Bargaining in Networks with Socially-Aware Agents

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Abstract. We introduce and characterize new stability notions in bargaining games over networks. Similar results were already known for networks induced by simple graphs, and for bargaining games whose underlying combinatorial optimization problems are packing-type. Our results are threefold. First, we study bargaining games whose underlying combinatorial optimization problems are *covering-type*. Second, we extend the study of stability notions when the networks are induced by *hypergraphs*, and we further extend the results to fully *weighted* instances where the objects that are negotiated have non-uniform value among the agents. Third, we introduce and characterize *new stability notions* that are naturally derived by polyhedral combinatorics and duality theory for Linear Programming. Interestingly, these new stability notions admit intuitive interpretations touching on *socially-aware* agents. Overall, our contributions are meant to identify natural and desirable bargaining outcomes as well as to characterize powerful positions in bargaining networks.

Keywords: Bargaining · Stable outcomes · Hypergraphs · Linear Programming

1 Introduction

Consider a set of agents, each of them demanding to receive a certain amount of service which can be offered by a number of available service providers. Choosing a specific service provider incurs some publicly known cost and serves a specific subset of the agents, possibly incurring different satisfaction to each of them. How would agents negotiate the cost distribution among them so as to agree on a global solution satisfying the demands of every player? Are there specific outcomes in which the cost of the services is fully covered by the agents, as well as agents' payment contributions are considered "fair"? We model this question as a General Covering Bargaining Game, and we characterize the existence of natural"fair" (or stable) outcomes. Our results are extensions to known stability notions for "Packing-Type Bargaining Games" in which the underlying graphs are simple, and agents treat all services (contracts) uniformly. Our findings further allow us to introduce and characterize new natural and relaxed notions of

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stability (fair outcomes), whose interpretation is associated with socially-aware agents.

1.1 Related Work

Bargaining in networks has been studied extensively and for a long time, both in economics (as *cooperative games*) and sociology (as *network exchange theory*). The focus in economics has been the study of resource distributions (e.g. for two-sided markets [27,28]), while in sociology the objective has been to understand the behaviour of agents who interact aiming to form relations of mutual benefit. For example, consider a primitive model, in which two players negotiate as to how to share the profit, say of 1, of a commonly required service. Given that each of them has *outside option* α and β, the so-called Nash bargaining solution [25] introduces the notion of a "fair" outcome in which each player receives her outside option and the surplus $1-\alpha-\beta$ is divided evenly between the players.

As in the example above, the study of bargaining games entails the refinement of solution concepts with respect to notions of "fairness". Two such key notions are that of *stability* and *balance*. At a high level, an outcome is stable if the utility of every agent is at least as good as her outside option, i.e. the best utility an agent could have by deviating from a current agreement, say with another player, and by forming a new agreement. Balanced solutions were first introduced in [11,27], and are a generalization of the Nash bargaining solution to networks. Interestingly, balanced solutions have been shown to agree with experimental results [30], however the focus of the current work is only on stable outcomes.

The framework of network bargaining games that we use in this work was first introduced by Kleinberg and Tardos in [22]. Their focus was a basic packing-type problem (matching), in which each agent could form up to one contract with a neighbour over a network (induced by a simple graph). Among others, Kleinberg and Tardos showed that such games have balanced solutions whenever they have stable solutions, and that (as we do in this work) the existence of stable solution is characterized by the integrality of a basic linear program relaxation for the associated combinatorial optimization problem (i.e. if a linear program has no discrepancy when compared to the exact but primitive integer program for the problem). Later, Bateni et al. [2] extended the work of [22] to bipartite (still simple) graphs in which some agents can engage in more than one contracts. Moreover, they showed that stable outcomes correspond to allocations in the *core* of the underlying coalition game, exhibiting this was a link between network bargaining (in matching and assignment games, previously studied in [8,12,13,18,28]) and cooperative game theory. More recently, Farczadi et al. [15] extended the results of the Kleinberg-Tardos model for networks with agents with general capacities, see also [19]. Relevant to our work is also [16] which considered again packing-type (matching) problems in which agents can bargain over a network from distance.

