Stable Random Vector and Gaussian Copula for Stock Market Data

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Abstract. A lot of real-world data sets have heavy tailed distribution, while the calculations for these distributions in multidimensional case are complex. The paper shows a method to investigate data of multivariate heavy-tailed distributions. We show that for any given number $\alpha \in (0; 2]$, each Gaussian copula is also the copula of an α-stable random vector. Simultaneously, every random vector is α-stable if its marginals are α-stable and its copula is a Gaussian copula. The result is used to build up a formula representing density functions of α-stable random vectors with Gaussian copula. Adopting a new tool, the paper points out that in most of cases, vectors of daily returns in stock market data have multivariate stable distributions with Gaussian copulas and we propose a new method to choose an investment portfolio by computing the distribution of linear combination of components of stable random vector. Dataset of 4 stocks on HOSE was analyzed.

Keywords: Stable distribution · Gaussian copula · Stock market

1 Introduction

Until the 1970's, most of the statistical analysis methods were developed under normality assumptions, mainly for mathematical convenience. In applications, however, normality is only a poor approximation of reality. In particular, normal distributions do not allow heavy-tailed, which are so common, especially in finance and risk management studies [1,7,9,15,18,20]. Arising as solutions to central limit problems, stable distributions are natural heavy tailed extensions of normal distributions and have attracted a lot of attention [9,12,23,24].

While the univariate stable distributions are now mostly accessible by several methods to estimate stable parameters and reliable programs to compute stable densities, cumulative distribution functions, and quantiles for stable random variables [5,10,23,25], the use of the heavy-tailed models in practice has been restricted by the lack of the tools for multivariate stable distributions.

The main challenge of dealing with multivariate data with heavy-tailed distributions is that of ambiguous dependence between coordinates of a random vector. Whilst the dependence can be completely determined by covariance matrix

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for the case of multinormal data, the covariance matrix does not exist for heavy-tailed data. Fortunately, the problem can be solved by the tool of copula. The term copula was first introduced by Sklar [19], but was not of great interest until recent years. Copula functions describe the dependence structure connecting random variables, giving an opportunity to separate the dependence structure and marginal distributions.

Another way of parameterizing multivariate stable distributions is to use the well known univariate stable distribution results about one-dimensional projections of random vectors. However, in practice, this approach faces with challenging computational problems which have not been generally solved for multivariate stable distributions. The problems are caused by the complexity of the possible distributions with an uncountable set of parameters. In recent years, computations are more accessible for elliptically contoured stable distributions [28] which are scale mixtures of multivariate normal distributions. The tools for the very special class of stable distributions were applied in several empirical studies [8,12].

Although the method is available only for a narrow subclass of symmetric multivariate stable distributions, that approach stimulates researchers to create similar tools for other subclasses of general stable multidimensional distributions. We developed a new method for investigating the data of multivariate distributions with heavy tails, trying to decrease the complexity of stable copulas downwards to a more practicable case of Gaussian copulas. This method was applied for daily return data on Nasdaq [16]. We continue to use it for daily return data on HOSE in Vietnam.

The paper is organized as follows. Section 2 presents some auxiliary results on copulas and stable distributions. In Sect. 3 we give the main results of multidimensional stable distributions with Gaussian copulas, demonstrating that Gaussian copulas are also those of some multivariate stable distributions and a random vector is stable if it has Gaussian copula and all its marginals are stable. In the last section we formulate the density function of a stable random vector with Gaussian copula, which can be practically computed. Then the results are applied for study of stock market data of Vietnam.

2 Copulas and Stable Random Vectors

Given a random vector $\mathbf{X} = (X_1, ..., X_d)^T$ taking values in Euclidean space $\mathbb{R}^d$, its cumulative distribution function (cdf hereafter) and probability density function (pdf hereafter) are denoted by $F_{\mathbf{X}}$ and $f_{\mathbf{X}}$, respectively. The coordinates $X_1, ..., X_d$ are called marginals, simultaneously $F_{X_1}, ..., F_{X_d}$ and $f_{X_1}, ..., f_{X_d}$ are called marginal cdf's and marginal pdf's of $\mathbf{X}$, respectively.

Let a increasing function $G : \mathbb{R} \to [0, 1]$, the *generalized inverse* of G is

$$G^{\leftarrow}(y) = inf\{x : G(x) \geq y\}, y \in [0, 1]$$

If G is cdf of continous variable, $G^{\leftarrow}(y) = G^{-1}(y)$.

2.1 Copulas

Copulas are tools for modelling dependence of a random vector with any marginal distributions. In 1959, copula was firstly introduced by Abe Sklar. In 1999, Embrechts, McNeil, Straumann applied copulas in finance. In 2006, copulas were used as a risk management tool in insurance and bank. Up to now, copulas have been used in many fields as sea storm (Corbella and Stretch 2013), risk evaluation of droughts (Zhang et al. 2013).

Let a random vector $\mathbf{X} = (X_1, ..., X_d)$, a d-dimensional *copula* (or d-copula) of $\mathbf{X}$ is the function $C_{\mathbf{X}} : [0,1]^d \to [0,1]$ given by

$$C_{\mathbf{X}}(u_1, ..., u_d) = F_{\mathbf{X}}(F^{\leftarrow}_{X_1}(u_1), ..., F^{\leftarrow}_{X_d}(u_d)) \tag{1}$$

where $F_{\mathbf{X}}$ is the cdf, and $F^{\leftarrow}_{X_1}, ..., F^{\leftarrow}_{X_d}$ are generalized inverse functions of marginal distribution functions of $\mathbf{X}$ [27].

The famous Sklar's Theorem [19] confirms the relationship

$$F_{\mathbf{X}}(x_1, ..., x_d) = C_{\mathbf{X}}(F_{X_1}(x_1), ..., F_{X_d}(x_d)) \tag{2}$$

for $x_1, ..., x_d \in \bar{\mathbb{R}} = [-\infty, +\infty]$.

