From Rosenstock Model to Random Walk Based Routing in Wireless Sensor Networks

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ABSTRACT

In this paper, we investigate random walk based routing in wireless sensor networks (WSN) that contains sink nodes independently distributed at random. Our goal is to quantify the effectiveness of such techniques based on the derivation of closed-form expressions. In particular, we focus on the global delay taken for the random walk to deliver data packets from sensor to sink nodes and study its statistics through closed-form derivations. At low concentration of sink nodes, we analytically establish an approximate formula that provides the cumulative distribution function of the global delay. We also perform a series of simulations to validate the analytical results. These simulation studies clearly agree with the analytical results provided that the concentration of sink nodes remains small. The main result of this paper is that the performance of data gathering based on random walks in randomized deployment can be significantly enhanced if the concentration of sink nodes is well tuned.

Keywords

Wireless sensor networks, Routing, Random walk, Performance evaluation

1. INTRODUCTION

One challenging problem in wireless sensor networks is related to the routing of packets towards the intended destination. This is mainly due to the limited capabilities of the devices that compose WSN: limited amount of storage, processing power, communication bandwidth, and energy resources [1]. Traditional routing algorithms, developed usually for small size networks and/or more powerful nodes, become prohibitively complex in terms of both communication and computational complexity [2,3]. Indeed, due to the large number of sensor nodes that often composes WSN, it may be not possible to build and maintain a global addressing scheme for the deployment of a large number of sensor nodes as the overhead of ID maintenance is high. Furthermore, the lack of reliability in WSN makes the routing problem even harder because the information sent within the network can be then lost and any previously established route can break down due to node or communication failures.

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One common technique for overcoming such problems consists in the use of *randomization* to route packets in a WSN. More specifically, this approach consists in forwarding packets at random. Indeed, making appropriate decisions to forward packets depends readily on the amount of state information a node holds (*e.g.*, pointers to parent in a spanning tree, routing tables, *etc.*). So, to deal with the limited capabilities encountered in WSN, one option consists in reducing the dependence of the routing protocol on state information by constructing a data forwarding scheme without recurring to topology-control mechanisms, thereby solving the routing problem strictly locally. The idea of choosing forwarding nodes at random can be therefore applied in this context. As it is most likely that a route will consist of multiple wireless hops, and by identifying packets with particles and the network with the medium in which particles move, the resulting routing scheme induces a *random walk* on the graph of the WSN.

The issue we analyze in this paper can be seen as a natural extension to our previous works [7, 11] where we have studied random walk based routing in the special case of *pre-determined* deployment of sensor and sink nodes over *square lattices*. Motivated by the insight gained by such a configuration, we tackle now a more interesting and complex scenario. Indeed, in this paper we consider again square lattices but we assume that sensor and sink nodes are *independently distributed at random*. Such a random deployment can have several applications to real WSN, in particular, for agricultural applications and environmental monitoring like forest fire detection applications where the position of sink nodes need not be engineered or pre-determined [17]. Another example also arises in target tracking applications, where several hosts or sink nodes are randomly deployed inside the observed region to achieve a higher degree of coverage for tracking a specific event.

To put this network configuration into a simple mathematical model, the extensive literature on random walk theory suggests to draw a parallel between the sought routing scheme and the *Rosenstock model* [14, 15]. This model is a member of a class of models that have received much interest in statistical physics literature, and that by their simple description but complicated nature have become a challenge to theoreticians. The Rosenstock model, for instance, serves to describe many physical processes in which absorption and emission occur, such as the context of photosynthesis and charges transport in an anisotropic molecular crystal. This model consists of a random walk on a d-dimensional *infinite* lattice with irreversible traps, *i.e.*, a walker encountering a trap is killed there. Each site of the lattice can be in either of two different states: with probability c it is a trap and with probability $1-c$ it is a non-trapping point.

Many authors have studied various properties of this random trap model. Some quantities on which interest has centered are: the probability for the walker to survive a given number of steps and/or the average number of steps made until trapping. In spite of its conceptual clarity and simplicity, this model often defies exact mathematical analysis. So far, only few rigorous results have been obtained, except in one dimension where it has a complete solution. On the other hand, several approximative methods have been developed. With a few exceptions, the results obtained are valid for values of c that are either small or close to unity. An extensive discussion of many aspects of this problem has been given by in [5].

In this respect, our proposed routing scheme with randomly distributed sink nodes is obviously akin to the Rosenstock model in the sense that sink nodes can be modeled as irreversible traps. Therefore, to some extent, the analysis of this scheme relies closely on the theoretical tools developed in existing works devoted to the Rosenstock model. However, the key assumption that makes our model different from the Rosenstock model is the starting position of the random walk (only at trap-free sites, namely sensor nodes) and the periodic boundary conditions of the square lattice (the network graph) on which the random walk takes place.

Accordingly, the main goal of this paper is to evaluate the performance of such a routing scheme. In particular, we investigate the time delay required for a packet issued from a randomly chosen sensor node to be gathered at a sink node. The first step towards this objective is to make use of a simple consideration, which has been known for a long time in random walk theory. It involves the number of distinct nodes visited in the first n hops of the walk. As will be shown in next sections, a matter of particular convenience in the description of the envisioned routing scheme is that some of its properties are easily expressed in terms of properties of the random walk in the absence of sink nodes (traps). This is an important simplification.

The remainder of this paper is divided as follows. In Section 2, we describe the network model, give some theoretical elements of random walk theory, and formulate the proposed routing scheme. Section 3 is devoted to the study of the mean distinct nodes visited through generating function techniques. In particular, we derive an approximation formula for this quantity. In Sections 4, we investigate the performance of the random walk based routing in terms of delay properties. In Section 5, we resort to extensive simulations to validate the analytical study. We conclude this paper in Section 6 by a summary of main results and future perspective.

2. RANDOM WALK MODELING

2.1 Network Description

Let $\mathfrak{T}$ be a finite square lattice of size $N \times N$ with 4-nearest neighbor connectivity, where each site $\vec{\mathbf{r}}$ is labeled with (r_1, r_2): r_1 and r_2 are nonnegative integers such that

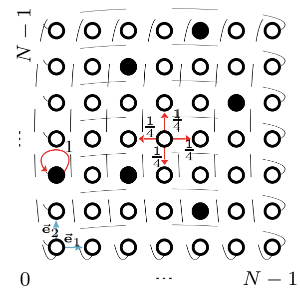

Figure 1: Randomized deployment of nodes over torus lattice $\mathfrak{T}$. The dark- filled circles are sink nodes while light-filled ones are sensor nodes.

$0 \leq r_1, r_2 \leq N-1$. To deal with boundary effects, we equip $\mathfrak{T}$ with periodic boundary conditions, which corresponds to connecting the opposite ends of $\mathfrak{T}$. The resulting structure is then equivalent to a torus lattice. Moreover, we suppose that a set of sensor and sink nodes are independently scattered at random over $\mathfrak{T}$. We denote by c the concentration of sink nodes, or equivalently the probability that a site is a sink node. We are concerned here with a particular routing scheme based on an unbiased random walk defined as follows. Whenever a packet is generated at a given sensor node, it performs a random motion until reaching the *first* encountered sink node, where it is trapped. At this moment, we consider that the data gathering process has occurred with success, and then we no longer care about the outcome of the walk. With 4-nearest neighbor connectivity, hops onto any nearest neighbor are permitted and equally probable with transition probability $\frac{1}{4}$. The resulting structure is depicted in Figure 1. Now, the fundamental question that arises can be summarized in a single question, namely "how long does it take for a packet generated at a randomly chosen sensor node to be gathered by the first encountered sink node?".