Variations of bargaining games have been studied extensively over the last decade. In [21] authors studied network bargaining games with general capacities.

Local dynamics in network bargaining games have been considered in [1,5,14]. [6] and [7] considered packing-type bargaining games with no capacity constraints, but with agents' utilities being nonlinear. [3] considered a local dynamical model of a one-sided exchange network (market) with transferable utilities and studied the dynamics of bargaining in such a market. In [9], the authors introduced alternative models for network bargaining based on instances with no stable outcomes, and in which players are both the negotiators and the negotiated objects. Finally, when bargaining instances do not admit stable solutions, the problem of minimally modifying the graph so as to inject stability was studied in [4,20].

1.2 Our Contributions and Paper Organization

In this paper we follow the techniques and further generalize part of the work of [2,15,22]. The common underlying bargaining games in these results pertained to a specific packing-type problem (matching), defined over simple graphs (players interactions were only binary), and where the subject of bargaining had a uniform value for all agents (contracts were worth the same to all agents). In contrast, we study covering-type games, where agents are competing to receive services, and we provide, as in the previous papers, a characterization of the existence of *stable* solutions. Our model is more general, in that we allow that services are of non-uniform value, i.e. the same service may provide different satisfaction to each agent. More importantly, our bargaining games are defined over networks induced by *hypergraphs*, i.e. services (or contracts in the previous packing-type problems) are not binary relations. As a consequence, we also introduce natural families of *relaxed notions of stability* based on cutting planes for the linear program formulations of the underlying combinatorial optimization problems. Interestingly, these new notions of stability admit an intuitive interpretation pertaining to *socially-aware* agents. Our findings also find applications to bargaining games whose combinatorial optimization problems do not admit IP formulations where constraints are associated only with agents. In particular, our relaxed stability notions apply also when not all vertices of a network are agents, rather they are present only to facilitate agreements. Notable, none of the previously known results were able to address stability in such networks.

In Sect. 2 we introduce the general covering-type problems we study in this work. In Sect. 2.1 we provide the game-theoretic perspective of these problems and we define the standard notions of feasibility and stability in bargaining outcomes. Then in Sect. 2.2 we give the combinatorial perspective of the covering-type problems, as well as we review the tools from Linear Programming that are used extensively in our work. Section 3 is devoted to the characterization of the existence of stable solutions. In particular, one of our main contributions, Theorem 1 is proven in Sects. 3.1 and 3.2. Then, in Sect. 4 we study relaxed notions of stability. In Sect. 4.1 we introduce the key concept of critical constraints upon which we will rely to relax stability. Section 4.2 introduces new notions of stability, and finally in Sect. 4.3 we provide a characterization of their existence, as well as we give a natural interpretation for them.

2 Covering-Type Problems in Hypergraphs; Bargaining Games vs Combinatorial Optimization

The purpose of this section is to formally introduce bargaining games in hypergraphs and their counterparts, combinatorial optimization problems. Common to both is the underlying input, which formally speaking consists of a hypergraph $\mathcal{H} = (V, E)$, where $e \subseteq V$ for each $e \in E$. Set V will be called the set of agents (or players), and E will be referred to as the set of services. $\mathbb{Q}_{++}$ below denotes the set of positive rational numbers. We consider functions $d : V \mapsto \mathbb{Q}_{++}, c : E \mapsto \mathbb{Q}_{++}, \rho : V \times E \mapsto \mathbb{Q}_{++}$. For each $i \in V, e \in E$ we will commonly write $d_i, c_e, \rho_{i,e}$, instead of $d(i), c(e), \rho(i, e)$, respectively. Moreover, d_i will be called the demand of agent i, c_e will be called the cost of service e, and $\rho_{i,e}$ will be called the satisfaction of agent i for service e. Whenever $i \in e \in E$ we will say that service e covers i. For each $i \in V$ we denote by T_i the set of services covering i, i.e. $T_i = \{e \in E : i \in e\}$. Altogether, we will refer to the tuple $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$ as a *covering-type problem*. In order to avoid degenerate cases, the silent assumption in all covering-type problems is that for each $i \in V$, $\sum_{e \in T_i} \rho_{i,e} \geq d_i$. In other words, all services are enough to meet all players' demands.