The copula was simply the joint distribution function of random variables with uniform marginals. To manipulate copulas, specific copula types have been introduced, and can be divided into two groups: explicit copulas (Archimedean copulas) and implicit copulas (Gaussian copula,...). The Gaussian copula is the most popular one in applications. It is simply derived from the correlation matrix Σ and mean vector $\boldsymbol{\mu}$ of a multivariate Gaussian distribution function (without loss of generality, we can assume $\boldsymbol{\mu} = \mathbf{0}$) and is given by the following formula:

$$C(u_1, ..., u_d) = \int_{-\infty}^{\phi^{-1}(u_1)} ... \int_{-\infty}^{\phi^{-1}(u_d)} \frac{exp\left[\frac{-1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]}{\sqrt{(2\pi)^d |\Sigma|}} dx_d ... dx_1 \tag{3}$$

$u_i \in [0,1], i = 1, 2, ..., d$, $\mathbf{x} = (x_1, ..., x_d)^T$, ϕ is cdf of univariable Gaussian distribution. Moreover, the calculation of Gaussian copula is available on computer softwares, such as R software.

One nice property of copulas is that for strictly monotone transformations of the random variables, copulas are either invariant. In general, the copula functions are invariant under strictly increasing transformations. Especially, the following proposition given by Embrechts et al. [1] provides an useful tool for getting the main results of our study.

Proposition 1. *Let $(X_1, ..., X_d)^T$ be a vector of continuous random variables with copula C. If $T_1, ..., T_d : \mathbb{R} \to \mathbb{R}$ are strictly increasing on $RanX_1, ..., RanX_n$, resp, then $(T_1(X_1), ..., T_d(X_d))^T$ also has copula C.*

2.2 Stable Random Vectors

A random variable X is said to have a *stable distribution* if for any positive numbers A, B, there is a positive number C and a real number D such that

$$AX_1 + BX_2 \stackrel{D}{=} CX + D \tag{4}$$

where X_1 and X_2 are independent copies of X.

For any stable random variable X, there is a number $\alpha \in (0, 2]$ such that the numbers A, B, C in (4) satisfy the following formula $A^\alpha + B^\alpha = C^\alpha$. The number α is called the index of stability or characteristic exponent. The stable random variable X with index α is called α-stable.

The probability densities of α-stable random variables exist and are continuous but, with a few exceptions, they are not known in closed form [22]. Characteristic function is a useful tool for studying these variables. In particular, *characteristic function* of an α-stable random variable X has the form

$$\hat{X}(u) = \mathbb{E}(e^{iuX}) = \begin{cases} exp(-\gamma^\alpha |u|^\alpha [1 - i\beta(tan\frac{\pi\alpha}{2})sign(u)] + i\delta u) & \alpha \neq 1, \\ exp(-\gamma|u|[1 + i\beta(\frac{2}{\pi})sign(u)ln|u|] + i\delta u) & \alpha = 1 \end{cases} \tag{5}$$

where $\beta \in [-1, 1]$, $\gamma \in (0, +\infty)$, $\delta \in (-\infty, +\infty)$ and

$$sign(u) = \begin{cases} 1 & if \quad u > 0, \\ 0 & if \quad u = 0, \\ -1 & if \quad u < 0 \end{cases}$$

The parameters α, β, γ and δ uniquely determine the distribution of X, the symbol $X \sim S(\alpha, \beta, \gamma, \delta)$ can be used to refer to that situation. Usually, α is called the stable index, meanwhile β, γ and δ are named as the skewness, the scale and the location parameters of X, respectively.

Next, some basic linear properties of α-stable random variables are used in paper [17].

Proposition 2. *If $X \sim S(\alpha, \beta, \gamma, \delta)$, then*

$$aX + b \sim \begin{cases} S(\alpha, \beta sign(a), |a|\gamma, a\delta + b) & \alpha \neq 1, \\ S(\alpha, \beta sign(a), |a|\gamma, a\delta + b - \beta\gamma\frac{2}{\pi}aln|a|) & \alpha = 1 \end{cases} \tag{6}$$

$a \neq 0, b \in R$.

Proposition 3. *If $X_1 \sim S(\alpha, \beta_1, \gamma_1, \delta_1)$ and $X_2 \sim S(\alpha, \beta_2, \gamma_2, \delta_2)$ are independent then*

$$X_1 + X_2 \sim S(\alpha, \beta, \gamma, \delta) \tag{7}$$

where $\beta = \frac{\beta_1\gamma_1^\alpha + \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha, \delta = \delta_1 + \delta_2$.

Similar to a α-stable random variable in $\mathbb{R}$, stable random vector is expanded in $\mathbb{R}^d$. Characteristic functions are still used to describe the stable distribution in multivariate case, but they are more complicated and determined by three paramaters: stable index α, vecotr μ^0 and spectral measure Λ. Moreover, any linear combination of the components of $\mathbf{X}$ of the type $Y = \sum_{k=1}^{d} b_k X_k$ is an α-stable radom variable. An important property of stable distribution family is the stability of the linear combinations of its coordinates. It completely determines $\mathbf{X}$, as it can be seen from the next theorem.

Theorem 1. *Let $\boldsymbol{X}$ be an α-stable random vector, every linear combination $\boldsymbol{u}^T\boldsymbol{X}$ has the stable distribution $S(\alpha, \beta(\boldsymbol{u}), \gamma(\boldsymbol{u}), \delta(\boldsymbol{u}))$, then the parameter functions are written in the following form:*

$$\gamma(\boldsymbol{u}) = \left(\int_{\mathbb{S}_d} |\boldsymbol{u}^T\boldsymbol{s}|\Lambda(ds)\right)^{1/\alpha}$$

$$\beta(\boldsymbol{u}) = \gamma(\boldsymbol{u})^{-\alpha} \int_{\mathbb{S}_d} |\boldsymbol{u}^T\boldsymbol{s}| sign(\boldsymbol{u}^T\boldsymbol{s})\Lambda(ds)$$

$$\delta(\boldsymbol{u}) = \begin{cases} \boldsymbol{u}^T\boldsymbol{\mu}^0 & ,\alpha \neq 1 \\ \boldsymbol{u}^T\boldsymbol{\mu}^0 - \frac{2}{\pi}\int_{\mathbb{S}_d} |\boldsymbol{u}^T\boldsymbol{s}|.log(\boldsymbol{u}^T\boldsymbol{s})\Lambda(ds) & ,\alpha = 1 \end{cases}$$

3 Stable Random Vector and Gaussian Copula

This section shows main contents of the paper. From that, we apply for actual data in next section.

