IAA Spectral Estimation in the Selective Range

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Abstract. Recently, the iterative adaptive approach (IAA) has been proposed to be a high-resolution spectrum estimator. Its main idea is reformulating the nonlinear frequency estimation problem as a linear one, with parameters being updated iteratively according to weighted least squares. Since the derivation is based on grid searching in a fixed frequency range $[0, 2\pi)$, the accurate of the IAA is limited by the number of grid. In this paper, we proposing two generalized versions of IAA, which can work well in a flexible frequency range, so that the performance can be improved in the same grid points. Simulation results are included to demonstrate the superior of our proposed methods.

Keywords: Iterative adaptive approach (IAA) · Spectral estimation · Frequency domain

1 Introduction

Spectral analysis has been an important topic in science and engineering because many real-world signals are well described by the sinusoidal model. Basically, the frequency components of the observed data can be obtained by means of either parametric or nonparametric techniques [1]. In the parametric approach, the signal is assumed to satisfy a generating model with known functional form, which allows the derivation of the optimal spectral estimators. However, the performance of these methods degrades when there is a mismatch between the assumed and actual signal models. On the other hand, no assumptions are made about the data in the nonparametric approach. Among numerous non-parametric estimators developed in the literature, a conventional representative is the periodogram based on the Fourier transform, but its resolution is fundamentally limited by the available observation length. To improve the performance, several algorithms such as principal-singular-vector utilization for modal analysis (PUMA) [2], Capon [3], multiple signal classification (MUSIC) [4] have been proposed, which can provide high-resolution in the scenario of high signal-to-noise ratio (SNR) and large number of snapshots. In [5], amplitude and phase estimator (APES) was suggested to accurately estimate the power of the source

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signal, which can resolve sources as well. Although these methods can obtain high accurate estimation in the case of high SNR or numerous snapshots, their performance degrades when only a few snapshots are available. This is because that accurate implementation of covariance matrix in these methods requires a large number of snapshots.

In [6], a super-resolution method, namely, the iterative adaptive approach (IAA), is developed, which iteratively obtaining spectrum estimate using the weighted least squares (WLS) approach. According to the Markov estimate [7], the weighing matrix in IAA is in fact the covariance matrix of observations. To ensure the high resolution, IAA updates the covariance matrix using the estimate iteratively, and hence, accurate implementation of the IAA covariance matrix requires the estimates in full frequency ranges of $[0, 2\pi)$. That is to say, IAA can only work well in the fixed full range. However, in the case that the coarse arrival ranges of sources are known *a priori*, full range estimation of IAA is redundant and suffers from high computational cost. Although fast implementation of IAA [8]–[9] has been proposed, it is still not a good choice.

In this paper, two generalized version of IAA, referred to as selective IAA I (SIAA I) and selective IAA II (SIAA II), are devised, which can be work well in a flexible frequency range. To be employed in any selective azimuth range, two implementation criteria of the covariance matrix are suggested, where only the spectrum estimate in the interested azimuth range is required. For SIAA I, we divide the full frequency range into interested one and non-interested one. Then the covariance matrix is modified utilizing the spectrum estimate in the interested range as well as the variance estimates outside the selective range that can be obtained by the selective spectrum estimate. While in SIAA II, we redefine the mathematical model of observations as the noise-free and noisy component, where the former is described by the selective azimuth range. The covariance matrix of SIAA II is then defined by the spectrum estimate and the variance of noise term.

The rest of this paper is organized as follows. In Sect. 2, a brief review of IAA algorithm is given. In Sect. 3, the main idea of both SIAA I and SIAA II are provided. Computer simulations in Sect. 4 demonstrate the accurate of the proposed methods. Finally, conclusions are drawn in Sect. 5.