Next we view covering-type problems $\mathcal{B}$ under two different lenses, one game-theoretic and one combinatorial. At a high level, the game-theoretic problem will attempt to understand $\mathcal{B}$ from the perspective of selfish and rational players, set V, who are willing to cover (part of) the expenses for choosing enough many services (set E) so as to cover their needs/demands (values $d_i, i \in V$). The combinatorial lens will view $\mathcal{B}$ from the perspective of a central authority who is attempting to choose the least expensive set of contracts so as to satisfy all demands. Later on, we bridge the two perspectives by identifying polyhedral combinatorial properties of the combinatorial problem that characterize when "stable" solutions to the game-theoretic bargaining problem exist.

2.1 Cost Sharing in Bargaining Games over Networks

The purpose of this section is to introduce covering-type bargaining games in networks induced by hypergraphs, along with their solution concepts. Formally, a *general covering bargaining game* (or simply, bargaining game) is given by covering-type problem $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$.

Bargaining game $\mathcal{B}$ corresponds to a set of players V, each of them i looking to be serviced by services that provide at least d_i satisfaction (in the uniform case, where all satisfactions are 1, d_i is the number of services requested). Naturally, the set of agents, together with services induce a hypergraph $\mathcal{H}$, in which hyperedges $e \subseteq V$ are identified by the subset of players they serve. Given that service e incurs cost $c_e > 0$, we would like to understand the bargaining dynamics in the induced network, when it comes to choosing a collection of services and paying for them. The underlying assumption, as in any game-theoretic problem, is that agents are selfish and rational. In our case, agents cannot compromise on the number of the services each receives (indicated by their demands). As all agents will need to naturally cover the entire cost of the chosen services, each agent would like to minimize her contribution toward purchasing (choosing) any of the services. In this direction, we introduce the notion of a bargaining outcome, which will be central in identifying the bargaining dynamics.

Definition 1 (Bargaining Feasible Outcome). *A bargaining feasible outcome (or simply an outcome) of bargaining game $\mathcal{B}$ is a tuple $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$, with $P_{i,e} \in \mathbb{R}_+$, satisfying the following properties:*

- *(Demand Satisfaction) For every $i \in V$, $\sum_{e \in T_i \cap A} \rho_{i,e} \geq d_i$.*
- *(Cost Recovery) For every $e \in A$, $\sum_{i \in V} P_{i,e} = c_e$.*

Bargaining feasible outcome $\mathcal{F}$ of bargaining game $\mathcal{B}$ specifies exactly a subset A of the services that are chosen to meet the demands of all players. Intuitively, every player i needs to contribute some non-negative payments $P_{i,e}$ for each chosen service, and these payments should cover its cost. Note that we allow positive contributions $P_{i,e}$ for players i even when they are not covered by a service e (this will become relevant when we will introduce relaxed notions of stability).

Given outcome $\mathcal{F}$ of a bargaining game, we partition the set of agents in two disjoint sets; the set V_* of *oversaturated* agents i for which $\sum_{e \in T_i \cap A} \rho_{i,e} > d_i$, and the set of *tight* agents $V \setminus V_*$. Note that for each tight agent i, and by the definition of feasible outcomes, we have $\sum_{e \in T_i \cap A} \rho_{i,e} = d_i$, i.e. agent i meets her demand exactly, while oversaturated agents receive strictly more service satisfaction than their demands.