2.2 Definitions

We define, for $n \geq 1$, $P_n(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ the probability of being at node $\vec{\mathbf{s}}$ after n hops, given that the packet has been issued at node $\vec{\mathbf{r}}$. We also define $F_n(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ the probability of arriving at node $\vec{\mathbf{s}}$ for the *first* time on the nth hop, given that the walk started at node $\vec{\mathbf{r}}$. We shall refer to these probabilities as the *node occupation probability* and the *first-passage probability* respectively. By convention we have $P_0(\vec{\mathbf{r}}, \vec{\mathbf{s}}) = \delta_{\vec{\mathbf{r}}\vec{\mathbf{s}}}$ and $F_0(\vec{\mathbf{r}}, \vec{\mathbf{s}}) = 0$. We make use in this paper of the *generating function formalism* [10] to deal with a sequence $\{c_n\}_{n\in\mathbb{N}}$ by capturing all these coefficients into a formal infinite series defined as $C(z) = \sum_{n=0}^{\infty} c_n z^n$ where the complex variable z is small enough to ensure the convergence of this series. $C(z)$ is called the generating function associated with $\{c_n\}_{n\in\mathbb{N}}$.

In the following, we denote the generating functions associated with $\{P_n(\vec{\mathbf{r}}, \vec{\mathbf{s}})\}_{n\in\mathbb{N}}$ and $\{F_n(\vec{\mathbf{r}}, \vec{\mathbf{s}})\}_{n\in\mathbb{N}}$ as $P(\vec{\mathbf{r}}, \vec{\mathbf{s}}|z)$ and $F(\vec{\mathbf{r}}, \vec{\mathbf{s}}|z)$ respectively. Hereafter we present two well-known classical relations extensively used in random walk theory, and upon which the theoretical results derived in this paper rely.

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LEMMA 1. *$P(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z)$ satisfies the normalization condition*

$$\sum_{\vec{\mathbf{s}} \in \mathfrak{T}} P(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z) = \frac{1}{1-z}. \qquad (1)$$

PROOF. As long as there is no possibility that the packet is removed from $\mathfrak{T}$ at any time n (this is so for the walks considered here), $P_n(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ is a distribution over the nodes of $\mathfrak{T}$ at fixed n and $\vec{\mathbf{r}}$, which leads to $\sum_{\vec{\mathbf{s}} \in \mathfrak{T}} P_n(\vec{\mathbf{r}}, \vec{\mathbf{s}}) = 1$, for all $n \geq 0$. By taking the generating functions of both sides of this relation, (1) follows. □

LEMMA 2. *$F(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z)$ and $P(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z)$ are related to each other according to the relation*

$$F(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z) = \frac{P(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z) - \delta_{\vec{\mathbf{r}}\vec{\mathbf{s}}}}{P(\vec{\mathbf{s}}, \vec{\mathbf{s}} \mid z)}, \qquad \vec{\mathbf{r}}, \vec{\mathbf{s}} \in \mathfrak{T}. \qquad (2)$$

PROOF. The proof is based on the law of total probability and is given in [8]. □

2.3 Problem Formulation and Characterization

Assume now that a packet is initially generated at a randomly chosen sensor node and then performs a random walk motion until reaching a sink node. To facilitate the characterization of this random walk, let us introduce $\beta_N^{\ddagger}(n, c)$ to be the probability that the packet survives for at least n hops without having encountered a sink node. In random walk theory, this probability is sometimes called the *survival probability*. Our interest in the survival probability stems from the fact that it is a key quantity from which most of the performance properties of the proposed routing scheme can be derived. In what follows, we show that this probability can be exactly related to the probability distribution of the number of distinct nodes visited in an n-hop random walk taking place on torus lattice $\mathfrak{T}$ *devoid* of sink nodes, *i.e.*, considering all lattice sites are sensor nodes. This induced random walk is deemed *perfect*, in the sense that all lattice sites become topologically equivalent. Before determining the survival probability, let us first specify some quantities and relations for the induced perfect random walk that will be needed in our later analysis.

Let us write $P_n^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ and $F_n^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ for the node occupation and first-passage probabilities on the nth hop of the induced perfect random walk respectively, and $P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z)$ and $F^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \mid z)$ for their associated generating functions. We introduce also random variable $\mathcal{S}_{N,n}^{\dagger}$, the number of distinct nodes visited in the first n hops of the induced perfect random walk. For a particular realization of the random walk, since a visited virgin node carries a probability c to be a sink node or equivalently a probability $1-c$ to be a sensor node, one has a probability $(1-c)^{\mathcal{S}_{N,n}^{\dagger}-1}$ that encountering a sink node has not occurred up to the nth hop. This is valid as long as the starting position of the random walk is a sensor node. By averaging over all the realizations of the random walk, the survival probability can be then written as

$$\begin{aligned} \beta_N^{\ddagger}(n, c) &= \sum_{s=1}^{n+1} \mathbf{Pr}\{\mathcal{S}_{N,n}^{\dagger} = s\}(1-c)^{s-1} \\ &= \mathbf{E}\big((1-c)^{\mathcal{S}_{N,n}^{\dagger}-1}\big). \end{aligned} \qquad (3)$$

This formula is a very important and basic relation in the sense that the survival probability $\beta_N^{\ddagger}(n, c)$ can be exactly determined once the probability distribution function of the random variable $\mathcal{S}_{N,n}^{\dagger}$ is known. Moreover, depending only on n, c, and the distribution probability function of $\mathcal{S}_{N,n}^{\dagger}$, the obtained expression of the survival probability involves only the properties of the random walk in the absence of sink nodes, that is, the induced perfect random walk. This is an important simplification. However, the probability distribution function of $\mathcal{S}_{N,n}^{\dagger}$ has been known only in special cases and there is no complete general solution. In particular, it has been known in one dimension, but the only available information is the first and second moments in two dimensional infinite lattices, as well as the fact that it tends to Gaussian distribution as the dimensionality tends to infinity [5].

3. NUMBER OF DISTINCT NODES VISITED

In this section, we shall focus our attention primarily on random variable $\mathcal{S}_{N,n}^{\dagger}$, the set of distinct nodes visited by a random walker (a packet) on torus lattice $\mathfrak{T}$ free of sink nodes. By convention, the starting event is counted as a visit to the starting node, so that we have $\mathcal{S}_{N,0}^{\dagger} = 1$, and $\mathcal{S}_{N,n}^{\dagger}$ is non-decreasing and constrained to satisfy the inequality $1 \leq \mathcal{S}_{N,n}^{\dagger} \leq n+1$. These two relations have the obviously necessary property that when $n = 0$, the survival probability given by (3) is equal to 1, which is consistent with our assumption that only sensor nodes generate packets. In the mathematical literature, the random variable $\mathcal{S}_{N,n}^{\dagger}$ is sometimes misleadingly called the range of the random walk [16].