Lemma 1. *Let Y and Z be continuous random variables with pdf's f_Y and f_Z which are positive on the images $ran(Y)$ and $ran(Z)$ of Y and Z. Then there exists a strictly increasing function $g : ran(Y) \to ran(Z)$ such that the random variable $g \circ Y : \Omega \to ran(Z)$ has the same distribution as Z. Moreover, the function g has positive derivative g'.*

Proof. By assumption, the pdf's f_Y and f_Z are positive, therefore the cdf's F_Y and F_Z are strictly increasing on $ran(Y)$ and $ran(Z)$, respectively. Then the function $g : ran(Y) \to ran(Z)$ defined by $g(u) = F_Z^{-1}(F_Y(u))$ is welldetermined as a strictly increasing function. Besides, for each $u \in ran(Z)$,

$$g'(u) = \frac{f_Y(u)}{f_Z(g(u))}$$

which is a positive function. The identity implies

$$f_Z(g(u))g'(u)du = f_Y(u)du$$

that yields

$$F_Z(g(u)) = \int_{-\infty}^{g(u)} f_Z(g(u))g'(u)du = \int_{-\infty}^{g(u)} f_Y(u)du = F_Y(u) \tag{8}$$

On the other hand, for every $t \in \mathbb{R}$,

$$F_{g\circ Y}(t) = \mathbb{P}(\omega : g(Y(\omega)) \leq t) = \mathbb{P}(\omega : Y(\omega) \leq g^{-1}(t)) = F_Y(g^{-1}(t))$$

Compared the above with (8), putting $t = g(u)$ implies

$$F_Z(t) = F_{g\circ Y}(t)$$

This confirms the two random variables Z and $g \circ Y$ have the same distribution. The lemma is proved. □

Although the lemma seems to be quite trivial and simple, it can be useful in practical application. Namely in applied statistics, sometimes normalizing transformations that turn a given data set to a new form with normal distribution need to be used. One of those normalizing transformation for continuously distributed data is proposed in the following immediate consequence of the lemma.

Corollary 1. *Let X be continuous random variable with pdf f_X which is positive on $ran(X)$. Then there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that the random variable $g \circ X$ has normal distribution.*

Corollary 2. *Let X be a normal random variable and a positive number $\alpha \in (0, 2]$. Then there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that the random variable $g \circ X$ is α-stable.*

Corollary 1, 2 is obtained from Lemma 1 when Z is the normal random variable, α-stable, respectively.

It is evident that all marginals of a stable random vector are stable random variables. The inverse statement is not true, a random vector with all stable marginals is not always stable. However, as it is confirmed in the next lemma, the inverse statement is valid if those marginals are independent.

Lemma 2. *Let $\alpha \in (0, 2]$ and a random vector $\boldsymbol{U} = (U_1, ..., U_d)^T$ be given. Supposed that the marginals $U_1, ..., U_d$ are independent α-stable random variables, then $\boldsymbol{U}$ is an α-stable random vector.*

Proof. Let two random vectors $\mathbf{U}^1 = (U_1^1, ..., U_d^1)$ and $\mathbf{U}^2 = (U_1^2, ..., U_d^2)$ which independent and identically distributed with $\mathbf{U} = (U_1, ..., U_d)$. With two positive real numbers A, B, we will prove existence of a positive real number C and vector $\mathbf{D} \in \mathbb{R}^d$ such that $A\mathbf{U}^1 + B\mathbf{U}^2 \stackrel{D}{=} C\mathbf{U} + \mathbf{D}$.

Because $\mathbf{U}^1, \mathbf{U}^2$ independent and identically distributed, $U_1, ..., U_d$ are independent α-stable random variables, thus every pair of U_k^1, U_k^2 are independent and identically distributed α-stable, $k = 1, ..., d$. Consider two positive numbers A, B, with every pair of U_k^1, U_k^2, there exist positive number $C = (A^\alpha + B^\alpha)^{1/\alpha}$ and D_k such that

$$AU_k^1 + BU_k^2 \stackrel{D}{=} CU_k + D_k \tag{9}$$

This means every $t_k \in \mathbb{R}, F_{AU_k^1+BU_k^2}(t_k) = F_{CU_k+D_k}(t_k), k = 1, .., d$. Based on the assumption, $U_1, ..., U_d$ are independent, we have $U_1^1, ..., U_d^1$ and $U_1^2, ..., U_d^2$ also are independent. Combine this with (9), we have

$$\begin{aligned} F_{A\mathbf{U}^1+B\mathbf{U}^2}(t_1, ..., t_d) &= F_{AU_1^1+BU_1^2}(t_1) \times ... \times F_{AU_d^1+BU_d^2}(t_d) \\ &= F_{CU_1+D_1}(t_1) \times ... \times F_{CU_d+D_d}(t_d) \\ &= F_{C\mathbf{U}+\mathbf{D}}(t_1, ..., t_d) \end{aligned}$$

So, $A\mathbf{U}^1 + B\mathbf{U}^2 \stackrel{D}{=} C\mathbf{U} + \mathbf{D}$. This ensures that $\mathbf{U}^1 = (U_1^1, ..., U_d^1)$ is an α-stable random vector. □

Proposition 4. *Let $\boldsymbol{X} = (X_1, ..., X_d)$ be an d-dimensional normally distributed random vector, where $X_1, ..., X_d$ are independence. Then, every $\alpha \in (0, 2]$, there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that the random vector $\boldsymbol{Y} = (g \circ X_1, ..., g \circ X_d)$ is α-stable.*

Proof. Because $\mathbf{X} = (X_1, ..., X_d)$ is normal random vector, its components $X_1, ..., X_d$ have normal distribution. By Corollary 2, every $k = 1, ..., d$, there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that the random variable $g \circ X_k : \Omega \to \mathbb{R}$ is α-stable. Otherwise, $X_1, ..., X_d$ are independent and g is strictly increasing, so $g \circ X_1, ..., g \circ X_d$ are also independent. Then, Corollary 2 ensures the random vector $\mathbf{Y} = (g \circ X_1, ..., g \circ X_d)$ is α-stable. □

The Lemma 2 shows that the independence is a strong condition that guarantees the stability of a random vector with stable marginals. However, for every 2-stable random vector, it is always possible to rotate the space axes to get a new basis, in which the random vector has independent marginals. Namely, the standard Cholesky decomposition theorem immediately implies the following.

