Scaling properties of complex networks: Towards Wilsonian renormalization for complex networks

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***Abstract*—Recently, kinds of scaling schemes for general large scale complex networks have been developed and attracted much attention. We propose a new scaling scheme named "two-sites scaling" and investigate how the degree distribution of network changes in applying the proposed scheme to various networks. Notably, the results indicate that networks constructed by BA algorithm behave differently compared with networks commonly appearing in the real world. In addition, since an iterative scaling scheme could define a new renormalizing method, we argue about using our scheme for Wilsonian renormalization group theory for general complex networks and its application to analyzing the dynamics of complex networks.**

***Index Terms*—complex network, renormalization, BA network**

I. Introduction

The complex networks have many interesting cross-disciplinary natures. The notable fact pointed out by Barabasi et al. is that power-law and scale-free properties are universal, from cinema actors' costarring network and the Internet to the metabolic network of cells [1], [2]. The scale-free property can be explained using BA algorithm that is a simple growing network model based on the notion of preferential attachment [3]. Other than frequently discussed quantities such as the power of power-law distribution and cluster coefficient, we can define numerous quantities that characterize the structure of the network. However, it is difficult to know which quantities are essential for the classification of networks. Moreover, some of quantities are hard to calculate because of their high computational complexity. Thus, our goal is to find an essential and scalable characterizing quantity of complex networks.

Recently, the dynamics over complex networks has also been attracting considerable attention [4] beside their structures. It is desirable to understand the dynamics over the network from the viewpoint of application, because there are many demands to control the dynamics over various networks. However, these studies depend on the individual dynamical systems and algorithms to generate complex networks. Therefore, it is necessary to constitute a systematic and analytical approach for the dynamics over complex networks.

In this paper, we propose a scaling method called two-sites scaling that is essentially based on the Wilsonian renormalization group theory. Using the Wilsonian renormalization group theory, we can extract information about the dynamics such as critical exponents in a systematic manner. The extracted information is expected to possess universal nature and would enable us to classify the networks from the viewpoint of dynamics.

The organization of this paper is as follows. We review briefly Wilsonian renormaliztion group theory in Section II. In Section III, we propose two-sites scaling method that defines the action of the renormalization group. In Section IV, we apply our two-site scaling method iteratively to various networks: BA network, router-level network, AS-level network, network of actors' costarring and protein-protein interaction network. In Section V, we discuss the result and the prospect of Wilsonian renormalization theory for complex networks.

II. Wilsonian renormalization group theory

Wilsonian renormalization group theory [5], [6] is powerful and systematic approach in theoretical physics. Using the renormalization theory, we can analyze the flow of the parameters in the parameter space of dynamical systems. For example, in condensed matter physics, it is known that the critical exponents derived from the renormalization theory have strong "universality" [7], [6]. The universality shows that the systems would have same critical exponent if the dimension and the numbers of states are same. Although the renormalization theory is usually applied to the scaling of lattices, because of the above mentioned universality, it is expected to be a systematic analytical approach for the dynamics over complex networks.

Wilsonian renormalization group theory has an important procedure called "rescaling". This is a procedure to integrate the dynamics in the subgraphs which is contracted by scaling, that is an action of renormalization group, and represent their contributions by renormalizing the parameters of the dynamics such as the parameters of the Hamiltonian. However, in this paper, we do not discuss rescaling but focus only on how to define the scaling in the complex networks.

Thanks to the universality, the critical exponents of the dynamical systems over the networks would be good measures for the classification of the systems. The resulting classes are called universality classes. For example, the critical exponents for Ising models can be calculated numerically.

Renormalization approaches have been already applied to complex network such as Watts-Strogatz's small world network [8] and geographical embedded network [9]. These approaches renormalize structural quantities of the networks using the natural scaling methods for the networks embeded in Euclidean space. However, they have not been applied to the renormalization of the dynamical systems. Thus they are not Wilsonian renormalization procedures.

In general complex network, there is not such natural scaling method. Thus we should define how to scale them. In these years, several scaling methods for general complex networks have been proposed [10], [11], which could help to define renormalization group methods for general complex networks.

III. Two-sites scaling

A well-known approach to scaling general complex networks is "box covering method" (or its variants) proposed by Song-Havlin-Makse[10] and succeeding works (e.g. [11]). This method divides a network to subgraphs whose diameters are smaller than a given size. From the graph theoretical view, this procedure corresponds to contraction[1]. In contraction, a subgraph is regarded as one virtual node v and the links between inside of the subgraph and outside of the subgraph as a link connected to the node v. In other words, scaling methods divide a network into subgraphs called "box" and contract these boxes in general.