It is relatively easy to analyze the first moment of the number of distinct nodes visited in an n-step random walk taking place on infinite lattices, which are homogeneous in the usual sense that all sites are topologically equivalent. A statistic from which the mean value of number of distinct nodes visited can be deduced was first introduced in [13]. However, it is much harder to analyze the variance and the asymptotic probability distribution. Bounds on the variance were first exhibited by Dvoretzky and Erdös in [6], but a full analysis of the problem did not appear until two decades later, in a series of papers by Jain, Pruitt, and Orey [9], some of which also address the limiting probability distribution. A full analysis of this problem can be found in [8].

In our case where packets perform random walks on a finite lattice with periodic boundary conditions, and with multiple randomly distributed sink nodes, we give an *original* elementary analysis of the mean value of random variable $\mathcal{S}_{N,n}^{\dagger}$, denoted by $\mathbf{E}(\mathcal{S}_{N,n}^{\dagger})$. In Subsection 3.1, we develop a general theoretical framework to facilitate the analysis of $\mathbf{E}(\mathcal{S}_{N,n}^{\dagger})$. In Subsection 3.2, we then provide an approximation formula for $\mathbf{E}(\mathcal{S}_{N,n}^{\dagger})$ that holds for low concentration of sink nodes. This our main contribution in this paper, from which are derived all subsequent results.

3.1 Generating Function Analysis of $\mathbf{E}(\mathcal{S}_{N,n}^{\dagger})$

Suppose that at time $n = 0$ a packet is generated at a given sensor node $\vec{\mathbf{r}}$. We propose now to estimate the generating function associated with $\{\mathbf{E}(\mathcal{S}_{N,n}^{\dagger})\}_{n \in \mathbb{N}}$, the sequence of the mean number of distinct nodes visited by the packet in the first n hops of the walk. To facilitate this estimation, we introduce an auxiliary random variable $\Delta_{N,n}^{\dagger}$, defined to be the number of virgin nodes visited on the nth hop, so that $\Delta_{N,0}^{\dagger} = 1$, and for $n > 1$, $\Delta_{N,n}^{\dagger}$ takes the values 0 and 1. Let us denote by $\mathbf{E}(\Delta_{N,n}^{\dagger})$ the expectation of random variable

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$\Delta^{\dagger}_{N,n}$. Let us also write $\mathcal{S}^{\dagger}_{N}(z)$ and $\Delta^{\dagger}_{N}(z)$ for the generating functions associated with sequences $\{\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})\}_{n\in\mathbb{N}}$ and $\{\mathbf{E}(\Delta^{\dagger}_{N,n})\}_{n\in\mathbb{N}}$ respectively.

THEOREM 1. *$\mathcal{S}^{\dagger}_{N}(z)$ and $\Delta^{\dagger}_{N}(z)$ are related to each other according to the formula*

$$\mathcal{S}^{\dagger}_{N}(z) = \frac{\Delta^{\dagger}_{N}(z)}{(1-z)}. \tag{4}$$

PROOF. By definition, we have clearly for all $n \geq 0$

$$\mathcal{S}^{\dagger}_{N,n} = \sum_{j=0}^{n} \Delta^{\dagger}_{N,j}. \tag{5}$$

By taking expected values of both sides of (5), we have

$$\mathbf{E}(\mathcal{S}^{\dagger}_{N,n}) = \sum_{j=0}^{n} \mathbf{E}(\Delta^{\dagger}_{N,j}). \tag{6}$$

Multiplying both sides of (6) by z^n and summing over all $n \geq 0$ leads to

$$\begin{aligned}\mathcal{S}^{\dagger}_{N}(z) &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \mathbf{E}(\Delta^{\dagger}_{N,j}) = \sum_{j=0}^{\infty} \mathbf{E}(\Delta^{\dagger}_{N,j}) \sum_{n=j}^{\infty} z^n \\ &= \frac{1}{(1-z)} \sum_{j=0}^{\infty} \mathbf{E}(\Delta^{\dagger}_{N,j})\, z^j = \frac{\Delta^{\dagger}_{N}(z)}{(1-z)},\end{aligned} \tag{7}$$

which is the assertion of Theorem 1. □

One can observe in passing that random variables $\Delta^{\dagger}_{N,j}$ are not independent. This is what makes the determination of the probability distribution of $\mathcal{S}^{\dagger}_{N,n}$ difficult. It remains now to express generating function $\Delta^{\dagger}_{N}(z)$ in terms of the node occupation probability generating function of the induced perfect random walk, that is, $P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z)$. We can therefore deduce the related expression for generating function $\mathcal{S}^{\dagger}_{N}(z)$ later via Theorem 1.

THEOREM 2. *Generating function $\Delta^{\dagger}_{N}(z)$ can be expressed as*

$$\Delta^{\dagger}_{N}(z) = \frac{1}{(1-z)P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)}, \tag{8}$$

where $P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$ is the node occupation probability generating function of the induced perfect random walk that starts and ends at the origin.

PROOF. Since random variable $\Delta^{\dagger}_{N,n}$ takes the values 0 and 1, we have

$$\mathbf{E}(\Delta^{\dagger}_{N,n}) = \mathbf{Pr}\{\Delta^{\dagger}_{N,n} = 1\}. \tag{9}$$

Using the law of total probability leads to

$$\mathbf{Pr}\{\Delta^{\dagger}_{N,n} = 1\} = \sum_{\vec{\mathbf{s}} \neq \vec{\mathbf{r}}} F^{\dagger}_{n}(\vec{\mathbf{r}}, \vec{\mathbf{s}}), \quad \text{for } n > 0, \tag{10}$$

while $\mathbf{E}(\Delta^{\dagger}_{N,0}) = 1$. We multiply both sides of (10) by z^n and we sum over all $n > 0$, finding that

$$\Delta^{\dagger}_{N}(z) = 1 + \sum_{\vec{\mathbf{s}} \neq \vec{\mathbf{r}}} F^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z). \tag{11}$$

We can use now Lemma 2 to eliminate the first-passage probability generating function $F^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z)$ in favor of the node occupation generating function $P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z)$. We obtain

$$\Delta^{\dagger}_{N}(z) = 1 + \sum_{\vec{\mathbf{s}} \neq \vec{\mathbf{r}}} \frac{P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z)}{P^{\dagger}(\vec{\mathbf{s}}, \vec{\mathbf{s}} \,|\, z)}. \tag{12}$$

For a the induced perfect random walk one can note that $P^{\dagger}(\vec{\mathbf{s}}, \vec{\mathbf{s}} \,|\, z) = P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$ (due to homogeneity property), thus

$$\begin{aligned}\Delta^{\dagger}_{N}(z) &= 1 + \frac{1}{P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)} \sum_{\vec{\mathbf{s}} \neq \vec{\mathbf{r}}} P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z) \\ &= 1 + \frac{1}{P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)} \left\{ \sum_{\vec{\mathbf{s}} \in \mathfrak{T}} P^{\dagger}(\vec{\mathbf{r}}, \vec{\mathbf{s}} \,|\, z) - P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z) \right\}\end{aligned} \tag{13}$$

Using the normalization condition stated by Lemma 1, Theorem 2 follows. □

By substituting for $\Delta^{\dagger}_{N}(z)$ from (8) into (4), we obtain the expression of generating function $\mathcal{S}^{\dagger}_{N}(z)$ in terms of the node occupation probability generating function of the induced perfect random walk which starts and ends at the origin. Thus, we obtain the following corollary.