As our goal is to identify "desirable" outcomes as well as the powerful positions in a bargaining network, the notion of feasible outcomes can be refined as follows.

Definition 2 (Stable Outcome). *A bargaining feasible outcome $\mathcal{F}$ of bargaining game $\mathcal{B}$ is called stable if the following properties are satisfied:*

- *(Greed) $P_{i,e} > 0$, implies that agent i is tight, service e is chosen and $i \in e$, i.e. $i \in V \setminus V_*$ and $i \in e \in A$.*
- *(Envy-Free) For every $f = \{i_1, \ldots, i_l\} \notin A$, and for every $e_j \in A \cap T_{i_j}, j = 1, \ldots, l$,*

$$\sum_{j=1}^{l} \frac{\rho_{i_j,f}}{\rho_{i_j,e_j}} P_{i_j,e_j} \leq c_f. \tag{1}$$

Stable outcomes are meant to propose a refinement of feasible outcomes that are desirable by the agents (meaning that a bargaining process may converge to such an outcome) by requiring fair payments. Indeed, an agent should never pay for a service that does not serve her, or a service that is not chosen. Every service $e \in E$ can be thought as a potential coalition among $e \subseteq V$ who choose to pay for service $e \in A$, and hence $f \notin A$ can be thought as coalitions that are not formed. A positive payment may be required only by tight agents, as otherwise an oversaturated agent might feel in a powerful negotiation power when a solution is proposed that oversatisfies her demands. Finally, for any unformed coalition $f \notin A$ (i.e. a service not in the solution), and every agent $i \in f$ consider the maximum payment P_{i,e_i} that i makes over all services in A. For the sake of the argument, assume that satisfaction rates are uniform. If it happened that $\sum_{i \in f} P_{i,e_i} > c_f$, then each of the agents i would like to leave coalition e_i so that all of them form coalition f, effectively sharing the cost c_f and reducing their previous maximum payments. The reader may think of the latter requirement as the standard stability notion of coalition games with transferable payoffs. Now, if the satisfaction rates are not uniform, as it is the case in our model, then the same argument holds for normalized payments $\rho_{i_j,f} \cdot P_{i_j,e_j}/\rho_{i_j,e_j}$. The latest expression is interesting in its own right, as it provides a form of satisfaction conversion between services f, e_j using the relative payment per unit of satisfaction of player i_j for service e_j. Finally, it is important to notice that expression (1) can be used to introduce (and hence generalize existing) notions of agents' *outside options* for our covering-type games.

2.2 The Underlying Covering-Type Combinatorial Optimization Problem

Now we turn our attention to the underlying combinatorial perspective of a covering-type problem $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$. We interpret $\mathcal{B}$ as a combinatorial optimization problem, in which a central authority is trying to find a feasible collection of services $A \subseteq E$ so as to satisfy all demands d_i of each agent i. Moreover, among the set of feasible collection of services, one would like to identify the least costly, i.e. to minimize the sum of costs c_e for services $e \in A$.

Combinatorial problem $\mathcal{B}$ admits a natural formulation as an Integer Program (IP). We introduce an indicator, 0-1, variable x_e for every $e \in E$ which is thought as 1 if and only if e is chosen in a feasible solution. Requiring that each agent i receives at least as many services as her demand d_i (with respect to her satisfactions) and minimizing the overall induced cost, we obtain the following exact formulation of $\mathcal{B}$.