We propose a scaling method called two-sites scaling that divides a network into boxes, each of which is either randomly selected pair of adjacent nodes or a fragmentary single node. However, if a network has many degree-1 nodes (called "leaf" hereinafter), the randomly selected pairs tend to be pairs consisting of a leaf and its adjacent node, and contraction is reduced to just removing one leaf. This makes contraction too inhomogeneous and is undesirable in order to extract information on dynamics and homogeneity of the whole network, which is our intention to develop two-sites scaling method. To cope with this issue, we just ignore leaves and apply the scaling procedure to only nodes with degree more than or equal to 2. For degree-1 nodes (leaves), a set of leaves which have an identical adjacent node are contracted.

More precisely, the proposed procedure is defined as follows:

1) count the number of degree-1 nodes
2) remove all degree-1 nodes

[1] Although in the above-mentioned papers the authors only count the number of subgraphs and this graph theoretical interpretation is not essential

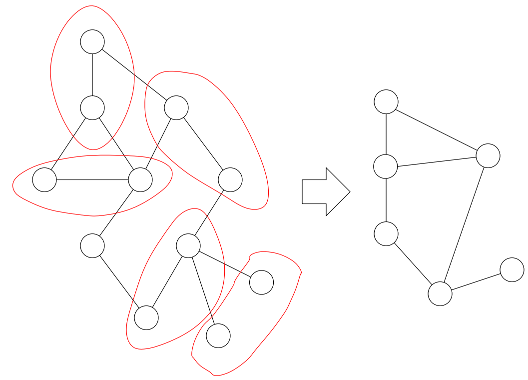

Fig. 1. example of two-sites scaling

3) cover the network by randomly selected pairs of adjacent nodes
4) contract those pairs.
5) add a half number of removed degree-1 nodes

This algorithm makes the size (the number of nodes) of the network approximately be half. By applying this algorithm iteratively, we can scale a network half by half, while the previous scaling methods could not control the number of nodes.

A. Comparison with other scaling methods

The method defined above resembles to the renormalization by block spin transformation(BST) in condensed matter physics. For d dimensional lattice, a usual method is to divide the lattice to d dimensional cubes and contracts the d dimensional cube as one site.

However, since in a general complex network we cannot expect d-dimensional cube appears homogeneously, we need another methods. Box-covering method proposed by [10] divides a network by boxes where distance between any pair of nodes in a box is less then ℓ_S. A simplified version of this method is proposed by [10], [11]. The method consist of a box that is composed of randomly selected node v and nodes whose distance from v is less than ℓ_B.

The box covering method resembles to the scaling method to calculate fractal dimension. Actually, it is intended not to define renormalization scheme but to calculate fractal dimension of complex networks. Thus, there are some problems such as dispersion of the box sizes, if we employ box covering method as a scaling method for the renormalization.

Another approach to define a renormalization scheme is to embed a complex networks into a 1-dimensional lattice or d-dimensional lattice and contract d dimensional cube[8], [12], [13], [9]. Although this method can be regarded as a natural extension of renormalization, it cannot be applied to general complex networks.

B. Features of two-sites scaling

One of the notable characteristics of two-sites scaling is that one step of iteration of two-sites scaling is homogeneous contraction, where almost all nodes are included in the boxes

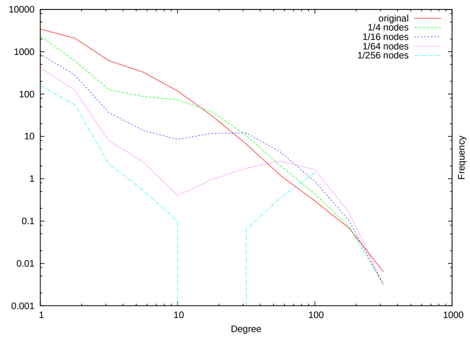

Fig. 2. scaling of network by BA algorithm: scaling property of degree distribution by every two times of two-sites scaling