COROLLARY 1. *Generating function $\mathcal{S}^{\dagger}_{N}(z)$ can be expressed as*

$$\mathcal{S}^{\dagger}_{N}(z) = \frac{1}{(1-z)^2 P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)}. \tag{14}$$

Remark: It remains now to explicitly express $P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$. This was carried out in our previous works [7, 11] where, based on the homogeneity property of the induced perfect random walk, a complete analysis of $P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$ is provided. In this paper, we only give the explicit expression of $P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$ as follows:

$$P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z) = \frac{1}{N^2} \sum_{\vec{\mathbf{m}} \in \mathfrak{T}} \frac{1}{1 - z\,\lambda^{\dagger}(\vec{\mathbf{m}})}, \tag{15}$$

where $\lambda^{\dagger}(\vec{\mathbf{m}}) = \frac{1}{2}\left(\cos(\frac{2\pi}{N}m_1) + \cos(\frac{2\pi}{N}m_2)\right)$ called the structure function of the walk [8].

3.2 Approximation of the Mean Number of Distinct Nodes

The previous analysis allows us to directly handle generating function $\mathcal{S}^{\dagger}_{N}(z)$ rather than sequence $\{\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})\}_{n\in\mathbb{N}}$. Once $\mathcal{S}^{\dagger}_{N}(z)$ is explicitly determined, all the coefficients $\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})$ can be then theoretically recovered. However, this is not a simple matter since Corollary 1 only enables us to express $\mathcal{S}^{\dagger}_{N}(z)$ as a function of $P^{\dagger}(\mathbf{0},\mathbf{0} \,|\, z)$. It may still be possible to extract an approximation formula for coefficients $\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})$ at large n using a powerful technique which we shall refer here as the *Method of Darboux* [12]. This method enables one to deduce the asymptotic behavior of the coefficients in a power series from the behavior of the power series in the neighborhood of a singular point.

Method of Darboux. *Let $f(z)$ be the generating function associated with sequence $\{f_n\}_{n\in\mathbb{N}}$. Assume that ρ is the distance from the origin of the nearest singularity of $f(z)$ and suppose that we can find a "comparison" function $g(z)$ having the following properties:*
(i) $g(z)$ is holomorphic in $0 < |z| < \rho$.

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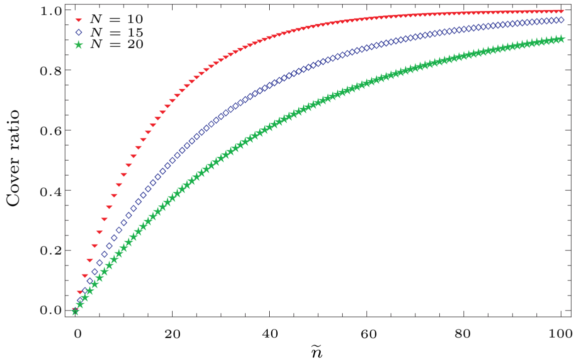

Figure 2: Cover ratio as a function of the number of hops normalized to N for different network sizes.

(ii) $f(z) - g(z)$ *is continuous in* $0 < |z| \le \rho$.
(iii) The coefficients g_n *in the power series expansion* $g(z) = \sum_{n=0}^{\infty} g_n z^n$ *have known asymptotic behavior.*
Then, between the coefficients f_n *and* g_n *holds the relation*

$$f_n = g_n + o(\rho^{-n}) \qquad as \qquad n \to \infty. \tag{16}$$

We can now apply the Method of Darboux to derive the following result.

Result 1. $\boldsymbol{E}\big(\mathcal{S}^{\dagger}_{N,n}\big)$ *has the following asymptotic behavior as* $n \to \infty$

$$\frac{\boldsymbol{E}\big(\mathcal{S}^{\dagger}_{N,n}\big)}{N^2} = 1 - \Big[1 + \frac{3}{N^2\big(3\,\varphi_N(1) + 1\big) - 1}\Big]^{-(n+1)} + o(1) \tag{17}$$

where $\varphi_N(1)$ *is an* N*-dependent series defined by (31) in Appendix A.*

Proof. See Appendix B. ☐

Note that at large time n, $\mathbf{E}\big(\mathcal{S}^{\dagger}_{N,n}\big)$ tends to N^2, which is consistent with the idea that at long-time the random walk tends to cover the entire network. The main question that arises now is at which rate nodes are visited? To answer this question, let us study the time evolution of the ratio between the mean number of distinct nodes visited and the total number of nodes, that is, $\mathbf{E}\big(\mathcal{S}^{\dagger}_{N,n}\big)/N^2$. We call this quantity the *cover ratio*. To find an identical description of this quantity for different network sizes, it is convenient to rescale the time step n to the variable $\widetilde{n} = n/N$. The reason for this variable rescaling is the fact that N is the characteristic distance in terms of number of hops between the two furthest-apart nodes in torus lattice $\mathfrak{T}$. It is also the time during which the entire network is covered by flooding.

Many key features can be drawn from Figure 2, where the cover ratio is plotted against the scaling variable $\widetilde{n}$ for $N = 10$, 15, and 20. First, note that the cover ratio grows initially with a high rate, but after a crossover value the growth is almost sublinear. The early time slope means that the random walker initially escapes its starting position and diffuses around the network covering new territories. After the first few hops, revisitation of the same locations starts and becomes more emphasized by the finite-size effect of the torus lattice. This results in the sublinear growth of the cover ratio. Looking more closely, we find that, for $N = 10$, 40% of the nodes are visited during the first 85 hops, whereas during the next 300 hops only 50% are newly explored. Second, notice that the larger the network size, the slower the cover ratio. Third, compared to flooding, random walks are slower in respect to the same covered space. For example, for $N = 10$, we find that to cover 90% of the network, random walks are nearly 30 times slower than flooding. However, the higher coverage speed of flooding is achieved at the expense of a large overhead in terms of the number of packets.

4. PERFORMANCE ANALYSIS OF THE GLOBAL DELAY

In this section, we investigate the performance of the envisioned routing scheme in terms of delay properties. We deal primarily with the global behavior that emerges when an event of interest is detected by a randomly chosen sensor node in torus lattice $\mathfrak{T}$ containing sink nodes at concentration c. This event should be then reported to one of the sink nodes through random walks. In particular, we study the delay time in number of hops experienced by the packet that bears that event to be trapped by the first encountered sink node. This delay time is referred to as the *global delay* and is denoted by $\mathcal{D}^{\ddagger}_N(c)$. This quantity is a discrete random variable taking nonnegative integer values. Since the survival probability $\beta^{\ddagger}_N(n,c)$ introduced in Subsection 2.3 is the probability that the packet has not been trapped after a random walk of n hops, we obtain in terms of probability notation

$$\mathbf{Pr}\{\mathcal{D}^{\ddagger}_N(c) > n\} = \beta^{\ddagger}_N(n,c), \qquad n \ge 0.$$

Therefore, if we denote by $\mathbf{F}^{\ddagger}_N(n,c)$ the cumulative distribution function of the global delay, we can then write

$$\begin{aligned}\mathbf{F}^{\ddagger}_N(n,c) &= \mathbf{Pr}\{\mathcal{D}^{\ddagger}_N(c) \le n\} = 1 - \mathbf{Pr}\{\mathcal{D}^{\ddagger}_N(c) > n\} \\ &= 1 - \beta^{\ddagger}_N(n,c).\end{aligned} \tag{18}$$

Recall that according to (3), the survival probability $\beta^{\ddagger}_N(n,c)$ depends on $\mathcal{S}^{\dagger}_{N,n}$, the number of distinct nodes visited in the first n hops. However, the probability distribution of random variable $\mathcal{S}^{\dagger}_{N,n}$ is not tractable and the only information available is about the approximation of its mean value, previously evaluated in Section 3. Since in WSN the number of sensor nodes usually exceeds largely that of sink nodes, we suppose in what follows that sink nodes are deployed on torus lattice $\mathfrak{T}$ at low concentration. Thus, we can write $c \ll 1$. Under this condition, we obtain the following result.