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e \, x_e && (F_{IP}(\mathcal{B})) \\
\text{s.t.} \quad & \sum_{e \in T_i} \rho_{i,e} x_e \geq d_i, && \forall i \in V \\
& -\mathbf{x} \geq -\mathbf{1}, \\
& \mathbf{x} \geq \mathbf{0}, \\
& \mathbf{x} \in \mathbb{Z}^{|E|}
\end{aligned}
$$

In what follows, we refer to this formulation as F_{IP}. Given as input $\mathcal{B}$, we denote by $optF_{IP}(\mathcal{B})$ its optimal value (note that the IP is feasible and bounded with rational coefficients, and hence it always admits and optimal solution). Next we overview some standard tools from combinatorial optimization that will be useful later on. First we introduce the so-called linear program (LP) relaxation of the IP above, which is obtained by dropping the integrality condition $\mathbf{x} \in \mathbb{Z}^{|E|}$. We will denote the resulting LP by F_{LP}. Given input covering-type problem $\mathcal{B}$, we denote by $optF_{LP}(\mathcal{B})$ its optimal solution (the LP is feasible and bounded, hence it always attains an optimal solution). By definition, and for every covering-type problem $\mathcal{B}$, we have that $optF_{LP}(\mathcal{B}) \leq optF_{IP}(\mathcal{B})$. In particular, the ratio $\frac{optF_{LP}(\mathcal{B})}{optF_{IP}(\mathcal{B})}$ is known as the *integrality gap* of F_{LP} on input $\mathcal{B}$ and measures the discrepancy between F_{LP} and F_{IP} for the same instance. The notion of the so-called *integrality gap* of the LP relaxation measures the worst case discrepancy between an IP and its LP relaxation, over all instances;

$$\inf_{\mathcal{B}} \frac{optF_{LP}(\mathcal{B})}{optF_{IP}(\mathcal{B})}.$$

It follows that the integrality gap of F_{LP} on input $\mathcal{B}$ is at most 1, and it is equal to 1 if and only if there exists an integral optimal solution to F_{LP} with input $\mathcal{B}$. Similarly, the integrality gap of F_{LP} is 1 if and only if F_{LP} admits an integral optimal solution for every input $\mathcal{B}$.

Example 1 (Vertex Cover). In VERTEX-COVER one is given a simple graph $G = (V_0, E_0)$. Feasible solutions are subsets of the vertices $S \subseteq V_0$ with the property that for every edge $e = \{i, j\}$, at least one of the endpoints i, j lies in S (the chosen subset of the vertices is called a vertex cover). Let $\mathcal{H}$ denote the line graph of G. Then, *Vertex-Cover* is the covering-type problem $\mathcal{B} = (\mathcal{H}, d, c, \rho)$, where $d_e = 1$ for all $e \in E_0$, $c_i = 1$ for all $i \in V_0$, and $\rho_{e,i} = 1$ for all $i \in V_0$ and $e \in E_0$. Note that the set of agents are the edges E_0 and the set of vertices V_0 forms the set of services. Each agent $e = \{i, j\}$ needs at least one service chosen among i, j. The LP relaxation of the exact IP formulation of the problem reads as

$$\begin{aligned} \min \quad & \sum_{i \in V_0} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1, \qquad \forall \{i, j\} \in E_0 \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}. \end{aligned} \tag{2}$$

It is well known that the integrality gap of the above LP is 1 if G is bipartite, while in general the integrality gap can be as small as 1/2.

3 Characterization of Stable Outcomes in Hypergraphs

In this section we characterize the existence of stable solutions of general bargaining games $\mathcal{B}$. Note that our result is an extension of known results when the underlying network of $\mathcal{B}$ is a simple graph, and hence every service can potentially serve exactly two agents. The extension to hypergraphs will be essential when we will introduce natural relaxed notions of stability in Sect. 4.