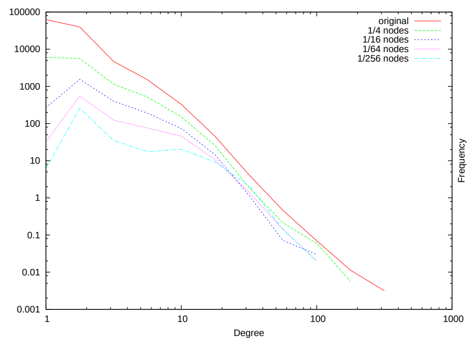

Fig. 3. scaling of the router network of the internet

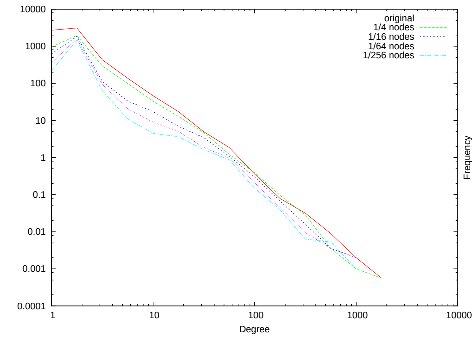

Fig. 4. scaling of the AS network of the internet

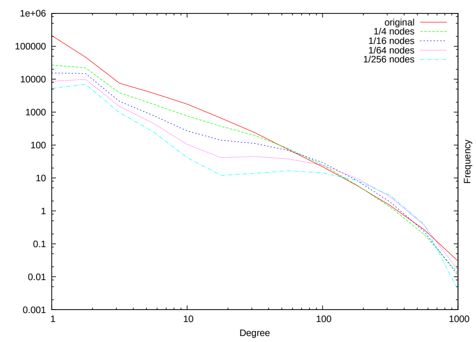

Fig. 5. scaling of actors' costarring network

whose size are the same, say two, which is minimum number of contraction. This characteristics minimizes the loss of fine link structure. Therefore we can have a coarse grained network with a demanded size without the significant loss of fine structure of the network. On the other hand, a special treatment for degree-1 nodes may cause a problem. However, without this exceptional treatment, we cannot scale network homogeneously.

IV. Scaling of Degree Distribution of Various Networks

Figures 2, 3, 4, 5, and 6 show the scaling property of degree distribution of network constructed by BA algorithm, the Internet topologies of router level and AS level, actor's costarring network and protein-protein interaction network, respectively. Two-sites scaling method acts to these networks iteratively. For every two (one for protein protein networks) times of iteration, degree distributions of these network are plotted.

As is clear in figure 2, degree distribution of network constructed by BA algorithm shows rapid transition from power-law distribution to bimodal distribution. On the other hand, other networks observed in nature keep power law distribution.

V. Discussion

Two-sites scaling method, which is inspired from block spin transformation in renormalization theory, gives new quantities to complex networks. We would like to emphasize that quantities given from renormalization theory would influence the dynamics of networks especially in critical phenomena. Although we have not tried to calculate critical exponents in this paper, these critical exponents are expected to have universal property, and would be used to classify the universal classes of dynamical systems over the complex networks.

A. Peculiarity of BA networks

We found that the degree distribution of the networks constructed by BA algorithm transforms from power-law distribution into bimodal distribution. It is peculiar behavior compared with the networks seen in the nature like the Internet, actor network, protein interaction network, which keeps power law by iteration of two-sites scaling. In other words, the networks seen in the nature would have more strong scale invariance in

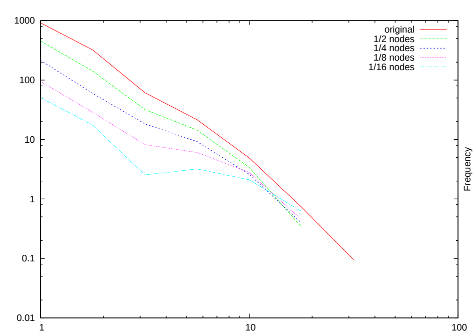

Fig. 6. scaling of protein-protein interaction network

terms of not only in degree distribution but also in two-site scaling method.

As far as we have considered, the reason of this peculiarity of BA networks is still not unknown. Analysis of this phenomena should be done in future work.

B. Towards Wilsonian renormalization group method

Studies of complex networks are often accompanied by the problem of computational complexity. It is because many elemental problems such as isomorphism problem is known to be NP[14]. Renormalization method with scaling of the complex network would provide scalable theory of the dynamics of the complex networks.

Two-site scaling method is resemble to block spin transformation in condensed matter physics. There is attempt to define fractal dimension of complex networks[15]. However the dimension of the complex networks can not be defined straightforwardly. Therefore more deep investigation is needed to verify whether two-site scaling method can define renormalization group action.

The renormalization scheme defined from box covering method, which is originally proposed to define fractal dimension, would be robust to the difference of fractal dimension. However, since the iteration of box covering contracts rapidly large network to one node, it is difficult to get some information such as critical exponents from the few iterations of box-covering method. In other words, these computation of renormalization is executed numerically,and numerical error of critical exponent become large.

In the future works, we should verify that the renormalization scheme defined by two-sites scaling functions like other block spin transformations. It is a difficult problem that the regular lattice is not necessarily mapped to the regular lattice but to the complex networks in our scaling scheme. For this reason, we should employ numerical methods such as Monte Carlo Renormalization. It may be good to verify that the problem will be overcomed by applying this scheme to the regular lattices, whose universality classes are known. If we have same critical exponent from two-sites scaling method as other renormalization schemes, we can expect these exponents provide the classification of dynamics and underlying networks.

Moreover the fact that networks constructed by BA algorithm have peculiar scaling property compared with the natural networks would mean that BA networks have different structure from the natural networks. Also, such difference may affect the dynamics over BA networks especially in critical phenomena.

VI. Conclusion

We have proposed a scaling method called “two-sites scaling”. This scaling scheme keeps the shape of degree distribution of various networks commonly appearing in the real world. However, the degree distribution of BA networks changes drastically by applying this scaling method. This implies that these networks show more strong scale invariance than scale-free degree distribution. Finally, we discussed the application of our scaling method to Wilsonian renormalization. To complete Wilsonian renormalization theory for complex networks, there are many things to do.

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