Lemma 3. *Let $\boldsymbol{X}$ be an d-dimensional normally distributed random vector with expextation $\boldsymbol{\mu}$ and positive defined covariance matrix Σ, $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$. Then there exists an orthogonal $d \times d$ matrix $A = (a_{ij})$ such that the r.v. $\boldsymbol{Y} = A\boldsymbol{X}$ has independent normally distributed marginals, where $A : \mathbb{R}^d \to \mathbb{R}^d$ is the linear transformation defined by $AX = AX^T$.*

Proof. We have matrix Σ positive defined, so there exists orthogonal $d \times d$ matrix $A = (a_{ij})$ such that $A^{-1}\Sigma A$ is diagonal matrix.

Set $\mathbf{Z} = (Z_1, ..., Z_d) = A\mathbf{X}$. Because $\mathbf{X} = (X_1, ..., X_d)$ is an d-dimensional normally distributed random vector, we have $X_1, ..., X_d$ are normal variables, so every random variable $Z_k = \sum_{j=1}^{d} a_{kj}X_j$ has normal distribution.

Overwise,

$$\begin{aligned} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) &= (A^{-1}\mathbf{z} - A^{-1}A\boldsymbol{\mu})^T \Sigma^{-1} (A^{-1}\mathbf{z} - A^{-1}A\boldsymbol{\mu}) \\ &= (\mathbf{z} - A\boldsymbol{\mu})^T (A^{-1})^T \Sigma^{-1} (A^{-1})(\mathbf{z} - A\boldsymbol{\mu}) \\ &= (\mathbf{z} - A\boldsymbol{\mu})^T A\Sigma^{-1}A^{-1}(\mathbf{z} - A\boldsymbol{\mu}) \\ &= (\mathbf{z} - \boldsymbol{\mu})^T (A^{-1}\Sigma A)^{-1}(\mathbf{z} - \boldsymbol{\mu}) \end{aligned}$$

This implies $\mathbf{Z}$ has normal distribution with covariance matrix $A^{-1}\Sigma A$ which is diagonal matrix. Therefore, $\mathbf{Z}$'s components $Z_1, ..., Z_d$ do not correlate, means independent. □

Proposition 5. *Let $\boldsymbol{X} = (X_1, ..., X_d)$ be an d-dimensional normally distributed random vector and $\alpha \in (0, 2]$. Then there exists an orthogonal $d \times d$ matrix $A = (a_{ij})$ and a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $\boldsymbol{Y} = (g \circ \sum_{j=1}^{d} a_{1j}X_j, ..., g \circ \sum_{j=1}^{d} a_{dj}X_j)$ is α-stable.*

Proof. With $\mathbf{X}$ is normal random vector, by Lemma 3, there exists an orthogonal $d \times d$ matrix $A = (a_{ij})$ such that $\mathbf{Z} = A\mathbf{X}$ has independent normally distributed marginals. Let $\alpha \in (0, 2]$, by Proposition 4, there exists the strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that random vector $\mathbf{Y} = (g \circ \sum_{j=1}^{d} a_{1j}X_j, ..., g \circ \sum_{j=1}^{d} a_{dj}X_j)$ is α-stable. Proposition is proven. □

We are now ready to state the main result.

Theorem 2. *Let C be a Gaussian copula of a normally distributed random vector $\boldsymbol{X}$ with positive defined covariance matrix. Then for every number $\alpha \in (0, 2]$ there exists an α-stable random vector $\boldsymbol{W}$ such that C is also the copula of $\boldsymbol{W}$.*

Proof. Because both addition and multiplication by positive numbers are strictly increasing transformations in $\mathbb{R}$, by virtue of Proposition 1 it can be supposed that all marginals $X_1, ..., X_d$ of the random vector $\mathbf{X}$ are standard normal random variables, $X_k \sim N(0, 1)$ for $k = 1, ..., d$. Based on the assumption, the covariance matrix Σ of $\mathbf{X}$ is positive defined, Lemma 3 implies the existence of an orthogonal $d \times d$ matrix $A = (a_{ij})$; $A^{-1} = A^T$, such that the normal random vector $\mathbf{Y} = A\mathbf{X}$ has independent marginals $Y_1, ..., Y_d$ and diagonal covariance matrix $A\Sigma A^T$ with diagonal elements consist of all eigenvalues of Σ.

Now α-stable random variables $S_1, ..., S_d$ are concerned. Lemma 1 ensures that, for each $k = 1, ..., d$, there exists a strictly increasing function $g_k : \mathbb{R} \to ran(S_k)$ such that the random variable $U_k = g_k \circ Y_k$ has the same α-stable distribution as S_k. Simultaneously, the independence of marginals $Y_1, ..., Y_d$ implies the independence of random variables $U_1, ..., U_d$. Thus, the random vector $\mathbf{U} = (U_1, ..., U_d)^T$ has independent α-stable marginals, it must be an α-stable random vector as the conclusion of Lemma 2. Let define a new random vector $\mathbf{W} = (W_1, ..., W_d)^T = A^{-1}U$. Then it is clear that $\mathbf{W}$ is also an α-stable random vector as $\mathbf{U}$. We attempt to point out that $\mathbf{W}$ has the same copula C as $\mathbf{X}$. Firstly, the α-stability of marginals $W_1, ..., W_n$ of the random vector $\mathbf{W}$ together with Lemma 1 ensures that for each $k = 1, ..., d$, there exist strictly increasing function hk such that the random variable $Z_k = h_k \circ W_k$ has standard normal distribution as X_k. Consequently, due to Proposition 1, the random vector's $\mathbf{W}$ and $\mathbf{Z} = (Z_1, ..., , Z_d)^T$ have the same copula, $C_{\mathbf{W}} = C_{\mathbf{Z}}$. Therefore, to complete the proof, it is sufficient to show that $C_{\mathbf{Z}} = C_{\mathbf{X}}$, which is equivalent to