Result 2. *At large* n, $\boldsymbol{F}^{\ddagger}_N(n,c)$ *can be approximately expressed as*

$$\boldsymbol{F}^{\ddagger}_N(n,c) \simeq \frac{cN^2}{2}\,(2+3c)\Big\{1 - \Big[1 + \frac{3}{N^2\big(3\,\varphi_N(1)+1\big) - 1}\Big]^{-(n+1)}\Big\} - c\,(1+c) \tag{19}$$

Proof. From (3), at low concentration of sink nodes, we can expand $\beta^{\ddagger}_N(n,c)$ as

$$\begin{aligned}\beta^{\ddagger}_N(n,c) &= \mathbf{E}\big((1-c)^{\mathcal{S}^{\dagger}_{N,n}-1}\big) \\ &= 1 - c\big(\mathbf{E}(\mathcal{S}^{\dagger}_{N,n} - 1)\big) + \frac{c^2}{2}\mathbf{E}\big((\mathcal{S}^{\dagger}_{N,n} - 1)(\mathcal{S}^{\dagger}_{N,n} - 2)\big) + \cdots .\end{aligned}$$

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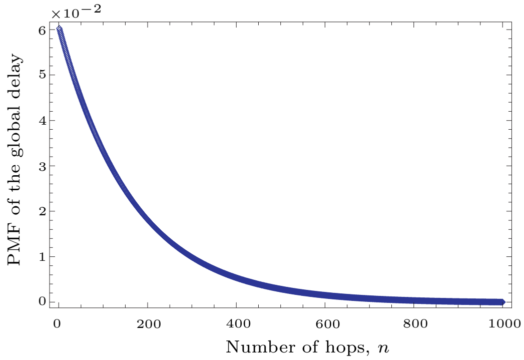

Figure 3: PMF of the global delay for $N = 10$ and $c = 1\%$.

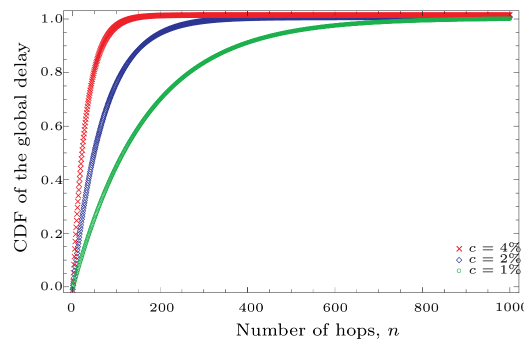

Figure 4: CDF of the global delay for different concentrations. $N = 10$.

If we truncate this series at the second order term, we find that the survival probability $\beta^{\ddagger}_N(n, c)$ can be approximately related to the first and second moments of $\mathcal{S}^{\dagger}_{N,n}$ as

$$\beta^{\ddagger}_N(n, c) \simeq 1 - c\big(\mathbf{E}(\mathcal{S}^{\dagger}_{N,n} - 1)\big) + \frac{c^2}{2}\big(\mathbf{E}({\mathcal{S}^{\dagger}}^2_{N,n}) - 3\mathbf{E}(\mathcal{S}^{\dagger}_{N,n}) + 2\big). \qquad (20)$$

Unfortunately, this requires calculation of $\mathbf{E}({\mathcal{S}^{\dagger}}^2_{N,n})$ which requires a much more sophisticated approach than that taken here. As $\mathbf{E}({\mathcal{S}^{\dagger}}^2_{N,n})$ is involved only in the second order term, we can neglect it while keeping the other coefficients, so that (20) becomes

$$\beta^{\ddagger}_N(n, c) \simeq 1 + c\,(1 + c) - \frac{c}{2}\,(2 + 3c)\,\mathbf{E}(\mathcal{S}^{\dagger}_{N,n}). \qquad (21)$$

Substituting $\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})$ from (17) into (21) and then $\beta^{\ddagger}_N(n, c)$ into (18), Result 2 follows. □

Discussion. By determining the cumulative distribution function, the global delay is now completely characterized. Therefore, the extraction of useful performance properties is straightforward. In particular, this enables us to infer the probability mass function of the global delay, the mean value, and the dispersion. The probability mass function of the global delay can be then derived as

$$\mathbf{Pr}\{\mathcal{D}^{\ddagger}_N(c) = n\} = \mathbf{F}^{\ddagger}_N(n, c) - \mathbf{F}^{\ddagger}_N(n - 1, c).$$

In Figure 3, we draw this probability mass function as a function of n for a torus lattice of size 10×10 and a single sink node ($c = 1\%$). The obtained curve suggests that the global delay may follow an exponential probability distribution. Computing the mean value and the standard deviation of the global delay for this configuration, we find respectively $\mathbf{E}\big(\mathcal{D}^{\dagger}_N(c)\big) = 168.47$ and $\mathbf{SD}\big(\mathcal{D}^{\dagger}_N(c)\big) = 166.43$ leading to a coefficient of variation very close to unity. This property can be also verified for other values of N provided that a single sink node is randomly deployed. Therefore, this property is a good evidence for a nearly exponential distribution. Weiss and den Hollander have also obtained this exponential behavior but for long-time regime using completely different arguments [4].

More interestingly than in case of the pre-determined deployment studied in our previous works [7, 11], where only we have evaluated the central tendency and the dispersion properties of the global delay, the analytical derivation of the cumulative distribution function enables us to calculate with accuracy the percentile the global delay is below a given threshold for a given concentration of sink nodes. Once again, for a torus lattice of size 10×10 and a single sink node, we find that there is 63% chance that the global delay is below its mean value. For the 95th percentile, we find a global delay of 485 hops. This is 97 times less efficient than when the shortest path routing is used. However, for this configuration, we can identify a scenario where the use of random walks achieves similar results to the shortest path routing. We reproduce the same scenario proposed in [7,11] where random walks are used to query each sensor node in order to update some information. Assume that we want to find the minimum temperature in the network. Using random walks, just one packet would be reasonably sufficient to scan the whole network. This is attributed to the high cover ratio achieved by random walks. For example, for the 95th percentile value obtained above, the cover ratio is about 95%. To produce this cover ratio, the shortest path routing would require nearly 500 hops, which is comparable to the performance of the random walk (485 hops).

