however, it will be shown further that, as is typically the case, for the conservative systems $p_0 = 1 - \rho$).

A system of equations (1)–(4) can be reduced to a single equation by using the apparatus of generating functions (GF). By entering GF

$$\pi(z,i) = \sum_{n=1}^{\infty} z^n p_n(i), \quad i \geq 1,$$

$$\pi(z) = \sum_{n=0}^{\infty} z^n p_n = p_0 + \sum_{i=1}^{\infty} \pi(z,i),$$

we have from the equations (1) and (2)

$$a\pi(z) = \left(\frac{\overline{a}}{z} + a\right)\pi(z,1), \tag{6}$$

and from the equations (3) and (4) —

$$\pi(z,i) = ab_i z p_0 + ab_i \sum_{j=1}^{\infty}[z\overline{d}(i,j) + d(i,j)]\pi(z,j+1) +$$

$$+ \left(\overline{a} + a\sum_{j=1}^{\infty} b_j[zd(j,i) + \overline{d}(j,i)]\right)\pi(z,i+1) +$$

$$+ ab_i\pi(z,1), \quad i \geq 1. \tag{7}$$

Remarks on solving a system of equations (6)–(7) are the same as with regard to solving a system (1)–(4).

Supposing that $z = 1$ in the formulae (6) and (7), we get after having made simple transformations:

$$\pi(1,1) = \sum_{n=1}^{\infty} p_n(1) = a,$$

$$\sum_{i=1}^{\infty} \pi(1,i) = a\sum_{i=1}^{\infty} B_i = \rho.$$

We have from here:

$$p_0 = 1 - \rho.$$

It is not likely that a system of equations (6)–(7) can be used for calculating stationary probabilities of the states, but as it'll be shown now by the example of mathematical expectation, through its help one can easily find the moments of stationary distribution of the number of customers in the system.

Using the relations (6) and (7), let's calculate the derivatives $\pi'(z)$ and $\pi'(z,i)$:

$$a\pi(z) + az\pi'(z) = a\pi(z,1) + (\overline{a} + az)\pi'(z,1),$$

$$\pi'(z,i) = ab_i p_0 + ab_i \sum_{j=1}^{\infty} \pi(z,j+1)\overline{d}(i,j) +$$

$$+ ab_i \sum_{j=1}^{\infty} z\pi'(z,j+1)\overline{d}(i,j) + a\pi(z,i+1)\sum_{j=1}^{\infty} b_j d(j,i) +$$

$$+ az\pi'(z,i+1)\sum_{j=1}^{\infty} b_j d(j,i) + \pi'(z,i+1)\overline{a} +$$

$$+ \pi'(z,1)ab_i + ab_i \sum_{j=1}^{\infty} \pi'(z,j+1)d(i,j) +$$

$$+ a\pi'(z,i+1)\sum_{j=1}^{\infty} b_j\overline{d}(j,i), \quad i \geq 1.$$

Supposing that in these formulae $z = 1$, we have:

$$\pi'(1,1) = a\overline{a} + a\pi'(1),$$

$$\pi'(1,i) = ab_i p_0 + ab_i\pi'(1) + a^2 b_i \sum_{j=1}^{\infty} \overline{d}(i,j)B_{j+1} +$$

$$+ a^2 \sum_{j=1}^{\infty} b_j d(j,i)B_{i+1} + \pi'(1,i+1), \quad i \geq 1.$$

Then, by expressing sequentially $\pi'(1,i)$ first via $\pi'(1,i-1)$, and then via $\pi'(1,i-2)$ etc., we arrive at the equality

$$\pi'(1,i+1) = aB_{i+1}\pi'(1) +$$

$$+ \sum_{l=i+1}^{\infty}\left(ab_l p_0 + \right.$$

$$\left. + a^2 b_l \sum_{j=1}^{\infty} \overline{d}(l,j)B_{j+1} + a^2\sum_{j=1}^{\infty} b_j d(j,l)B_{l+1}\right), \quad i \geq 1.$$

By summing this equation over i from 1 to ∞, we get

$$\pi'(1)(1-\rho) = a\overline{a} + \sum_{i=1}^{\infty}\sum_{l=i+1}^{\infty}\left(ab_l p_0 + \right.$$

$$\left. + a^2 b_l \sum_{j=1}^{\infty} \overline{d}(l,j)B_{j+1} + a^2\sum_{j=1}^{\infty} b_j d(j,l)B_{l+1}\right).$$

Thus, the stationary mean number $N = \pi'(1)$ of customers in the system is defined by the formula

$$N = \frac{1}{1-\rho}\left[a\overline{a} + \sum_{i=1}^{\infty}\sum_{l=i+1}^{\infty}\left(ab_l p_0 + \right.\right.$$

$$\left.\left. + a^2 b_l \sum_{j=1}^{\infty} \overline{d}(l,j)B_{j+1} + a^2\sum_{j=1}^{\infty} b_j d(j,l)B_{l+1}\right)\right]$$

In like manner, one can calculate the moments of higher orders of stationary distribution of the number of customers in the system.