$$c_{\mathbf{X}} = c_{\mathbf{Z}} \tag{10}$$

However, for $(u_1, ..., u_n) \in [0,1]^d$, it implies from (1) that

$$\begin{aligned} c_{\mathbf{X}}(u_1, ..., u_d) &= \frac{f_{\mathbf{X}}(F_{X_1}^{-1}(u_1), ..., F_{X_d}^{-1}(u_d))}{f_{X_1}(F_{X_1}^{-1}(u_1))...f_{X_d}(F_{X_d}^{-1}(u_d))} \\ &= \frac{f_{\mathbf{X}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))}{\varphi(\Phi^{-1}(u_1))...\varphi(\Phi^{-1}(u_d))} \end{aligned} \tag{11}$$

and

$$\begin{aligned} c_{\mathbf{Z}}(u_1, ..., u_d) &= \frac{f_{\mathbf{Z}}(F_{Z_1}^{-1}(u_1), ..., F_{Z_d}^{-1}(u_d))}{f_{Z_1}(F_{Z_1}^{-1}(u_1))...f_{Z_d}(F_{Z_d}^{-1}(u_d))} \\ &= \frac{f_{\mathbf{Z}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))}{\varphi(\Phi^{-1}(u_1))...\varphi(\Phi^{-1}(u_d))} \end{aligned} \tag{12}$$

From (11) and (12), it is evident that (10) is equivalent to

$$f_{\mathbf{X}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) = f_{\mathbf{Z}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) \tag{13}$$

Denoting $\mathbf{g} := (g_1, ..., g_d)$, $\mathbf{h} := (h_1, ..., h_d)$ and

$$Q := A^{-1} \circ \mathbf{g} \circ A$$

we see that $\mathbf{Z} = \mathbf{h}(\mathbf{W})$ and $\mathbf{W} = Q(\mathbf{X})$. Then, due to $h_k = \Phi^{-1} \circ F_{W_k}$ and the independence of $Y_1, ..., Y_d$, setting $x_1 = F_{W_1}^{-1}(u_1), ..., x_d = F_{W_d}^{-1}(u_n)$, the right hand side of (13) equals to

$$\begin{aligned} f_{\mathbf{Z}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) &= f_{\mathbf{h}\circ\mathbf{W}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) \\ &= f_{\mathbf{W}}(\mathbf{h}^{-1}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))) \\ &= f_{Q\circ\mathbf{X}}(h_1^{-1}(\Phi^{-1}(u_1)), ..., h_d^{-1}(\Phi^{-1}(u_d))) \\ &= f_{\mathbf{X}}(Q^{-1}(h_1^{-1}(\Phi^{-1}(u_1)), ..., h_d^{-1}(\Phi^{-1}(u_d)))) \\ &= f_{\mathbf{X}}(A^{-1} \circ \mathbf{g}^{-1} \circ A(h_1^{-1}(\Phi^{-1}(u_1)), ..., h_d^{-1}(\Phi^{-1}(u_d)))) \\ &= f_{\mathbf{X}}(A^{-1} \circ \mathbf{g}^{-1} \circ A(h_1^{-1}(F_{W_1}^{-1}(u_1)), ..., h_d^{-1}(F_{W_d}^{-1}(u_d)))) \\ &= f_{A^{-1}\circ\mathbf{Y}}(A^{-1} \circ \mathbf{g}^{-1} \circ A(x_1, ..., x_d)) \\ &= f_{\mathbf{Y}}(A \circ A^{-1} \circ \mathbf{g}^{-1} \circ A(x_1, ..., x_d)) \\ &= f_{Y_1}(g_1^{-1}(\sum_{j=1}^{d} a_{1j}x_j) \times ... \times f_{Y_d}(g_d^{-1}(\sum_{j=1}^{d} a_{dj}x_j) \end{aligned}$$

or

$$f_{\mathbf{Z}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) = f_{U_1}(\sum_{j=1}^{d} a_{1j}x_j) \times ... \times f_{U_d}(\sum_{j=1}^{d} a_{dj}x_j) \tag{14}$$

Simultaneously, setting $y_1 = F_{U_1}^{-1}(u_1), ..., y_d = F_{U_d}^{-1}(u_d)$, the left hand side of (13) is equal to

$$\begin{aligned}
f_{A^{-1}\circ\mathbf{Y}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) &= f_{\mathbf{Y}}(A(\Phi^{-1}(F_{U_1}(u_1)), ..., \Phi^{-1}(F_{U_d}(u_d))) \\
&= f_{\mathbf{Y}}(A(g_1^{-1}(y_1), ..., g_d^{-1}(y_d))) \\
&= f_{Y_1}(\sum_{j=1}^{d} a_{1j} g_j^{-1}(y_j)) \times ... \times f_{Y_d}(\sum_{j=1}^{d} a_{dj} g_j^{-1}(y_j)) \\
&= \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}[\sum_{j=1}^{d} a_{1j} g_j^{-1}(y_j)]^2) \times ... \\
&\qquad ... \times \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}[\sum_{j=1}^{d} a_{dj} g_j^{-1}(y_j)]^2) \\
&= (\frac{1}{\sqrt{2\pi}})^d exp(-\frac{1}{2} \sum_{k=1}^{d} [\sum_{j=1}^{d} a_{kj} g_j^{-1}(y_j)]^2) \\
&= (\frac{1}{\sqrt{2\pi}})^d exp(-\frac{1}{2} \sum_{k=1}^{d} \sum_{j=1}^{d} a_{kj} g_j^{-1}(y_j) \sum_{i=1}^{d} a_{ki} g_i^{-1}(y_i) \\
&= (\frac{1}{\sqrt{2\pi}})^d exp(-\frac{1}{2} \sum_{k=1}^{d} \sum_{j=1}^{d} \sum_{i=1}^{d} a_{kj} a_{ki} g_j^{-1}(y_j) g_i^{-1}(y_i) \\
&= (\frac{1}{\sqrt{2\pi}})^d exp(-\frac{1}{2} \sum_{j=1}^{d} \sum_{i=1}^{d} \delta_{ij} g_j^{-1}(x_j) g_i^{-1}(x_i) \\
&= (\frac{1}{\sqrt{2\pi}})^d exp(-\frac{1}{2} \sum_{j=1}^{d} (g_j^{-1}(x_j))^2) \\
&= \varphi(g_1^{-1}(x_1)) \times ... \times \varphi(g_d^{-1}(x_d)) \\
&= f_{Y_1}(g_1^{-1}(y_1)) \times ... \times f_{Y_d}(g_d^{-1}(y_d))
\end{aligned}$$

or

$$f_{A^{-1}\circ\mathbf{Y}}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)) = f_{U_1}(y_1) \times ... \times f_{U_d}(y_d) \tag{15}$$

where $\delta_{kk} = 1$ for $k = 1, ..., d, \delta_{ki} = 0$ for $k \neq i = 1, ..., d$. Comparing (14) to (15), with $y_1 = \sum_{j=1}^{d} a_{1j} x_j, ..., y_d = a_{dj} x_j$, we can conclude (3) is true, that means (10) holds, the proof completes. □

The next theorem presents Gaussian copula as a sufficient condition for a random vector with stable marginals to have stable distribution.

