An Action of the Picard Group on Generalized Euler Classes

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Abstract. In this paper, we construct an action of Picard group on the generalized Euler classes defined in Chow-Witt groups $\widetilde{CH^d(X,\mathscr{L})}$ with twisted line bundle $\mathscr{L}$, which is the generalization of the Euler classes defined in Chow groups. This result gives a new method to compare the generalized Euler classes with different twisted bundle.

Keywords: Picard group · Euler classes · Chow groups

1 Introduction

Let X =Spec(A) be a smooth affine Noetherian scheme of dimension d over a field k, $\mathscr{E}$ be a vector bundle over X of rank d with determinant isomorphic to a line bundle $\mathscr{L}$. One could give an obstruction class in algebraic geometry, such that its vanishing is the only obstruction to $\mathscr{E}$ splitting as the direct sum of a rank $d-1$ vector bundle and a trivial line bundle. Or more precisely, to give an obstruction classes in algebraic geometry which is an analog of the Euler classes in algebraic topology. Historically there are mainly three different theories be constructed for this problem. One theory comes from the work of S. Bhatwadekar and R. Sridharan [10], which are based upon the ideal of M.V. Nori [16]. This theory defines the obstruction in a pure algebraically constructed group named generalized Euler class group $E^d(X;\mathscr{L})$. Second theory comes from the lecture of F. Morel [20], where one constructs an Euler class as the primary obstruction to existence of a non-vanishing section of $\mathscr{E}$. Let Gr_d denotes the infinite Grassmannian, and γ_d is the universal rank d vector bundle on Gr_d, the first non-trivial stage of the Moore-Postnikov factorization in A^1-homotopy theory of the map $Gr_{d-1}\to Gr_d$ gives rise to a canonical morphism

$$Gr_d \to K^{G_m}(\mathbf{K}_d^{MW},\ d)$$

This gives a canonical cohomology class

$$o_d \in H^d_{Nis}(Gr_d, \mathbf{K}_d^{MW}(det(\gamma_g^{vee})))$$

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And further more, given any smooth scheme X and vector bundle $\mathscr{E}$ as above. The map $\zeta : X \to Gr_d$ which classifying $\mathscr{E}$ defines a class

$$e_m(\mathscr{E}) := \zeta^*(o_d) \in H^d_{Nis}(X, \mathbf{K}^{MW}_d(det(\mathscr{E}))) = \widetilde{CH^d(X, (\mathscr{E})}$$

This definition of obstruction class is evidently more homotypical.

The third method is constructed by J. Barge and F. Morel in [9], which is a "cohomological" version of the obstruction. Or more precisely, they give an cohomological class

$$e_c(\mathscr{E}) \in H^d_{Nis}(X, \mathbf{K}^{MW}_d(det(\mathscr{E}))) = \widetilde{CH^d(X, (\mathscr{E})}$$

for the vector bundle $\mathscr{E}$ using push-forward operation in cohomology theory of sheaf of Milnor-Witt groups. The associated obstruction theory is called by A. Asok and J. Fasel in paper [6] as the "cohomological obstruction theory".

By the works of A. Asok and J. Fasel in papers [6] and [4], and the work of Yang in paper [26], we know that all of the theories above are equivalent, and all of the Euler classes (obstruction classes) defined by the above theories are the same under these equivalence.

By the results in paper [11] of S. M. Bhatwadekar and Raja Sridharan, we know that for different line bundles $\mathscr{L}$ and $\mathscr{L}'$, the Euler class groups $E^d(X;\mathscr{L})$ and $E^d(X;\mathscr{L}')$, or equivalently Chow-witt groups $\widetilde{CH^d(X, \mathscr{L})}$ and $\widetilde{CH^d(X, \mathscr{L}')}$, are not isomorphic generally. Now for any pair of line bundles $\mathscr{L}$ and $\mathscr{L}'$, we could regard them as the elements in the Picard group $\mathrm{Pic}(X)$ of X, and there is an element $\alpha = \mathscr{L}^{-1}\mathscr{L}'$, such that $\alpha \cdot \mathscr{L} = \mathscr{L}'$. In this paper, we will give a construction for a lifting of this action to an action of Picard on the generalized Euler class groups. More precisely, for any element $\alpha \in \mathrm{Pic}(X)$ and element $\rho \in E^d(X;\mathscr{L})$ we could define an element $\alpha \cdot \rho \in E^d(X; \alpha \cdot \mathscr{L})$ which defines an action of group on sets.