2 Review of IAA

Here we just consider a 1-D uniformly sampled sequence of N samples. IAA is based on a uniform frequency grid with K points in full range $[0, 2\pi)$ and the frequency bin: $\omega_k = 2\pi\frac{k}{K}, k = 0, 1, \ldots, K-1$. Then the frequency components can be expressed as $\mathbf{A} = [\mathbf{a}(\omega_0) \quad \mathbf{a}(\omega_1) \quad \ldots \quad \mathbf{a}(\omega_k) \quad \ldots \quad \mathbf{a}(\omega_{K-1})]$ with $\mathbf{a}(\omega_k) = [1 \quad e^{j\omega} \quad \ldots \quad e^{jn\omega} \quad \ldots \quad e^{j(N-1)\omega}]^T$ standing for the steering vector. Then the data model can be written as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{q} \tag{1}$$

where $\mathbf{y} = [y_0 \;\; y_1 \;\; \ldots \;\; y_{N-1}]^T$ is observed data and $\mathbf{x} = [x_0 \;\; x_1 \;\; \ldots \;\; x_{K-1}]^T$ is the amplitude corresponding to each frequency bin with x_k denoting the complex amplitude corresponds to the kth bin, and $\mathbf{q}$ is noise term.

IAA can solve (1) by minimize the cost function:

$$||\mathbf{y} - \mathbf{a}(\omega_k)x_k||^2_{Q^{-1}} \quad k = 0, 1, \ldots, K-1 \tag{2}$$

where $\|\mathbf{x}\|_{\mathbf{Q}^{-1}} = \mathbf{x}^H\mathbf{Q}^{-1}\mathbf{x}$ and

$$\begin{aligned}\mathbf{Q} &= E\{(\mathbf{y} - \mathbf{a}(\omega_k)x_k)(\mathbf{y} - \mathbf{a}(\omega_k)x_k)^H\} \\ &= \mathbf{R} - p_k\mathbf{a}(\omega_k)\mathbf{a}^H(\omega_k)\end{aligned} \tag{3}$$

is the weighting matrix which is also IAA interference and noise covariance matrix. Where $p_k = |x_k|^2$ stands for the signal power at frequency ω_k. Introducing a definition of IAA covariance matrix which is:

$$\mathbf{R} = E\{\mathbf{y}\mathbf{y}^H\} = \mathbf{APA}^H \tag{4}$$

where $\mathbf{P}$ is a diagonal matrix with diagonal elements from power vector $\mathbf{p} = [p_0 \quad p_1 \quad \ldots \quad p_{K-1}]^T$. Then we can minimize (2) with respect x_k yields [10]

$$x_k^{IAA} = \frac{\mathbf{a}^H(\omega_k)\mathbf{Q}^{-1}\mathbf{y}}{\mathbf{a}^H(\omega_k)\mathbf{Q}^{-1}\mathbf{a}(\omega_k)} \quad k = 0, 1, \ldots, K-1 \tag{5}$$

Using matrix inverse lemma we can see:

$$\mathbf{a}^H(\omega_k)\mathbf{Q}^{-1} = \frac{\mathbf{a}^H(\omega_k)\mathbf{R}^{-1}}{1 - p_k\mathbf{a}^H(\omega_k)\mathbf{R}^{-1}\mathbf{a}(\omega_k)} \tag{6}$$

Then (5) can be simplified as:

$$x_k^{IAA} = \frac{\mathbf{a}^H(\omega_k)\mathbf{R}^{-1}\mathbf{y}}{\mathbf{a}^H(\omega_k)\mathbf{R}^{-1}\mathbf{a}(\omega_k)} \quad k = 0, 1, \ldots, K-1 \tag{7}$$

(7) avoids the computation of p_k for each bin, so we usually use (7) replace (5) as solution of IAA.

3 Proposed Method

Although IAA estimate amplitude x_k one by one, we cannot just change the range of $\mathbf{A}$ because it will have a singular problem when we compute inverse of covariance matrix. Even when we solve the singular problem, the result is wrong so the covariance should utilize the information in full range.

The spectrum of an observed signal is composed of noise-free signal and noise. We just consider the additive Gaussian white noise here. If we have already known the locate range of frequency, ripples outside of this range are just noise which is Gaussian distribution. So in order to reduce the computation cost, we can just use IAA estimate amplitude in locate range. Here we suppose locate range is $[t_1, t_2)$.