Theorem 3. *For given $\alpha \in (0, 2]$, let $\boldsymbol{X}$ be a Gaussian random vector with positive defined covariance matrix such that the matrix $A = (a_{ij})$ defined in the Lemma 3 satisfies $det|(a_{ij})|^\alpha \neq 0$. Suppose that $\boldsymbol{W}^\star = (W_1^\star, ..., W_d^\star)^T$ is a random vector with α-stable marginals $W_1^\star, ..., W_d^\star$ such that the copula of $\boldsymbol{W}^\star$ and of $\boldsymbol{X}$ are equal, $C_{\boldsymbol{W}^\star} = C_{\boldsymbol{X}}$. Then $W^\star$ is an α-stable random vector.*

Proof. Let $\beta_1^\star, ..., \beta_d^\star \in [-1, 1]$, $\gamma_1^\star, ..., \gamma_d^\star$ and $\delta_1^\star, ..., \delta_d^\star \in \mathbb{R}$ be the stable parameters and $W_1^\star \sim S(\alpha, \beta_1^\star, \gamma_1^\star, \delta_1^\star), ..., W_d^\star \sim S(\alpha, \beta_d^\star, \gamma_d^\star, \delta_d^\star)$. Then, since α-stability of a random vector is unchanged after invertible linear transformation

$$(x_1, ..., x_d) \mapsto (a_1 x_1 + b_1, ..., a_d x_d + b_d)$$

for any $a_1 > 0, ..., a_d > 0$ and $(b_1, ..., b_d) \in \mathbb{R}^d$, without loss of generality, it can be supposed that $\gamma_1^\star = ... = \gamma_d^\star = 1$ and $\delta_1^\star = ... = \delta_d^\star = 0$

Let matrix $A = (a_{ij})$ and the random vector $Y = (Y_1, ..., Y_d)^T = AX$ be determined as in Lemma 3. We attempt to determine α-stable random variables $S_1 \sim S(\alpha, \beta_1, \gamma_1, 0), ..., S_d \sim S(\alpha, \beta_d, \gamma_d, 0)$ satisfying the equations

$$\begin{cases} 1 = (\gamma_1^\star)^\alpha = |a_{11}|^\alpha \gamma_1^\alpha + ... + |a_{d1}|^\alpha \gamma_d^\alpha \\ ... \\ 1 = (\gamma_1^\star)^\alpha = |a_{1d}|^\alpha \gamma_1^\alpha + ... + |a_{dd}|^\alpha \gamma_d^\alpha \end{cases} \tag{16}$$

and

$$\begin{cases} \beta_1^\star = \frac{\beta_1 |a_{11}|^\alpha \gamma_1^\alpha + ... + \beta_d |a_{d1}|^\alpha \gamma_d^\alpha}{|a_{11}|^\alpha \gamma_1^\alpha + ... + |a_{d1}|^\alpha \gamma_d^\alpha} = |a_{11}|^\alpha \gamma_1^\alpha \beta_1 + ... + |a_{d1}|^\alpha \gamma_d^\alpha \beta_d \\ ... \\ \beta_d^\star = \frac{\beta_1 |a_{1d}|^\alpha \gamma_1^\alpha + ... + \beta_d |a_{dd}|^\alpha \gamma_d^\alpha}{|a_{1d}|^\alpha \gamma_1^\alpha + ... + |a_{dd}|^\alpha \gamma_d^\alpha} = |a_{1d}|^\alpha \gamma_1^\alpha \beta_1 + ... + |a_{dd}|^\alpha \gamma_d^\alpha \beta_d \end{cases} \tag{17}$$

with unknowns $\gamma_1^\alpha, ..., \gamma_d^\alpha$ and $\gamma_1^\alpha \beta_1, ..., \gamma_d^\alpha \beta_d$.

From the assumption, $det|(a_{ij})|^\alpha \neq 0$, it is clear that the Eqs. (16) and (17) are solved and the α-stable random variables $S_1 \sim S(\alpha, \beta_1, \gamma_1, 0), ..., S_d \sim S(\alpha, \beta_d, \gamma_d, 0)$ are completely defined. By virtue of Corollary 2, for each $k = 1, ..., d$, there exists a strictly increasing function $g_k : \mathbb{R} \to rank(S_k)$ such that the random variable $U_k = g_k \circ Y_k$ has the same α-stable distribution as S_k. Simultaneously, the independence of marginals $Y_1, ..., Y_d$ implies the independence of α-stable random variables $U_1, ..., U_d$.

With $\mathbf{W} = (W_1, ..., W_d)^T = A^{-1}\mathbf{U}$, by the same argument of Theorem 2, it is certain that $\mathbf{W}$ is an α-stable random vector that has the same copula as $\mathbf{X}$; $C_\mathbf{W} = C_\mathbf{X} = C_{\mathbf{W}^\star}$. On the other hand, the Eqs. (16) and (17) together with Proposition 1 imply the equality in distribution of all the marginals of $\mathbf{W}$ to correspondent marginals of $\mathbf{W}^\star$. In particular, $W_1 \sim S(\alpha, \beta_1^\star, \gamma_1^\star, \delta_1^\star), ..., W_d \sim S(\alpha, \beta_d^\star, \gamma_d^\star, \delta_d^\star)$. Consequently, $\mathbf{W}^\star \overset{D}{=} \mathbf{W}$ and $\mathbf{W}^\star$ is an α-stable random vector, the proof is fulfilled. □

The above theorem suggests a procedure to check whether a data set can be fitted to any stable random vector or not, with details as follows:

Step 1. To estimate stable parameters of data marginals and to check if the all marginals have α-stable distributions with a common suitably chosen α.