Notations

In this paper, a base field k is always denoted a (infinite) perfect field. We write Sm_k for the category of separated, smooth and finite type schemes over $\mathrm{Spec}(k)$. $sPre(Sm_k)$ denotes the category of simplicial presheaves over Sm_k, objects of which will be called $k-$spaces. We equip the category $sPre(Sm_k)$ with its usual injective Nisnevich local model structure [13]. The associated homotopy category is denoted by $\mathscr{H}_{Nis}(k)$. If $\mathcal{X}$ and $\mathcal{Y}$ are spaces in $\mathscr{H}_{Nis}(k)$, then we write $[\mathcal{X}, \mathcal{Y}]_{Nis}$ for the set $Hom_{\mathscr{H}_{Nis}(k)}(\mathcal{X}, \mathcal{Y})$.

Using a left Bousfield localization of $\mathscr{H}_{Nis}(k)$, one could get the Morel-Voevodsky A^1-homotopy category $\mathscr{H}(k)$. In particular, there is an endo-functor $\mathscr{L}_{A^1}$ of the category $sPre(Sm_k)$, together with a natural transformation $\theta : Id \to \mathscr{L}_{A^1}$ such that for any space $\mathcal{X}$, the map $\mathcal{X} \to \mathscr{L}_{A^1}(\mathcal{X})$ is an trivial A^1-cofibration and $\mathscr{L}_{A^1}(\mathcal{X})$ is simplicially fibrant and A^1-local. We refer the reader to [1] for a convenient summary of properties of the A^1-localization functor and note in passing that $\mathscr{L}_{A^1}$ commutes with the formation of finite products.

We set $[\mathcal{X},\ \mathcal{Y}]_{A^1} := Hom_{\mathscr{H}_k}(\mathcal{X},\ \mathcal{Y})$. Beside these, we write $S^{i+j,j} = S^i \wedge \mathbf{G}_m^{\wedge j}$ for the bigrade spheres in $\mathscr{H}(k)$. Particular S^i denotes the constant presheave associated with the $i-$th simplicial sphere. Further, unless we mention otherwise, sheaf cohomology will always be taken with respect to Nisnevich topology.

Beside these, in this paper, we sometimes use the letters with an arrow hat $\boldsymbol{X}$ or $\boldsymbol{a}$ to denote the sequences such as $(x_1, x_2, \cdots, x_n) \in A^n$, and use $<x_1, x_2, \cdots, x_n>$ to denote the ideal of A generated by the elements

$$\{x_1, x_2, \cdots, x_n\} \subset A,$$

where A is a commutative ring with identity.

2 The Representation of $E^d(X;\mathscr{L})$

In this section we will recall the construction of the representation of group $E^d(X;\mathscr{L})$ in the paper [26], which is the base point of this paper.

First, Assume

$$Q_{2(n+d)} = Spec\frac{k[x, x^{'}, \cdots, x^d, x_2, \cdots, x_n, y, y^{'}, \cdots, y^d, y_2, \cdots, y_n, s]}{<\Sigma x^j y^j + \Sigma x_i y_i + s^2 = 1>}$$

which is the affine quadric hypersurface defined by A. Asok, D, Brent and J. Fasel in the paper [5]. Now let $\widehat{Q_{2(n+d)}}$ be the blow-up of $Q_{2(n+d)}$ at the subscheme defined by the ideal $< x, x^{'}, \cdots, x^d >$. So we have a canonical map

$$\phi:\ \widehat{Q_{2(n+d)}} \rightarrow P^d$$

where P^d =Proj$k[t_0, \cdots, t_d]$ is the projective space of dimension d. By the analysis in the paper [26] we know that this construction give a motivic sphere ($P^{\wedge n}$ or $S^n \wedge G_m^{\wedge n}$) bundle over P^d.

Let A be a regular Noetherian ring of dimension d, smooth over the field k. Let X =Spec(A) be the affine scheme, and $[X,\ \widehat{Q_{2(n+d)}}]_{A^1}$ be the homotopy mapping classes in $\mathscr{H}(k)$, i.e. $[X,\ \widehat{Q_{2(n+d)}}]_{A^1} = Hom_{\mathscr{H}(k)}(X,\ \widehat{Q_{2(n+d)}})$.