4. STATIONARY DISTRIBUTION OF CUSTOMER SOJOURN TIME IN THE SYSTEM

Now, let's determine stationary distribution of customer sojourn time in the system.

Note instantly that the system busy period (BP) opened by a customer of the length i has the GF

$$\gamma_i(z) = (z[a\gamma(z) + \overline{a}])^i, \quad i \geq 1, \tag{8}$$

where $\gamma(z)$ — is GF opened by an arbitrary customer satisfying the equation

$$\gamma(z) = \beta(z[a\gamma(z) + \overline{a}]), \tag{9}$$

where $\beta(z) = \sum_{i=1}^{\infty} z^i b_i$ — GF of customer service time.

Stationary distribution of waiting for service time (further waiting time) for a customer of the length i has the GF

$$\psi_i(z) = p_0 + \sum_{n=1}^{\infty}\left(p_n(1) + \right.$$

$$\left. + \sum_{j=1}^{\infty}[d(i,j) + \overline{d}(i,j)\gamma_j(z)]p_n(j+1)\right), \quad i \geq 1. \tag{10}$$

Distribution of the total time of servicing a customer of the length i (i.e. a period of time from the moment of first arrival of a customer on the server till its exit from the system) has the GF

$$\delta_1(z) = z, \tag{11}$$

$$\delta_i(z) = z\left(a\sum_{j=1}^{\infty} b_j[d(j,i-1)\gamma_j(z) + \overline{d}(j,i-1)] + \right.$$

$$\left. + \,\overline{a}\right)\delta_{i-1}(z), \quad i \geq 2. \qquad (12)$$

Finally, stationary distribution of the sojourn time of a customer of the length i in the system has the GF

$$\varphi_i(z) = \psi_i(z)\delta_i(z). \qquad (13)$$

Corresponding unconditional characteristics are determined by averaging over the length of a customer. For instance, stationary distribution of the sojourn time of an arbitrary customer in the system has the GF

$$\varphi(z) = \sum_{i=1}^{\infty} b_i\varphi_i(z). \qquad (14)$$

By differentiating the formulae (8)–(14), one can get expressions for the moments of the corresponding stationary characteristics. In particular, the mean length of the BP and the mean length of the BP opened by a customer of the length i are defined by formulae

$$\overline{g} = \frac{\overline{b}}{1-\rho}, \quad \overline{g}_i = \frac{i}{1-\rho}.$$

Mathematical expectation of stationary distribution of waiting time of a customer of the length i starts and mathematical expectation of the total time of servicing a customer of the length i are defined by the formulae

$$\overline{v}_i = \frac{1}{1-\rho}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} j\overline{d}(i,j)p_n(j+1), \quad i \geq 1,$$

$$\overline{d}_1 = 1, \quad \overline{d}_i = 1 + \frac{a}{1-\rho}\sum_{j=1}^{\infty} jb_j d(j,i-1) + \overline{d}_{i-1}, \quad i \geq 2.$$

Mathematical expectations of stationary distributions of the sojourn times of a customer of the length i and an arbitrary customer in the system are defined by the formulae

$$\overline{w}_i = \overline{v}_i + \overline{d}_i, \quad i \geq 1, \quad \overline{w} = \sum_{i=1}^{\infty} \overline{w}_i b_i.$$

5. PARTICULAR CASES

Let's consider some particular cases dealing with the system under study. For this purpose we'll demonstrate only a convenient method of calculating stationary distribution of the number of customers in the system. The task of finding distributions related to customer's sojourn time in the system does not require any special commentaries.

5.1 Example 1

Let $\overline{d}(i,j) \equiv 0$. Then we have an inversive order of servicing without interruption of servicing, under this order stationary probabilities of the states comply with stationary probabilities of the states under a standard discipline FCFS (First Come FirstServed). In this case the GF of stationary distribution of the number of customers in the system has the form of

$$\pi(z) = \frac{1-z}{\beta(\overline{a}+az)-z}(1-\rho).$$

5.2 Example 2

In the case when $\overline{d}(i,j) \equiv 1$, we arrive at inversive order of servicing with interruption of servicing. Then

$$p_n = \left(\frac{a(\overline{b}-1)}{\overline{a}}\right)^{n-1}\frac{\rho}{\overline{a}}(1-\rho), \quad n \geq 1.$$

We note that similarly to an analogous continuous-time system, when a fixed mean length of a customer is $\overline{b}$, the stationary distribution of the number of customers in the system does not depend on the customer length distribution. However, unlike a continuous-time system, firstly, the stationary distribution of the number of customers in the discrete-time system depends not only on the load ρ but also on the parameters a and $\overline{b}$ and, secondly, is defined by geometrical progression starting only from the probability p_1.