What is the impact of introducing more sink nodes on the global delay at fixed network size, or equivalently what is the impact of increasing the concentration of sink nodes? For answering this question, let us plot the cumulative distribution function as a function of the number of hops for different concentrations of sink nodes, $c = 1\%$, $c = 2\%$, and $c = 4\%$ respectively. We set here $N = 10$. This is illustrated in Figure 4. As it can be observed in this figure, larger concentrations on sink nodes yield a significant lower global delay. More specifically, we observe a large reduction in both the mean global delay and the dispersion. Intuitively, this can be explained by the fact that the higher number of sink nodes at fixed N, the more likely for a packet to be trapped soon. For example, increasing the concentration by two orders of magnitude (from $c = 1\%$ to $c = 2\%$ or from $c = 2\%$ to $c = 4\%$) leads to a nearly 50% decrease in the mean global delay. A heuristic reason for this behavior is that by doubling the number of sink nodes, sink-free regions become twice smaller, thereby reducing the number of hops to be trapped by half.

5. NUMERICAL RESULTS

We perform in this section a series of simulations to validate the obtained analytical results and to assess the accu-

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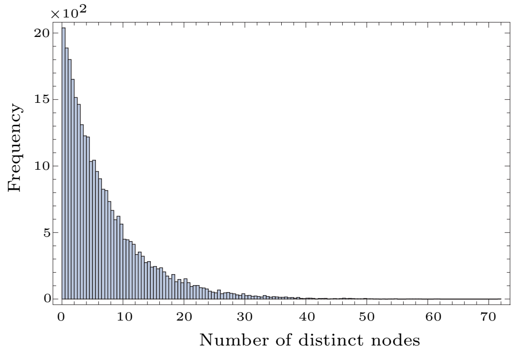

Figure 5: Histogram $H_1(n,s)$. Frequency of the number of distinct nodes visited in the first 100 hops for a torus lattice of size 10×10.

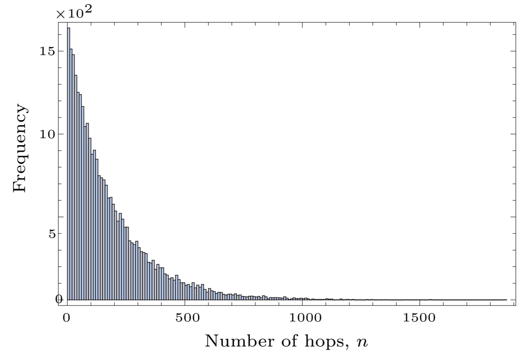

Figure 6: Histogram $H_2(n)$. Frequency of walks that last exactly n hops before trapping, for a torus lattice of size 10×10 with a concentration of sink nodes $c = 1\%$.

racy of the two approximate formulas derived for the cover ratio achieved by the packet in the first n hops given by (17), and the cumulative distribution function of the global delay given by (19). The simulations are performed using functional programming supported by MATHEMATICA [18].

5.1 Methodology

We use two procedures for simulations. The first one aims at evaluating the mean number of distinct nodes visited. It consists in calculating the probability mass function of $\mathcal{S}^{\dagger}_{N,n}$, the distinct nodes visited by the packet in the first n hops. Based on (3), we can also calculate the survival probability or equivalently the cumulative distribution function of the global delay. The second procedure consists in directly calculating the global delay for different realizations.

Procedure 1. In this procedure, all the sites of the torus lattice are equivalent, *i.e.*, no trap sites (sink nodes). So, at the beginning of the experiment, we fix the torus lattice size, for example $N = 10$. We should also fix n_{max}, the maximum length (in number of hops) of the generated random walk in order to terminate the process. The random walker starts at the origin. One has to simply record the number of distinct sites visited after n hops for each n in the range $[1, n_{max}]$. Upon repeating the same procedure many times, N_s, we construct a histogram $H_1(n,s)$ of the number of distinct sites visited after n hops for $1 \leq s \leq n+1$ (see Figure 5). Then, when $H_1(n,s)$ is divided by the number of realizations, we obtain an approximation of the probability $\mathbf{Pr}\{\mathcal{S}^{\dagger}_{N,n} = s\}$. The mean number of distinct nodes visited in the first n hops of the walk can be approximately expressed as

$$\begin{aligned}\frac{\mathbf{E}(\mathcal{S}^{\dagger}_{N,n})}{N^2} &= \frac{1}{N^2}\sum_{s=1}^{n+1} s\,\mathbf{Pr}\{\mathcal{S}^{\dagger}_{N,n} = s\} \\ &= \frac{1}{MN^2}\sum_{s=1}^{n+1} s\, H_1(n,s).\end{aligned} \tag{22}$$

We can also calculate the cumulative distribution function of the global delay based on (3), that is,

$$\begin{aligned}\mathbf{F}^{\ddagger}_N(n,c) &= 1 - \beta^{\ddagger}_N(n,c) \\ &= \frac{1}{M}\sum_{s=1}^{n+1} H_1(n,s)(1-c)^{s-1}.\end{aligned} \tag{23}$$

Procedure 2. This procedure serves only to directly calculate the cumulative distribution function of the global. To this end, we randomly choose a percentage c of the network nodes and designate them as sink nodes. A random walker (a packet) is placed on a random sensor node and performs a random walk until it meets a sink node. In this procedure, we do not need to fix the maximum length of the generated random walk. However, this random walk terminates whenever the walker steps on a sink node. In this procedure, the time (number of hops) to trapping is recorded. We repeat the same procedure for independent random walkers and different networks, and we construct a histogram $H_2(n)$ of the number of random walks that last exactly n hops before the first encounter of a sink node (see Figure 6). Then, the cumulative distribution function of the global delay is simply given by

$$\mathbf{F}^{\ddagger}_N(n) = \frac{1}{M}\sum_{s=1}^{n} H_2(s). \tag{24}$$

5.2 Results and Discussion

We present here the results of simulations for a torus lattice of size 10×10. For the first procedure, we fix $n_{max} = 1000$ hops. For the second procedure, we run a series of simulations for different concentrations of sink nodes. To establish accurate simulation results, we perform multiple independent replications of each experiment: we set $N_s = 30\,000$.

To obtain insight into how often distinct nodes are visited during the walk, we present in Figure 5 one example of histogram $H_1(n,s)$. We set here $n = 100$ and plot the frequency with which a given number of distinct nodes visited in the first 100 hops occurs during simulations. The key feature of this plot is the fact that small values of the number of distinct nodes are more likely than moderate or large values. A closer inspection of this plot indicates also that the maximum number of distinct nodes visited in the first 100 hops of the walk does not exceed the value of 39. For a torus lattice of size 10×10, this corresponds to a cover ratio of 39%. This can be explained by the fact that at early time, the random walker initially escapes its starting position and diffuses around the torus lattice covering quickly new territories. However, after this initial phase, revisitation of the same locations becomes important, and hence discovering

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new virgin locations slows down.

In Figure 6, we present one example of histogram $H_2(n)$ for a torus lattice of size 10×10 and a concentration of sink nodes $c = 1\%$. This histogram corresponds to the number of random walks that last exactly n hops before the first encounter of a sink node. The key observation from this plot is the fact that the histogram has the same form of the analytical curve of the probability mass function of the global delay depicted in Figure 3.