For any $\rho_{\mathscr{L}} \in Hom_{Sm_k}(X,\ P^d)$, define

$$E_{\rho_{\mathscr{L}}}(X) \subseteq Hom_{\mathscr{H}(k)}(X,\ \widehat{Q_{2(n+d)}})$$

to be the subset of $Hom_{\mathscr{H}(k)}(X,\ \widehat{Q_{2(n+d)}})$ consisting of elements $f \in Hom_{\mathscr{H}(k)}$ $(X,\ \widehat{Q_{2(n+d)}})$ such that $f \circ \phi = \rho_{\mathscr{L}} \in Hom_{Sm_k}(X,\ P^d)$. In the paper [26], with the condition $d \leq 2n-2$, we construction a monoid structure on $E_{\rho_{\mathscr{L}}}(X)$, and prove that there is an isomorphism between the generalized Euler class group and $E_{\rho_{\mathscr{L}}}(X)$. More precisely we have the following theorem

Theorem 1. *Let A be a regular Noetherian ring of dimension d, smooth over the field k, Spec$(A) = X$ be the associated affine scheme. Let $E_n(A,\mathscr{L})$ be the* *generalized Euler Class Group where $\mathscr{L}$ is an invertible ideal and $d \leq 2n-2$. Then the Segre Class Map*

$$S(\ ,\ ):\ E_n(A, \mathscr{L}) \to E_{\rho_{\mathscr{L}}}(X)$$

is an isomorphism. Thus we get that the Abelian monoid $E_{\rho_{\mathscr{L}}}(X)$ is a group isomorphic to the Euler Class Group $E_n(A, \mathscr{L})$.

Furthermore, by the analysis in the first section of the paper [26], we know that for any affine scheme Spec$(A) = X$, any element $f \in E_{\rho_{\mathscr{L}}}(X)$ could be defined by a concrete map $f \in Hom_{Sch}(X, \widehat{Q_{2(n+d)}})$. Thus we could use concrete map $f \in Hom_{Sch}(X, \widehat{Q_{2(n+d)}})$ to represent the Euler classes in $E_n(A, \mathscr{L})$.

3 Action of Picard Group on $E^d(X; \mathscr{L})$

Let Pic(X) be the Picard group of X. Now for any element $mathscrL' \in$Pic(X) there is a map in $Hom_{Sch}(X, P^d)$ represent it which is noted by α. The element α defines an action on Pic(X) by the group operation.

Now for any element $\rho \in E_n(A, \mathscr{L})$ by the analysis above we have a map $\rho \in Hom_{Sch}(X, \widehat{Q_{2(n+d)}})$ representing it. The composition $\phi \circ \rho:\ X \to P^d$ represent the line bundle $\mathscr{L}$. Now α acts on the element $\mathscr{L}$ to get the line bundle $\mathscr{L}\mathscr{L}'$. Or more precisely, we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "\rho"] & \widehat{Q_{2(n+d)}} \arrow[d, "\phi"] & \widehat{Q_{2(n+d)}} \arrow[d, "\phi"] & (*) \\
 & P^d \arrow[r, "\cdot\alpha"] & P^d &
\end{tikzcd}$$

Let $Gr_{d+1}(\mathscr{O}^N)$ be the Grassmannian of subbundles of dimension $d+1$ in $\mathscr{O}^N$ and $\mathscr{M}_{N\times(d+1)}$ be the scheme over k defined by the $(N \times (d+1))$ matrixes over k of rank $d+1$. Of course we have a map

$$\tau:\ \mathscr{M}_{N\times d+1} \to Gr_{d+1}(\mathscr{O}^N).$$

By the analysis in paper [27], we get that this is an A^1-fibration.

Lemma 1. *The above action α could be lifted to an element in $\mathscr{M}_{N\times(d+1)}(X)$.*

Proof. We know that the action α comes from the Segre Embedding $S:\ P^d \times P^d \to P^{N-1}$. Or equivalently the action α could be defined by the composition

$$X \xrightarrow{(\phi\circ\rho,\alpha)} P^d \times P^d \xrightarrow{S} P^{N-1}$$

Now for any point $p \in P^d$, we could define the subbundle $\mathscr{V}_p \subset \mathscr{O}^N$ as the bundle generated by

$$\{S(e_1, p), S(e_2, p), \ldots, S(e_{d+1}, p)\}$$

in $\mathscr{O}^N$, where e_i are the standard basis of the bundle $\mathscr{O}^{d+1}$. Thus we define a map

$$S: \ P^d \rightarrow Gr_{d+1}(\mathscr{O}^N).$$

Thus we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "(\phi\circ\rho,\alpha)"] & P^d \times P^d \arrow[r, "S"] \arrow[d, "(Id_{P^d}\times S)"'] & P^{N-1} \arrow[dd] \\
& P^d \times Gr_{d+1}(\mathscr{O}^N) & \\
& P^d \times \mathscr{M}_{N\times(d+1)} \arrow[u, "(Id_{P^d}\times\tau)"] \arrow[r] & P^{N-1}
\end{tikzcd} \tag{$**$}$$

Now since the map τ is an A^1-fibration and is $d+1-$connected. By the obstruction arguments in the paper [20], the map $(Id_{P^d \times S})(\phi \circ \rho, \alpha)$ could be lifted uniquely to a map $(\phi \circ \rho \times \gamma) : X \rightarrow P^d \times \mathscr{M}_{N\times(d+1)}$.