Step 2. To estimate a Gaussian copula for the transformed data having normal distributed marginals, and to check if the original data are fitted to that Gaussian copula.

If the two steps are satisfied, it can be concluded the data vector has stable distribution with the stability index α.

Moreover, if a random vector is α-stable, we can easily consider that vector through by linear combination, in particular, calculation density and quantitile values. Although, most of the density functions of stable random variables are not closed form, many computer software now support this calculation by numerical methods. However, calculations for joint density function of stable random vectors are still difficult. Nolan [12] represented an approach to determine density functions of stable random vectors belonging to a specific class of random vectors with elliptically contoured stable distributions, an α-stable radially symmetric (around 0) random vector $\mathbf{X}$.

However, it is clear that all marginals of an α-stable radially symmetric random vector are identically distributed and all marginals of an elliptically contoured stable random vector are symmetric (around respective location parameter). Those properties of data are quite rarely met in practice. In the following part, we propose a method of computing density functions of stable random vectors having Gaussian copulas. Whilst marginals of a stable random vector with Gaussian copula are not necessarily symmetric, this model of multivariate stable distributions may be more acceptable in the practical data analysis.

Let $\mathbf{G} \sim N(0, \Sigma)$ be a Gaussian random vector with covariance matrix $\Sigma = (\sigma_{ij})$ and $\sigma_{ii} = 1$ for all $i = 1, ..., d$. It is evident that Σ also is the correlation matrix of $\mathbf{G}$ and all marginals of this random vector are standard normal random variables. Then $\mathbf{G}$ has the density function

$$f_{\mathbf{G}}(\mathbf{x}) = (2\pi)^{-d/2}|\Sigma|^{-1/2} exp(-\frac{1}{2}\mathbf{x}\Sigma^{-1}\mathbf{x}^T)$$

for $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$, and its copula can be computed by

$$C_{\mathbf{G}}(\mathbf{u}) = (2\pi)^{-d/2}|\Sigma|^{-1/2} \int_{-\infty}^{\phi^{-1}(u_1)} ... \int_{-\infty}^{\phi^{-1}(u_d)} exp(-\frac{1}{2}\mathbf{x}\Sigma^{-1}\mathbf{x}^T) dx_d ... dx_1$$

for $\mathbf{u} = (u_1, ..., u_d) \in [0, 1]^d$. Moreover from (1), its copula density is determined by

$$\begin{aligned} c_{\mathbf{G}}(\mathbf{t}) &= \frac{f_{\mathbf{G}}(\phi^{-1}(t_1), ..., \phi^{-1}(t_d))}{\varphi(\Phi^{-1}(t_1)) ... \varphi(\Phi^{-1}(t_d))} \\ &= \frac{exp(-\frac{1}{2}(\phi^{-1}(t_1), ..., \phi^{-1}(t_d))\Sigma^{-1}(\phi^{-1}(t_1), ..., \phi^{-1}(t_d))^T}{(2\pi)^{d/2}|\Sigma|^{1/2}\varphi(\Phi^{-1}(t_1)) ... \varphi(\Phi^{-1}(t_d))} \end{aligned} \tag{18}$$

for $\mathbf{t} = (t_1, ..., t_d) \in [0, 1]^d$, where φ, ϕ are pdf and cdf of standard normal random variable $N(0, 1)$, respectively:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2}), \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp(\frac{-t^2}{2}) dt,$$

For fixed $\alpha \in (0, 2)$, let $S_1 \sim S(\alpha, \beta_1, \gamma_1, \delta_1), ..., S_d \sim S(\alpha, \beta_d, \gamma_d, \delta_d)$ be certain α-stable random variables with density functions $f_{S_1}, ..., f_{S_d}$ and cumulative

distribution functions $F_{S_1}, ..., F_{S_d}$ respectively. Then formula (2) from Sklar's Theorem ensures the function

$$F_{\mathbf{Y}}(y_1, ..., y_d) = C_{\mathbf{G}}(F_{S_1}(y_1), ..., F_{S_d}(y_d))$$

for $y_1, ..., y_d \in \overline{\mathbb{R}} = [-\infty, +\infty]$, defines the cumulative distribution function of some random vector $\mathbf{Y}$ with copula $C_{\mathbf{G}}$ and marginals $S_1, ..., S_d$. In the case when the condition of Theorem 3 fulfilled for $A = \Sigma^{1/2}$, the random vector $\mathbf{Y}$ is truly an α-stable random vector, and by virtue of formula (1.4), its density function has the following

$$f_{\mathbf{Y}}(y_1, ..., y_d) = c_{\mathbf{G}}(F_{S_1}(y_1), ..., F_{S_d}(y_d)) \times f_{S_1}(y_1)...f_{S_d}(y_d)$$

$$= \frac{exp(-\frac{1}{2}(\phi^{-1}(F_{S_1}(y_1)), ..., \phi^{-1}(F_{S_d}(y_d)))\Sigma^{-1}(\phi^{-1}(F_{S_1}(y_1)), ..., \phi^{-1}(F_{S_d}(y_d)))^T}{(2\pi)^{d/2}|\Sigma|^{1/2}\varphi(\Phi^{-1}(F_{S_1}(y_1)))...\varphi(\Phi^{-1}(F_{S_d}(y_d)))}$$

$$\times f_{S_1}(y_1)...f_{S_d}(y_d) \tag{19}$$

It is evident that (19) can be calculated by ordinary computing without symmetry assumption on the α-stable random variables $S_1, ..., S_d$. Therefore, the approach represented here is more flexible than the one of Nolan. The next section will show this application in stock market data of Vietnam.

4 Application for Vietnam Stock Market

In this section, we analyze data of stable distribution with Gaussian copula by using the results given in the previous section. The special structure of Gaussian copulas allows researchers to combine well - known computational tools for one-dimensional stable distributions and Gaussian copulas to compute density functions and quantile values of data which follow stable distributions with Gaussian copulas.

Dataset includes daily return of 4 stocks: BID (BIDVbank), VCB (Vietcombank), FLC (FLC Group), VNM (Vinamilk), with a sample from July 24, 2017 to October 14, 2019 to imply observations downloaded from Bao Viet Securities website. These are stocks on Ho Chi Minh Stock Exchange (abbreviated HOSE). Continuously compounded percentage returns are considered, i.e. daily returns are measured by log-differences of closing pricing multiplied by 100. Descriptive statistics together with Kolmogorov-Smirnov test for normal distribution of the univariate series are shown in Table 1 and the result for the univariate stable model estimation are presented in Table 2.