Let us now focus on the fundamental question raised at the beginning of this section, as to what extent the two approximate analytical formulas derived for the cover ratio and the cumulative distribution function of the global delay given respectively by (17) and (19) sound accurate. The plot described in Figure 7, where both the empirical cover ratio and the analytical one are plotted as a function of the rescaled variable $\widetilde{n} = n/N$, shows a good adequation between the analytical study and the simulation results. Indeed, note from Figure 7 that the empirical curve reasonably coincides with the analytical one at small and large values of $\widetilde{n}$. For moderate values of $\widetilde{n}$, the empirical curve deviates a little from the theoretical curves. However, one can observe that for a fixed value of the rescaled variable $\widetilde{n}$, the analytical study slightly overestimates the cover ratio as compared with simulation results. We find that the maximum relative error resulting from the use of the approximate formula of the cover ratio given by (17) is about 7.4%.

As regards the global delay, we compare in Figures 8 its analytical cumulative distribution function based on the approximate formula stated by Result 2 and the empirical ones obtained by a series of simulations based on the above procedures. Inspection of this figure provides many key features. First, observe that the empirical curves obtained by the two procedures closely coincide with each other, which confirms the reliability of our simulations. Second, observe that the empirical curves coincides with the theoretical ones at small and moderate values of n. For large values of n, the empirical curves deviates from the theoretical curves. This deviation becomes pronounced when the concentration of sink nodes passes from $c = 1\%$ to $c = 5\%$. This behavior is due to the fact that when we established the proof of Result 2, we have neglected the second order term involved in the asymptotic expansion of the survival delay. This is valid as long as c remains small and at small and moderate value of n. Finally, we can conclude that the numerical evaluation of the cumulative distribution function of the global delay gives good agreement with analytical results as long as the concentration of sink nodes remains relatively small.

6. SUMMARY

In this paper, we investigated a random walk based routing scheme in a wireless sensor network where sink nodes are independently distributed at random with a concentration c. When a packet is generated from a randomly chosen sensor node, it performs an unbiased random motion until reaching a sink node where it is trapped. The primary objective was to analytically evaluate the cumulative distribution function of the global delay, which is the time experienced by a packet to be trapped at a sink node. At low concentration of sink nodes, which often characterizes many WSN application, we derived an approximate expression for the cumulative distribution function of the global delay. We have also performed a series of simulations based on two different procedures to validate the analytical results concerning the cover ration and the global delay. These numerical studies clearly agree with the analytical results provided that the concentration of sink nodes remains small and for short and moderate times

There are many lines along which this work could proceed

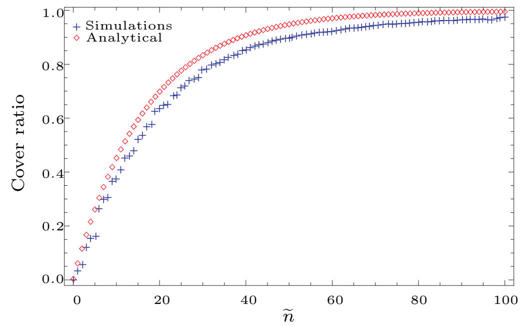

Figure 7: Comparison between the simulation results and the analytical study. Cover ratio as a function of the number of hops normalized to N for a torus lattice of size 10×10.

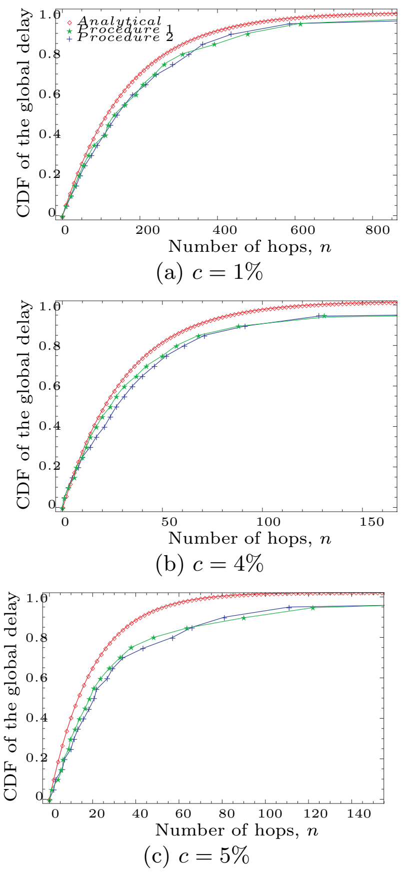

Figure 8: Comparison between the analytical and empirical CDF of the global delay for a torus lattice of size $N = 10$ and different concentrations of sink nodes.

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further. One consists in extending the basic model considered in this paper to more realistic models, such as random geometric graphs. Although this is certainly a necessary step, we have chosen to investigate square lattices with periodic boundary conditions for the simple reason that the main ideas we wanted to explore could be derived through closed-form derivations allowing us to have a deep insight. Those random graphs, although certainly much more interesting, are not amenable to a completely analytical solution and we must resort to simulation methods. So now that we have a good understanding about square lattice networks, it does make sense to consider more general and interesting cases.

APPENDIX

A. ASYMPTOTIC EXPANSION OF $P(\mathbf{0},\mathbf{0}\,|\,Z)$ AS $Z \to 1^-$

Before calculating the asymptotic expansion of $P(\mathbf{0},\mathbf{0}\,|\,z)$ as $z \to 1^-$, we propose first to simplify the expression of $P(\mathbf{0},\mathbf{0}\,|\,z)$ given by (15) and to study its singularity at $z = 1$. By factorizing the denominator of the summand and using the addition theorems of trigonometric functions, we obtain

$$\begin{aligned} P(\mathbf{0},\mathbf{0}\,|\,z) &= \frac{1}{N^2}\sum_{m_1=0}^{N-1}\sum_{m_2=0}^{N-1}\left\{\frac{1}{1-\frac{z}{2}\cos(\frac{2\pi}{N}m_1)}\right. \\ &\qquad \left.\times\frac{1}{1-c_{m_1}(z)\cos(\frac{2\pi}{N}m_2)}\right\} \\ &= \frac{1}{N^2}\sum_{m_1=0}^{N-1}\left\{\frac{1}{1-\frac{z}{2}\cos(\frac{2\pi}{N}m_1)}\times\mathcal{S}_{m_1}(z)\right\} \end{aligned} \tag{25}$$

where functions $c_{m_1}(z)$ and $\mathcal{S}_{m_1}(z)$ are defined as

$$c_{m_1}(z) = \frac{z}{2-z\cos(\frac{2\pi}{N}m_1)} \tag{26a}$$

$$\mathcal{S}_{m_1}(z) = \sum_{m_2=0}^{N-1}\frac{1}{1-c_{m_1}(z)\cos(\frac{2\pi}{N}m_2)}. \tag{26b}$$