The following lemma comes from the paper [27], which could be proved by the main theorem of A. Asok, M. Hoyois and M. Wendt about the localization of A^1-fibration sequences.

Lemma 2. *The following sequence is an A^1-fibration sequence*

$$SL_{N-(d+1)} \longrightarrow SL_N \longrightarrow \mathscr{M}_{N\times(d+1)}$$

Now the map $\gamma : X \rightarrow \mathscr{M}_{N\times(d+1)}$ could be lifted to a map $\Gamma : X \rightarrow SL_N$ if and only if the Euler Class defined by the map γ in $E_N(A)$ is zero. But since the dimension of A is d, so this element must be zero by the Bass Cancellation Theorem.

Now we could get the following theorem about the action of the Picard group on the Euler classes.

Theorem 2. *Let the notations be as above. By the analysis in Sect. 1, for the any element $\rho \in E_n(A, \mathscr{L})$, there is a map $\rho \in Hom_{Sch}(X, \widehat{Q_{2(n+d)}})$. Now for any element $\alpha \in$Pic(X), we have an action of α on $\phi \circ \rho$. Now combining the* *diagram (*) and (**) we have the following digram consisting of concrete arrows*

$$\begin{tikzcd}
X \arrow[r, "\rho"] & \widehat{Q_{2(n+d)}} \times P^d \arrow[r, dotted] \arrow[d, "\phi"] & \widehat{Q_{2(n+N)}} \arrow[d, "\phi"] \\
 & P^d \times P^d \arrow[r, "\cdot\alpha"] \arrow[d, "(Id_{P^d} \times S)"'] & P^{N-1} \arrow[dd] \\
 & P^d \times Gr_{d+1}(\mathscr{O}^N) & \\
 & P^d \times \mathscr{M}_{N \times (d+1)} \arrow[u, "(Id_{P^d} \times \tau)"] \arrow[r] & P^{N-1} \arrow[d] \\
 & P^d \times SL_N \arrow[u] \arrow[r, "\cdot\Gamma"] & P^{N-1}
\end{tikzcd}$$

Now we could complete the diagram to a commutative diagram with a dot arrow. That is to say, we could define an action of $\alpha \in$Pic(X) on the Euler class group $E_n(A, \mathscr{L})$.

Proof. On one hand, by the Lemma 2.1 and Lemma 2.2 the lifting of the map ρ is defined by an invertible matrix $SL_N(A)$. On another hand, since $\dim(A) = d$, any invertible matrix in $SL_N(A)$ must be a product of some elementary matrix. So in order to prove the theorem, we just need to prove that for elementary matrix δ, the action Γ could be lifted to the $\widehat{Q_{2(n+N)}}$. But this is the conclusion in the last section of the paper [12].

Corollary 1. *For any line bundle $\mathscr{L}$ and $\mathscr{L}'$ in Pic(X), there is a map $E_n(A, \mathscr{L}) \to E_n(A, \mathscr{L}')$ such that if the bundle $\mathscr{L}'$ is trivial the map is identity.*

Corollary 2. *For any line bundle $\mathscr{L}$ and $\mathscr{L}'$ in Pic(X), then there is an isomorphism $E_n(A, \mathscr{L}) \cong E_n(A, \mathscr{L}')$.*

Proof. For line bundles $\mathscr{L}$ and $\mathscr{L}'$ in Pic(X), we always have $\mathscr{L}^{-1}\mathscr{L}'$ and $\mathscr{L}\mathscr{L}'^{-1}$ in Pic(X). Using the action of $\mathscr{L}^{-1}\mathscr{L}'$ on $E_n(A, \mathscr{L})$ and action of $\mathscr{L}\mathscr{L}'^{-1}$ on $E_n(A, \mathscr{L}')$, we can construct the isomorphisms between $E_n(A, \mathscr{L})$ and $E_n(A, \mathscr{L}')$ by the above lemma.

Conclusion

From Theorem 1, we get that the Abelian monoid $E_{\rho_{\mathscr{L}}}(X)$ is a group isomorphic to the Euler Class Group $E_n(A, \mathscr{L})$. Thus, we could use a concrete map $f \in Hom_{Sch}(X, \widehat{Q_{2(n+d)}})$ to represent the Euler classes in $E_n(A, \mathscr{L})$. From Theorem 2, we could define an action of $\alpha \in$Pic(X) on the Euler class group $E_n(A, \mathscr{L})$. This result gives a new method to compare the generalized Euler classes with different twisted bundle.