Table 1. Normal distribution test for HOSE daily returns data

Daily return	Size	Mean	SD	Skew	Kurtosis	p-value
BID	497	0.0015	0.03	0.42	3.74	0.00455
VCB	497	0.0021	0.02	0.34	3.85	0.00068
FLC	497	–0.0015	0.02	–1.18	13.55	0.00014
VNM	497	–0.0003	0.02	–2.47	43.59	0.00018

Table 2. K-S test for univariate stability of HOSE daily returns data

Daily return	α	$\bar{\alpha}$	β	γ	δ	p-value
BID	1.464,	1.416	0.008	0.0143	–0.0015	0.5046
VCB	1.446	1.416	0.164	0.0101	–0.0021	0.2884
FLC	1.256	1.416	0.185	0.0093	–0.0007	0.2233
VNM	1.498	1.416	0.072	0.0082	0.0002	0.1952

In Table 1, all p-values smaller than 5% confirm the significant divergence from normal distribution of the 4-dimensional vector. Simultaneously, the greater than 5% p-values of Kolmogorov-Smirnov tests in Table 2 are crucial arguments to conclude all the daily returns series of BID, VCB, FLC and VNM have univariate stable distributions with common stable index $\bar{\alpha} = 1.416$ (the average number of α's of those returns series).

After the above conclusion, we guess the 4-component vector of series of BID, VCB, FLC and VNM has multivariate stable distributions. To check this, we will use model of multivariate stable distributions with Gaussian copula. In the first step, the correlation matrices of daily returns (after normalizing by respective functions determined in Corollary 1) were calculated, with results given in Table 3 (showing only upper diagonal part of each correlation matrix). Then, the function named *gofCopula* in the copula package of R software was used to test the hypotheses of having Gaussian copula for 4-coordinates vector of daily returns, p-value is 0.2552. The result shows that the copula of daily returns vector is significantly Gaussian copula. Thus, the vector of 4 stocks BID, VCB, FLC and VNM has multivariate stable distributions with stable index $\bar{\alpha}$.

Table 3. Correlation matrices of yearly HOSE daily returns

Daily return	BID	VCB	FLC	VNM
BID	1	0.05001	0.00232	0.01380
VCB	-	1	0.04964	0.05349
FLC	-	-	1	0.00167
VNM	-	-	-	1

Consequently, (19) can be applied to determine the density functions of 4-coordinate vector of daily returns. In particular, the density functions are defined by the following explicit form:

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) =$$

$$\frac{exp(-\frac{1}{2}(\Phi^{-1}(F_{X_1}(x_1)), ..., \Phi^{-1}(F_{X_4}(x_4)))\Sigma^{-1}(\Phi^{-1}(F_{X_1}(x_1)), ..., \Phi^{-1}(F_{X_4}(x_4)))^T}{(2\pi)^2|\Sigma_i|^{1/2}\varphi(\Phi^{-1}(F_{X_1}(x_1)))...\varphi(\Phi^{-1}(F_{X_4}(x_4)))}$$

$$\times f_{X_1}(x_1)...f_{X_4}(x_4)$$

where $\mathbf{X} = (X_1, X_2, X_3, X_4)$ with 4-coordinate valued daily returns of stocks BID, VCB, FLC, and VNM respectively.

An investment portfolio is a set of financial assets owned by an investor that may include bonds, stocks, currencies, cash and cash equivalents, and commodities. Further, it refers to a group of investments that an investor uses in order to earn a profit while making sure that capital or assets are preserved, to achieve high returns and reduce the overall risk of the investment. To create a good investment portfolio, an investor or financial manager has to determine the objective of the portfolio. Markowitz model is a portfolio optimization model which was introducted in 1952 by Harry Markowitz [11]. In this model, Markowitz based on mean and variance of securities to choose securities give in investment portfolio. However, with multidimensional and stable distributed data, Markowitz model has complexity in computation and it is difficultly to decide. We use the feature which linear properties of α-stable random vector is univariate α-stable as a new tool which aims choosing optimal investment portfolio, that is calculating percentile values with given probabilities, comparing these values and choosing the maximum value.

In particular, we consider α-stable random vector $\mathbf{X} = (X_1, X_2, X_3, X_4)$ with 4-coordinate valued daily returns of stocks BID, VCB, FLC, and VNM respectively. According to [15], $\mathbf{X}$ is α-stable random vector, then every linear combination of coordinates $P = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$ of vector $\mathbf{X}$ is also α-stable random variable. Thus, every investment portfolio $P = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, a_1 + a_2 + a_3 + a_4 = 1$ has stable distribution with index $\bar{\alpha} = 1.416$. This result helps choosing investment portfolios more conveniently based on calculation percentile values of stable distribution. Table 4 shows percentile values with probabilities $1\%, 2\%, 3\%, 5\%$ in some investment portfolios.

Table 4. Percentile values of investment portfolios

Portfolio	(a_1, a_2, a_3, a_4)	1%	2%	3%	5%
1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	0.05258	0.03351	0.026087	0.01930
2	$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	0.08569	0.05400	0.04152	0.03001
3	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6})$	0.06852	0.04320	0.03325	0.02408
4	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6})$	0.06227	0.03918	0.03010	0.02172
5	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$	0.05180	0.03281	0.02540	0.01862

In Table 4, with the same probability, the 2^{nd} investment portfolio has the maximum return value. Thus, the 2^{nd} option is optimal for investment.

In short, the multivariate stable density functions can be directly used to compute the implied distribution of any portfolio of 4 assets BID, VCB, FLC and VNM. As the joint distribution of the vector of asset-returns is a multivariate stable distribution, the univariate distribution of returns of any portfolio of these assets is also stable. This approach can be used to solve many problems related to portfolio selection.

5 Conclusion and Future Works

Stable random vectors are increasingly being applied to real-world data, especially heavy-tailed data, while the calculation on these vectors is quite complicated. Besides the connection role, the Gaussian copula is also a tool to test vectors with stable distributions easily. However, this is just a rendezvous of the family of stable random vectors. Research can be extended to special classes on stable random vectors like sub-Gaussian stable distribution.

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