Then we can decompose the original data model (1) into:

$$\begin{aligned}
\mathbf{y} &= \mathbf{s} + \mathbf{q} \\
&= \mathbf{A}(\Psi + \Phi) \\
&= \mathbf{A}_s(\Psi + \Phi_1) + \mathbf{A}_{os}\Phi_2 \\
&= \mathbf{s}_{new} + \mathbf{q}_{new} \\
&= \mathbf{A}_s\mathbf{x}_s + \mathbf{A}_{os}[\mathbf{x}_{os1} \quad \mathbf{x}_{os2}]
\end{aligned} \tag{8}$$

Where $\mathbf{s}$ is the noise-free signal and $\mathbf{q}$ is noise with Ψ being the amplitude vector of noise-free signal and Φ denoting the 'amplitude' vector of noise. We divide the frequency bin into two range: $\mathbf{A}_s$ in frequency location range $[t_1, t_2)$ and $\mathbf{A}_{os}$ in range other than frequency range $[0, t_1)$ and $[t_2, 2\pi)$. And the corresponding amplitude vector can also be divided into two parts: $\Psi + \Phi_1$ and Φ_2. From the decomposition we reconstruct a new noise-free signal $\mathbf{s}_{new}$ and a new noise $\mathbf{q}_{new}$. Here the new noise is also Gaussian distribution.

Then from the new definition of signal and noise, we can divide the whole range $[0, 2\pi)$ into $[0, t_1)$, $[t_1, t_2)$and $[t_2, 2\pi)$. And corresponding amplitude vectors can also be broken into $\mathbf{x}_{os1}$,$\mathbf{x}_s$ and $\mathbf{x}_{os2}$, which means the amplitude distributed in $[0, 2\pi)$ can be expressed as $[\mathbf{x}_{os1} \quad \mathbf{x}_s \quad \mathbf{x}_{os2}]$. We also introduce a steering vector $\mathbf{a}_s(\omega) = [1 \quad e^{j\omega} \quad \ldots \quad e^{j(N-1)\omega}]$ so Eq. (8) can be simplified as:

$$\begin{aligned}
\mathbf{y} &= \mathbf{s}_{new} + \mathbf{q}_{new} \\
&= \mathbf{A}_s\mathbf{x}_s + \mathbf{q}_{new}
\end{aligned} \tag{9}$$

where $\mathbf{A}_s = [\mathbf{a}_s(\omega_0) \quad \ldots \quad \mathbf{a}_s(\omega_{K-1})]$ with $\omega_k = t_1 + (t_2 - t_1)\frac{k}{K}$ and $\mathbf{x}_s = [x_{s_0} \quad x_{s_1} \ldots \quad x_{s_{K-1}}]$ is the corresponding amplitude vector.

If we have known $\mathbf{y}$ and $\mathbf{A}_s$, we can estimate $\mathbf{x}_s$ by minimizing cost function:

$$||\mathbf{y} - \mathbf{a}_s(\omega_k)x_{s_k}||^2_{\mathbf{R}^{-1}} \quad k = 0, 1, \ldots, K-1 \tag{10}$$

where $\mathbf{R}$ has the same definition with original IAA. And there are two method to compute $\mathbf{R}$ which will be shown later.

And the solution of Eq. (10) can be expressed:

$$x^{IAA}_{s_k} = \frac{\mathbf{a}^H_s(\omega_k)\mathbf{R}^{-1}\mathbf{y}}{\mathbf{a}^H_s(\omega_k)\mathbf{R}^{-1}\mathbf{a}_s(\omega_k)} \quad k = 0, 1, \ldots, K-1 \tag{11}$$

3.1 SEIAA I

This algorithm is based on the original definition of $\mathbf{R}$ in frequency domain. The definition of $\mathbf{R}$ request us to know the amplitude corresponding to $[0, 2\pi)$. As we just want to update the amplitude $\mathbf{x}_s$ of $[t_1, t_2)$, we should use another method to reconstruct information outside of this range. Only the power of each complex amplitude is useful in estimating $\mathbf{R}$. As ripples outside of $[t_1, t_2)$ are Gaussian distribution and the values of ripples are very small comparing with the peak of signal, we can assume all magnitudes outside $[t_1, t_2)$ are equal and this is the basic idea of SEIAA.