Before studying the singularity of $P(\mathbf{0},\mathbf{0}\,|\,z)$ at $z = 1$, we propose to simplify $\mathcal{S}_{m_1}(z)$. The first step is to see from (26a) that $0 < |c_{m_1}(z)| < 1$ for $0 < z < 1$. Using the exponential representation of trigonometric functions, $\mathcal{S}_{m_1}(z)$ can be written as

$$\mathcal{S}_{m_1}(z) = -\frac{2}{c_{m_1}(z)}\sum_{m_2=0}^{N-1}\left\{\frac{e^{i\frac{2\pi}{N}m_2}}{e^{i\frac{2\pi}{N}m_2}-\alpha_{m_1}(z)}\times\frac{1}{e^{i\frac{2\pi}{N}m_2}-\alpha_{m_1}^{-1}(z)}\right\}$$

where $\alpha_{m_1}(z)$ is the smaller root of the equation

$$X^2 - \frac{2}{c_{m_1}(z)}X + 1 = 0$$

whose discriminant is positive for $0 < z < 1$. Thus, we find

$$\alpha_{m_1}(z) = \frac{1-\sqrt{1-c_{m_1}^2(z)}}{c_{m_1}(z)}. \tag{27}$$

Now, using partial fraction decomposition, we can write $\mathcal{S}_{m_1}(z)$ as

$$\mathcal{S}_{m_1}(z) = \left(1-c_{m_1}^2(z)\right)^{-\frac{1}{2}}\sum_{m_2=0}^{N-1}\left\{\frac{1}{1-\alpha_{m_1}(z)\,e^{-i\frac{2\pi}{N}m_2}} + \frac{\alpha_{m_1}(z)\,e^{i\frac{2\pi}{N}m_2}}{1-\alpha_{m_1}(z)\,e^{i\frac{2\pi}{N}m_2}}\right\}.$$

Noting that $|\alpha_{m_1}(z)| < 1$, it is then possible to expand each sum involved in $\mathcal{S}_{m_1}(z)$ by using successively the expansion $1/(1-x) = \sum_{k=0}^{\infty} x^k$ and the identity

$$\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}mn} = \begin{cases} N & \text{for}\quad n = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise.}\end{cases}$$

which can be derived by remarking that the vectors $e^{i\frac{2\pi}{N}mn}$ form an orthogonal basis over the set of N-dimensional complex vectors. Therefore, we obtain

$$\mathcal{S}_{m_1}(z) = \frac{N}{\left(1-c_{m_1}^2(z)\right)^{\frac{1}{2}}}\times\frac{1+\alpha_{m_1}^N(z)}{1-\alpha_{m_1}^N(z)} \tag{28}$$

By substituting (28) into (25), we find

$$P(\mathbf{0},\mathbf{0}\,|\,z) = \frac{1}{N}\sum_{m_1=0}^{N-1}\left\{\frac{1}{\left(1-\frac{z}{2}\cos(\frac{2\pi}{N}m_1)\right)\left(1-c_{m_1}^2(z)\right)^{\frac{1}{2}}}\times\frac{1+\alpha_{m_1}^N(z)}{1-\alpha_{m_1}^N(z)}\right\}. \tag{29}$$

Note that the summand involved in (29) is holomorphic over $0 < z < 1$ for all $0 \le m_1 \le N-1$. However, it diverges at $z = 1$ if and only if $m_1 = 0$. Thus, the singularity of $P(\mathbf{0},\mathbf{0}\,|\,z)$ at $z = 1$ comes only from the first term of the sum given by (29). It is convenient therefore to separate out the singular and non-singular parts of $P(\vec{\mathbf{r}},\mathbf{0}\,|\,z)$ as follows

$$P(\mathbf{0},\mathbf{0}\,|\,z) = \frac{1}{N(1-z)^{\frac{1}{2}}}\times\frac{1+\alpha_0^N(z)}{1-\alpha_0^N(z)} + \varphi_N(z) \tag{30}$$

where

$$\varphi_N(z) = \frac{1}{N}\sum_{m_1=1}^{N-1}\left\{\frac{1}{\left(1-\frac{z}{2}\cos(\frac{2\pi}{N}m_1)\right)\left(1-c_{m_1}^2(z)\right)^{\frac{1}{2}}}\times\frac{1+\alpha_{m_1}^N(z)}{1-\alpha_{m_1}^N(z)}\right\} \tag{31}$$

is holomorphic at $z = 1$. The first term involved in (30) corresponds to the term $m_1 = 0$ in (29), and the second term, that is $\varphi_N(\vec{\mathbf{s}}, z)$, corresponds to the sum over the range $1 \le m_1 \le N-1$.

To obtain the zero-order asymptotic expansion of $P(\mathbf{0},\mathbf{0}\,|\,z)$ as $z \to 1^-$, let us successively expand the first term of $P(\mathbf{0},\mathbf{0}\,|\,z)$ involved in (30) and then function $\varphi_N(z)$ close to $z = 1$. Hence, we find

$$P(\mathbf{0},\mathbf{0}\,|\,z) = -\frac{1}{N^2(z-1)} + \frac{N^2\left(1+3\varphi_N(1)\right)-1}{3N^2} + o(1). \tag{32}$$

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B. PROOF OF RESULT 1

The proof is obtained by applying the Method of Darboux to function $f(z) = \mathcal{S}_N^\dagger(z)$. We start by calculating the asymptotic expansion of $P(\mathbf{0},\mathbf{0}\,|\,z)$ as $z \to 1^-$, which is given by (32) in Appendix A. Substituting $P^\dagger(\mathbf{0},\mathbf{0}\,|\,z)$ from (32) into (14), as $z \to 1^-$ we obtain

$$\mathcal{S}_N^\dagger(z) = \frac{1}{1-z} \times \frac{3N^2}{3 + \Big(N^2\big(1+3\varphi_N(1)\big) - 1\Big)(1-z)} + o(1). \tag{33}$$

Then, we can deduce that $\mathcal{S}_N^\dagger(z)$ is singular at $z = 1$. Let us now show that $\mathcal{S}_N^\dagger(z)$ is holomorphic for $|z| < 1$. First, recall by definition $P^\dagger(\mathbf{0},\mathbf{0}\,|\,z)$ is convergent for $|z| < 1$. Second, from (15), we have

$$P^\dagger(\mathbf{0},\mathbf{0}\,|\,z) = \frac{1}{N^2} \sum_{\vec{\mathbf{m}} \in \mathfrak{T}} \frac{1}{1 - z\,\lambda^\dagger(\vec{\mathbf{m}})}, \tag{34}$$

where $\lambda^\dagger(\vec{\mathbf{m}})$ is the structure function of the induced perfect random walk. From (15), it is easy to show that $|\lambda^\dagger(\vec{\mathbf{m}})| \le 1$. Therefore, from (34), $P^\dagger(\mathbf{0},\mathbf{0}\,|\,z)$ can have no zeros inside $|z| \le 1$. Thus, we conclude that $\mathcal{S}_N^\dagger(z)$ is holomorphic in $0 < |z| < 1$ and $z = 1$ is the only singularity on $|z| = 1$. So that $\rho = 1$. It remains now to find a comparison function $g(z)$ satisfying the properties stated in the Method of Darboux. Let us set

$$g(z) = \frac{1}{1-z} \times \frac{3N^2}{3 + \Big(N^2\big(1+3\varphi_N(1)\big) - 1\Big)(1-z)}. \tag{35}$$

Clearly, $g(z)$ is holomorphic in $0 < |z| < \rho$. In addition, from (33), we have

$$f(z) - g(z) = o(1) \qquad as \qquad z \to 1^-, \tag{36}$$

so $f(z) - g(z)$ is continuous in $0 < |z| \le \rho$. Finally, the last step is to expand $g(z)$ in terms of power series close to $z = 0$, and then by applying relation (16), Result 1 follows.

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