Theorem 1. *Bargaining game $\mathcal{B}$ admits a stable outcome if and only if the integrality gap of $F_{LP}(\mathcal{B})$ is 1.*

We comment throughout the exposition of our proof how one needs to extend/modify existing arguments (for simple graphs) to obtain our results for hypergraphs and for non-uniform satisfactions and demands. However, the key ingredient to establishing Theorem 1 is actually the introduction of a proper definition of stability, which is also tailored to the notion of over-saturation. In fact, our generalized/modified notions of stability and over-saturation simplify to the already studied notions of stability and over-saturation when all demands and satisfactions are 1 and all contracts are of size exactly 2. More specifically, our main goal was to establish a characterization of the existence of stable solutions for our generalized bargaining instances based on the integrality of natural LP relaxations. As such, the new proposed notion of stability (the one of Theorem 1) was derived mechanically, as it is tailored to the integrality of a certain LP. Whether the associated notions of over-saturation that one needs to consider is practical or not is outside the scope of this paper, nevertheless it is not difficult to argue, at a high level, why it is natural.

It is important to note that Theorem 1 is tailored to the exact formulation F_{IP}. Indeed, Theorem 1 characterizes the existence of stable solutions to the bargaining when the set of feasible solutions to the combinatorial optimization problem can be described by one linear constraint associated with each agent, i.e. when F_{IP} is an exact formulation to the covering-type problem. In particular, if the set of feasible solutions to the combinatorial optimization problem requires additional constraints in order to be determined, or if "redundant constraints" are added to the IP formulation (not necessarily redundant for the LP relaxation), then Theorem 1 does not apply. In other words, the existence of "natural" solution concepts (that of stable solutions as per Definition 2) to bargaining games are derived *mechanically* by structural properties in polyhedral combinatorics, at least in special cases.

Example 2. Consider the bargaining game induced by VERTEX-COVER of Example 1, for some input graph $G = (V_0, E_0)$. If G is bipartite, then the bargaining game admits a stable outcome. On the other hand, consider the simple graph

$$G_0 = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}, \{1, 3\}\})\,.$$

Setting $x_1 = x_2 = x_3 = 1/2$ is feasible to LP (2), hence the optimal value to the LP is at most 3/2. At the same time, no less than 2 services among {1, 2, 3} need to be chosen in any solution to the covering-type problem (independently of its cost). The integrality gap of the LP is not 1, hence, according to Theorem 1 the bargaining game admits no stable solution.

The driving force behind proving Theorem 1 is duality theory and complementary slackness conditions. First we propose the dual linear program of $F_{LP}(\mathcal{B})$, where $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$, which reads as follows;

$$\begin{aligned}\max \quad & \sum_{i\in V} d_i\ y_i - \sum_{e\in E} u_e & & (F_{LP}^D(\mathcal{B}))\\ \text{s.t.} \quad & \sum_{i\in e} \rho_{i,e} y_i - u_e \le c_e, & \forall e \in E &\\ & \mathbf{y}, \mathbf{u} \ge \mathbf{0}, & &\end{aligned}$$

We denote by $optF_{LP}^D(\mathcal{B})$ the optimal solution to the above LP.

Now consider a primal dual pair of feasible solutions $\bar{\mathbf{x}}, (\bar{\mathbf{y}}, \bar{\mathbf{u}})$ to $F_{LP}(\mathcal{B})$ and $F_{LP}^D(\mathcal{B})$, respectively. Since $F_{LP}(\mathcal{B})$ admits an optimal solution for every $\mathcal{B}$ such a pair always exists (by strong duality), and they are each optimal to the primal and dual LPs if and only if the so-called *complementary slackness conditions* hold true;

$$\left(\sum_{e\in T_i} \bar{\rho}_{i,e}\bar{x}_e - d_i\right)\bar{y}_i = 0,\ \forall i \in V \tag{3}$$

$$(\bar{x}_e - 1)\,\bar{u}_e = 0,\ \forall e \in E \tag{4}$$

$$\left(\sum_{i\in e} \rho_{i,e}\bar{y}_i - \bar{u}_e - c_e\right)\bar{x}_e = 0,\ \forall e \in E \tag{5}$$

3.1 Integrality from Stability

In this section we prove the "only if" claim of Theorem 1, that is we prove the statement below, whose proof follows closely known arguments for simple graphs (one needs to only normalize payments with respect to satisfaction rates).