We can use the energy conservation law and sample average method to compute new noise variance σ_{new}^2:

$$\begin{aligned}
\mathbf{y}^H\mathbf{y} &= t^2(\mathbf{x}_s^H\mathbf{x}_s + \sigma_{new}^2) \\
\sigma_{new}^2 &= \frac{1}{N}||\mathbf{y} - t\mathbf{A}_s\mathbf{x}_s||_2^2 \\
\sigma_{new}^2 &= \mathbf{x}_{os1}^H\mathbf{x}_{os1} + \mathbf{x}_{os2}^H\mathbf{x}_{os2} \\
\mathbf{x}_{os1}^H\mathbf{x}_{os1} &= \mathbf{x}_{os2}^H\mathbf{x}_{os2}
\end{aligned} \tag{12}$$

where t is coefficient to balance the relationship between $\mathbf{y}$ and $\mathbf{x}$ and $\mathbf{A}_s$ is shorten for the frequency bin of $[t_1, t_2)$. The detail of Eq. (12) can be shown in appendix A.

After we estimate the value of $\mathbf{x}_{os1}$ and $\mathbf{x}_{os2}$, we can reshape the power vector $\mathbf{p}$ as $\mathbf{p} = [|\mathbf{x}_{os1}|^2 \quad |\mathbf{x}_s|^2 \quad |\mathbf{x}_{os2}|^2]$ and then we can see the step of algorithm in Table 1.

Table 1. Steps of SEIAA I

Steps of Proposed Algorithm
1. Implementing the integrated frequency matrix $\mathbf{A}$ and frequency matrix $\mathbf{A}_s$;
2. Setting a initial value of $\mathbf{x}_s, \mathbf{x}_{os1}, \mathbf{x}_{os2}$, e.g., all set to 1;
3. Reshaping vector $\mathbf{p}$ and then $\mathbf{R}$ using Eq. (4);
4. Estimating $\mathbf{x}_s$ using solution (11);
5. Computing $\mathbf{x}_{os1}, \mathbf{x}_{os2}$ using Eq. (12);
6. Repeat steps 3 - 5 until $\frac{||\mathbf{x}^{t+1} - \mathbf{x}^t||_2}{||\mathbf{x}^{t+1}||_2}$ in tolerance;

3.2 SEIAA II

Combining the definition of $\mathbf{R}$ and Eq. (9) we can see:

$$\begin{aligned}
\mathbf{R} =& E\{\mathbf{y}\mathbf{y}^H\} \\
=& E\{(\mathbf{A}_s\mathbf{x} + \mathbf{q}_{new})(\mathbf{A}_s\mathbf{x} + \mathbf{q}_{new})^H\} \\
=& \mathbf{A}_s\mathbf{P}_s\mathbf{A}_s + E\{\mathbf{q}_{new}\mathbf{q}_{new}^H\} \\
=& \mathbf{A}_s\mathbf{P}_s\mathbf{A}_s + \sigma_{new}^2\mathbf{I}_N
\end{aligned} \tag{13}$$

where $\mathbf{P}_s$ is the diagonal matrix with diagonal entries from power vector $\mathbf{p}_s = [|x_{s_0}|^2, \ldots, |x_{s_{K-1}}|^2]$ and σ_{new}^2 is the variance of new noise $\mathbf{q}_{new}$. σ_{new}^2 is same with the Eq. (12). So Algorithm II can be shown in Table 2.

Table 2. Steps of SEIAA II

Steps of Proposed Algorithm
1. Implementing frequency matrix $\mathbf{A}_s$;
2. Setting a initial value of $\mathbf{x}_s$, e.g., all set to 1;
3. Estimating σ^2_{new} using Eq. (12);
4. Constructing vector $\mathbf{p}_s$ and then $\mathbf{R}$ using Eq. (13);
5. Estimating $\mathbf{x}_s$ using solution (11);
6. Repeat steps 3 - 5 until $\frac{\|\|\mathbf{x}^{t+1}-\mathbf{x}^t\|\|_2}{\|\|\mathbf{x}^{t+1}\|\|_2}$ in tolerance;