5.3 Example 3

Let

$$\overline{d}(i,j) = \begin{cases} 1, & i < j, \\ 0, & i \geq j. \end{cases}$$

Then service discipline consists in the following [15]. At the moment of a new customer arrival its length is compared with the length of a customer on the server, and the one having the minimal length enters on the server (or continues its servicing) and the second one occupies the first position in the queue. In this case the following recurrent procedure for n and j is valid:

$$p_1(1) = \frac{a}{\overline{a}}\,p_0,$$

$$p_1(i+1) = \frac{1}{1-aB_{i+1}}\{p_1(i) - [p_0 + P_1(i)]ab_i\}, \quad i \geq 1,$$

$$p_n(1) = \frac{a}{\overline{a}}[p_{n-1} - p_{n-1}(1)], \quad n \geq 2,$$

$$p_n(i+1) = \frac{1}{1-aB_{i+1}}\{p_n(i) - p_{n-1}(i+1)aB_i -$$

$$- [\overline{P}_{n-1}(i+1) + P_n(i)]ab_i\}, \quad n \geq 2, \quad i \geq 1.$$

5.4 Example 4

If

$$\overline{d}(i,j) = \begin{cases} 0, & i \leq j, \\ 1, & i > j. \end{cases}$$

then we deal with a discipline inverted toward the discipline that was considered in the previous paragraph: of the two customers under comparison the one of these that has the maximum length enters on the server while another one having the minimal length occupies the first position in the queue. Assuming that

$$c_i = (\overline{a} + aB_{i+1})^{-1}, \quad i \geq 1,$$

$$\widetilde{p}_1(i) = ab_i[p_0 + p_1(1)], \quad i \geq 1,$$

$$\widetilde{p}_n(i) = ab_i[P_{n-1}(i) - p_{n-1}(1) + p_n(1)] +$$

$$+ \,a\overline{B}_{i+1}p_{n-1}(i+1), \quad n \geq 2, \quad i \geq 1,$$

we have

$$p_n(i+1) = C_n(i) - D_n(i)p_n, \quad n \geq 1, \quad i \geq 1,$$

where $C_n(i)$ and $D_n(i)$ are defined with the help of numbers $E_n(i)$ and $F_n(i)$ from the recurrent relations

$$E_n(1) = p_n(1), \quad n \geq 1, \qquad F_n(1) = 0, \quad n \geq 1,$$

$$C_n(1) = c_1[(1+ab_i)p_n(1) - \widetilde{p}_n(1)], \quad n \geq 1,$$

$$D_n(1) = c_1 ab_1, \quad n \geq 1,$$

$$E_n(i) = E_n(i-1) + C_n(i-1), \quad n \geq 1, \quad i \geq 2,$$

$$F_n(i) = F_n(i-1) + D_n(i-1), \quad n \geq 1, \quad i \geq 2,$$

$$C_n(i) = c_i[C_n(i-1) - \widetilde{p}_n(i) + ab_i E_n(i)], \quad n \geq 1, \quad i \geq 2,$$

$$D_n(i) = c_i[D_n(i-1) + ab_i(1 + F_n(i))], \quad n \geq 1, \quad i \geq 2,$$

and p_n is calculated on condition that $p_n(i) \to 0$ when $i \to \infty$, i.e. from the formula

$$p_n = \lim_{i\to\infty} \frac{C_n(i)}{D_n(i)}.$$

5.5 Example 5

In conclusion, let's consider one more variant of the general discipline, under which calculation of stationary probabilities of the states can be done on the basis of solving a system of linear algebraic equations. Namely let us suppose that

$$d(i,j) = \sum_{k=1}^{K} d_k^{(1)}(i) d_k^{(2)}(j).$$

Then, by introducing the notations

$$x_{nk} = \sum_{j=1}^{\infty} p_n(j+1) d_k^{(2)}(j), \tag{15}$$

we get from (3) and (4):

$$p_1(i+1) = \Bigg(p_1(i) - p_0 ab_i - p_1(1)ab_i - $$

$$- ab_i \sum_{k=1}^{K} d_k^{(1)}(i) x_{nk}\Bigg)\Bigg(\overline{a} + a\sum_{j=1}^{\infty} b_j \overline{d}(j,i)\Bigg)^{-1}, \quad i \geq 1,$$

$$p_n(i+1) = \Bigg(p_n(i) - ab_i \sum_{j=1}^{\infty} p_{n-1}(j+1)\overline{d}(i,j) - $$

$$- ap_{n-1}(i+1)\sum_{j=1}^{\infty} b_j d(j,i) - p_n(1)ab_i - ab_i \sum_{k=1}^{K} d_k^{(1)}(i) x_{nk}\Bigg) \times$$

$$\times \Bigg(\overline{a} + a\sum_{j=1}^{\infty} b_j \overline{d}(j,i)\Bigg)^{-1}, \quad n \geq 2, \quad i \geq 1.$$

By inserting the computed expressions $p_n(i+1)$ into (15), we arrive in the system of linear algebraic equations with the aim of determining unknown coefficients x_{nk}.