References

1. Asok, A., Hoyois, M., Wendt, M.: Affine representability results in A^1-homotopy theory I: vector bundles. Duke Math. J. **166**(10), 1923–1953 (2017)
2. Asok, A., Hoyois, M., Wendt, M.: Affine representability results in A^1-homotopy theory II: principal bundles and homogeneous spaces. Geom. Top. **22**(2), 1181–1225 (2018)
3. Asok, A., Jean, F.: Splitting vector bundles outside the stable range and homotopy theory of punctured affine spaces. J. Amer. Math. Soc. **28**(4), 1031–1062 (2014)
4. Asok, A., Jean, F.: Euler class groups and motivic stable cohomotopy. ArXiv:1601.05723
5. Asok, A., Doran, B., Jean, F.: Smooth models of motivic spheres and the clutching construction. Int. Math. Res. Not. (6), 1890–1925 (2017)
6. Asok, A., Jean, F.: Comparing Euler classes. Q. J. Math **67**(4), 603–635 (2016)
7. Asok, A., Wickelgren, K., Williams, T.B.: The simplicial suspension sequence in A^1-homotopy. Geom. Top. **21**(4), 2093–2160 (2017)
8. Bass, H.: K-theory and stable algebra. Inst. Hautes tudes Sci. Publ. Math. **22**, 5–60 (1964)
9. et Fabien Morel, J.B.: Groupe de Chow des cycles orientes et classe d'Euler des fibres vectoriels, C.R. Acad. Sci. Paris, t. 330, Serie I, pp. 287–290 (2000)
10. Bhatwadekar, S.M., Sridharan, R.: Zero cycles and the Euler class groups of smooth real affine varieties. Inventiones mathematicae **136**(2), 287–322 (1999)
11. Bhatwadekar, S.M., Sridharan, R.: The Euler class group of a noetherian ring. Compositio Mathematica **122**(2), 183–222 (2000)
12. Fasel, J.: On the number of generators of ideals in polynomial rings. Ann. Math **184**(1), 315–331 (2016)
13. Jardine, J.F.: Local Homotopy Theory. Springer Monographs in Mathematics. Springer, New York (2015)
14. Jardine, J.F.: Documenta Mathematica **5**, 445–553 (2000)
15. Georgies, R., Strunk, F.: Documenta Mathematica **23**, 1757–1797 (2018)
16. Mandal, S.: On efficient generation of idealsProjective Modules and Complete Intersections. Inventiones mathematicae **75**(1), 59–67 (1984)
17. Mandal, S.: Projective Modules and Complete Intersections. LNM, vol. 1672. Springer, Heidelberg (1997). https://doi.org/10.1007/BFb0093560
18. Manda, S., Yang, Y.: Excision in algebraic obstruction theory. JPAA **216**, 2159–2169 (2012)
19. Mandal, S., Yang, Y.: Intersection theory of algebraic obstructions (with Yong Yang). JPAA **214**, 2279–2293 (2010)
20. Fabien Morel A^1-Algebraic Topology over a Field, LNM 2052, (2010)
21. Fabien Morel and Vladimir Voevodsky; A^1-homotopy theory of schemes. Publ. Math. IHES, (90), 45–143 (1999)
22. Mohan Kumar, N.: Complete intersections. J. Math. Kyoto Univ. **17**(3), 533–538 (1977)
23. Rezk, C.: Toposes and homotopy toposes (2005). http://www.math.uiuc.edu/?rezk/homotopy-topos-sketch.pdf
24. Strunk, F.: On motivic spherical bundles. Thesis Institut fur Mathematik Universit at Osnabruck (2012)

25. Suslin, A.A.: On the structure of the special linear group over polynomial rings. Math. USSR Izv. **11**, 221–238 (1977)
26. Yang, Y.: Generalized Euler Class Group and Chow-Witt Group with Twist, preprint
27. Schlichting, M., Tripathi, G.S.: Geometric models for higher Grothendieck−Witt groups in A^1-homotopy theory **362**, 1143–1167(2015)
28. Bergeron, N., Charollois, P., Garcia, L.E.: Transgressions of the Euler class and Eisenstein cohomology of GL N (Z). Jpn. J. Math. **15**(2), 311–379 (2020)
29. Naolekar, A.C., Singh, T.A.: Euler classes of vector bundles over iterated suspensions of real projective spaces. Mathematica Slovaca **68**(3), 677–684 (2018)