4 Simulation ResultS

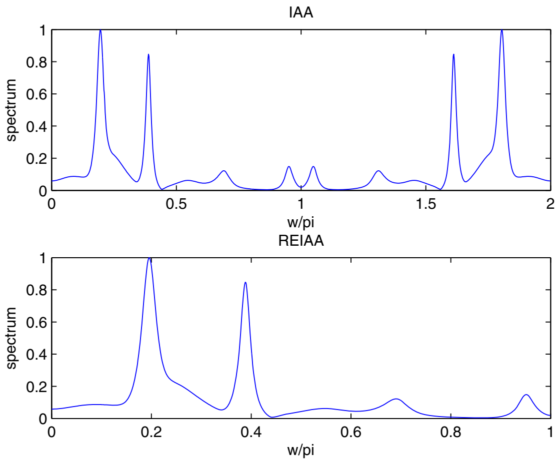

Fig. 1. Spectrum of two 1-D real tones using original IAA and REIAA

In this section, we examine the performance of the proposed algorithm. To have a directly conclusion, we set the number of grid points $K = 600$ for SEIAA and the interested range is $[0.1\pi, 0.5\pi)$, so we choose $K = 3000$ for original IAA. Meanwhile, when we estimate spectrum, SNR(signal to noise ratio) is 12 dB and te data length is $N = 40$.

Figure 1 is for a 1-D real tone. And the signal model is $y_n = \alpha_1 \cos(\omega_1 n + \phi_1) + \alpha_2 \cos(\omega_2 n + \phi_2) + q_n, \quad n = 0, 1, \ldots, N-1$ and parameters we set are $\alpha_1 = \alpha_2 = 1, \omega_1 = 0.2\pi, \omega_2 = 0.3\pi, \phi_1 = 0.1; \phi_2 = 0.12$. From Fig. 1 we can see, for real tones, IAA gives the spectrum which is symmetric by π. The two curves shows that the real-tone IAA (REIAA) gives the same spectrum with IAA in range $[0, \pi)$. So in order to save complexity, we use REIAA to replace IAA when estimate real tones.

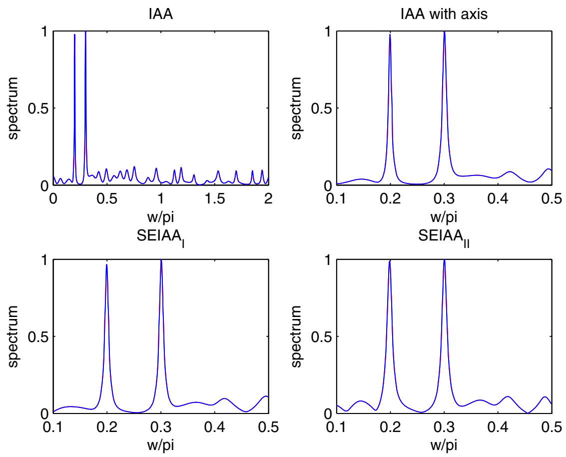

Fig. 2. Spectrum of two 1-D complex tones using original IAA and $\mathrm{SEIAA_I}, \mathrm{SEIAA_{II}}$

Table 3. Computation time of three methods

original IAA	3.9652
SEIAA I	1.2515
SEIAA II	0.2012

To simplify the problem, we use complex tones When we test SEIAA. the data model is $y_n = \alpha_1 e^{\omega_1 n + \phi_1} + \alpha_2 e^{\omega_2 n + \phi_2} + q_n, \quad, n = 0, 1, \ldots, N-1$ and the parameters we set are $\alpha_1 = \alpha_2 = 1, \phi_1 = 0.1; \phi_2 = 0.12$. Figure 2 gives four curves which is original IAA, original IAA in $[0.1\pi, 0.5\pi)$ (to compare easily),SEIAA I and SEIAA II, while the Table 3 shows the computation time of those three methods. From the curve 'IAA with axis' and 'SEIAA I', we can see the spectrum of the first two is approximately same but 'SEIAA I' runs faster than 'IAA'. For 'SEIAA II', it can resolve two peaks but it is not flat in range $[0.2\pi, 0.3\pi)$ compared with the former two methods. But 'SEIAA I' runs fastest in those three methods. So from the simulation result we can see, IAA gives the best spectrum but the highest complexity; SEIAA I gives the similar spectrum with IAA but the medium complexity in three method; SEIAA II gives a not so good spectrum but the computation time is fastest.

Figure 3 and Fig. 4 also use the same data model with Fig. 2. But here we suppose the number of grid points is same for three methods. Those two curves give us a shown that the accuracy of estimating frequency using original IAA and our methods comparing with crlb. all the test are dependent on 200 independent runs.