6. INEQUALITY RELATIONS FOR THE NUMBER OF CUSTOMERS IN THE SYSTEM

We'll give the boundaries into which the number of customers present in the system under study can change in dependence on initial parameters.

à Let us introduce the stochastic process

$$\nu(t) = \{\nu_1(t), \nu_2(t), \ldots, \quad t \geq 0\},$$

where $\nu_1(t)$ is the total length af all the customers in the system just before the moment t, $\nu_2(t)$ is the total length of all customers in the system before that moment beginning with the penultimate one, etc. It is evident that if before the moment t there are n customer in the system, then $\nu_m(t) = 0$ under $m > n$, and the number $\mu(t)$ of customers in the system can be defined by the formula

$$\mu(t) = \max\{n: \ \nu_n(t) > 0\}.$$

It is also clear that $\nu_{\mu(t)}(t)$ is the customer length on the server.

Let us introduce the ordering "$\prec$" of distribution functions. Let us denote $F_\xi(x) \prec F_\eta(x)$, if $F_\xi(x) \geq F_\eta(x)$ for all x. It is not too difficult to see that $F_\xi(x) \prec F_\eta(x)$ if and only if ξ and η have the defined on the same probabilistic space equivalent stochastic variables ξ' and η' such that $\xi' \leq \eta'$.

Let there be now the three queueing systems $Geo/G/1/\infty$, into which the same flow of customers arrive, but in the first one there was implemented an arbitrary inversive order of servicing with probabilistic priority (i.e. this is a system with arbitrary probabilities $d(i,j)$), and other two ones were considered in the Examples 3 and 4 of the previous section. Let us call the first QS a general one and the second and third ones — the 1st and 2nd systems. For the general system let us denote the above-introduced process by $\nu(t)$, and for the 1st and 2nd systems let us assign the indexes $\nu'(t)$ and $\nu''(t)$.

Theorem 1. *Under the same starting values the stochastic processes $\nu(t)$, $\nu'(t)$ and $\nu''(t)$ are related to each other by inequalities*

$$\nu'(t) \leq \nu(t) \leq \nu''(t), \quad t \geq 1. \tag{17}$$

Proof. Proof by induction in accordance with the slots t is not presented here. □

From the proved theorem it follows in particular that distributions $F_{\mu(t)}(x)$, $F_{\mu'(t)}(x)$ and $F_{\mu''(t)}(x)$ of the number of customers in the general, 1st and 2nd systems are interrelated in the sense of ordering "$\prec$" by the inequalities

$$F_{\mu'(t)}(x) \prec F_{\mu(t)}(x) \prec F_{\mu''(t)}(x).$$

Naturally, such ordering is also valid for stationary distributions of the number of customers in the same systems.

We note, that inequalities for the number of customers in the 1st system and system $Geo/G/1/\infty$ under some other service disciplines in the sense of ordering "$\prec$" are presented in the work [15].

7. CONCLUSION

In conclusion, let us touch on some interesting peculiarities of the results outlined in the present work.

First of all, it is necessary to point out that while constructing and solving the system of steady-state equations we used a non-Markovian process. But this in no way influenced the correctness of obtained results. Really, one could consider more complicated but Markovian process generated by remaining time of all customers, but not only by arriving and servicing customers. But then the Kolmogorov–Chapman equation can be summed over by "unessential" coordinates, which will lead to the SSE that we received.

Next, considering the QS variants under discrete time it is always interesting to clear up the difference between the obtained relations and their analogs for the QSs under continuous time. It is necessary to say that with regard to this given QS there is no significant difference between the methods of constructing the equations in both cases.

It should be also noted that an inversive order of servicing with probabilistic priority in continuous time has several analogs in discrete time. This is because in contrast with continuous-time systems a number of events related with arrival and departure of customers can occur simultaneously in discrete-time systems. In particular, we can consider a variant of inversive order of servicing with probabilistic priority under which if at the moment of a new customer arrival the customer that had already been served leaves the system, then the lengths of an arriving customer and the first customer in the queue are compared. However, constructing the SSE in this case it is necessary to take into account also the length of the first customer in the queue.

8. ACKNOWLEDGMENTS

This work was supported by the grants 06-07-89056 and 08-07-00152 of the Russian Foundation for Basic Research.

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