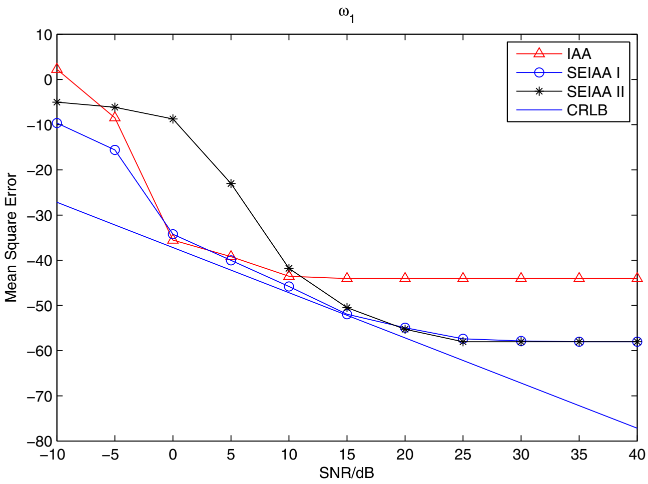

Fig. 3. Mean square error of ω_1 versus SNR

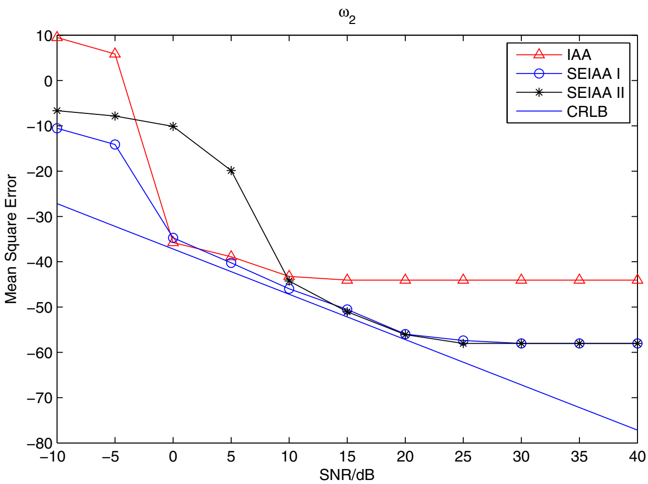

Fig. 4. Mean square error of ω_2 versus SNR

5 Conclusion

Based on the symmetric property of covariance and the symmetric property of magnitude for real tones, we simplify the IAA from $[0, 2\pi)$ to $[0, \pi)$. We also propose another two method: SEIAA I and SEIAA II to a selective range $[t_1, t_2)$ with $0 \leq t_1 < t_2 \leq 2\pi$. SEIAA I are based on assumption that all the value outside of interested range is equal and use energy conservation law to estimate those then reshape the covariance. SEIAA II reshape a new signal and new noise and use energy conservation law to compute variance of noise and then estimate covariance matrix according to its definition. And then simulation results prove the performance of SEIAA.

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Appendix: Noise Variance Relationship Between Time Domain and Frequency Domain

In IAA, If the input data is the noise $\mathbf{q}_n, \quad n = 0, \ldots, N-1$:

$$\mathbf{q} = \mathbf{A}\mathbf{x}_q \tag{14}$$

where $\mathbf{A}$ is the frequency bin in $[0, 2\pi]$ and $\mathbf{x}_q = [x_{q_0} \quad \ldots \quad x_{q_{K-1}}]$ so the variance of noise can be written:

$$\begin{aligned}\sigma^2 =& \frac{1}{N} E\{\mathbf{q}^H \mathbf{q}\} \\ =& \frac{1}{N} E\{\mathbf{x}_q^H (A^H A)\mathbf{x}_q\} \\ =& E\{|\mathbf{x}_q|^2\}\end{aligned} \tag{15}$$

where noise is i.i.d noise so $E\{\mathbf{x}_{q_k} \mathbf{x}_{q_t}\} = 0$ when $t \neq k$. From the original data model we can see and use sample average to express expectation:

$$\begin{aligned}\sigma^2 =& E\{\mathbf{q}^H \mathbf{q}\} \\ =& \frac{1}{N} ||\mathbf{y} - \mathbf{A}\mathbf{x}||_2^2\end{aligned} \tag{16}$$

$$E\{|\mathbf{x}_q|^2\} = \sum_{k=0}^{K-1} |x_{q_k}|^2 \tag{17}$$

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