Lemma 1. *If bargaining game $\mathcal{B}$ admits a stable outcome then the integrality gap of $F_{LP}(\mathcal{B})$ is 1.*

So, fix some bargaining game $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$ and a stable (and feasible) outcome $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i\in V, e\in E})$. We prove Lemma 1 by finding an *optimal solution* to $F_{LP}(\mathcal{B})$ which is also integral.

We define the following primal-dual pair $\bar{\mathbf{x}}, (\bar{\mathbf{y}}, \bar{\mathbf{u}})$ of feasible solutions (as we will shortly prove) to $F_{LP}(\mathcal{B})$ and $F_{LP}^D(\mathcal{B})$, respectively.

$$\bar{x}_e := \begin{cases} 1, \text{if } e \in A\\ 0, \text{if } e \notin A\end{cases} \tag{6}$$

$$\bar{y}_i := \begin{cases} \max\left\{\frac{P_{i,e}}{\rho_{i,e}} :\ e \in T_i \cap A\right\}, & \text{if } i \text{ is tight}\\ 0, & \text{if } i \text{ is oversaturated}\end{cases} \tag{7}$$

and

$$\bar{u}_e := \begin{cases} \sum_{i\in e} \rho_{i,e}\ \bar{y}_i - c_e, & \text{if } e \in A\\ 0, & \text{if } e \notin A\end{cases} \tag{8}$$

Lemma 2. $\bar{\mathbf{x}}, (\bar{\mathbf{y}}, \bar{\mathbf{u}})$ *are feasible to $F_{LP}(\mathcal{B})$ and $F_{LP}^D(\mathcal{B})$, respectively.*

Proof. $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$ is a feasible outcome of the bargaining game $\mathcal{B}$. As such, by Definition 1 we have $d_i \leq \sum_{e \in T_i \cap A} \rho_{i,e} = \sum_{e \in T_i} \rho_{i,e} \bar{x}_e$. Given also that $\mathbf{0} \leq \bar{\mathbf{x}} \leq \mathbf{1}$, we see that $\bar{\mathbf{x}}$ is feasible to $F_{LP}(\mathcal{B})$.

Now we show that $(\bar{\mathbf{y}}, \bar{\mathbf{u}})$ is feasible to $F^D_{LP}(\mathcal{B})$. First we argue that $\bar{\mathbf{y}}, \bar{\mathbf{u}}$ are non-negative vectors. First, $\bar{\mathbf{y}}$ follows by the non-negativity of payments $P_{i,e}$ (see Definition 1) and the fact that service satisfactions are strictly positive. Second, let $e \in E$ arbitrary. If $e \notin A$ we have $\bar{u}_e \geq 0$ by construction. Otherwise, consider some service $e \in A$. For every $i \in e$, denote by $f_i := \arg\max \left\{ \frac{P_{i,f}}{\rho_{i,f}} : \ f \in T_i \cap A \right\}$. Then,

$$\bar{u}_e = \sum_{i \in e} \rho_{i,e}\ \bar{y}_i - c_e = \sum_{i \in e} \rho_{i,e} \frac{P_{i,f_i}}{\rho_{i,f_i}} - c_e \geq \sum_{i \in e} \rho_{i,e} \frac{P_{i,e}}{\rho_{i,e}} - c_e = 0,$$

where the last equality follows from Definition 1 (Cost Recovery).

It remains to prove that for all $e \in E$ we have $\sum_{i: e \in T_i} \rho_{i,e} \bar{y}_i - \bar{u}_e \leq c_e$. We examine two cases. If $e \in A$, then note that by construction (see (7) and (8)), the constraint is satisfied tightly. Otherwise, if $e \notin A$, we rely on that the feasible outcome is stable, and indeed we have

$$\sum_{i: e \in T_i} \rho_{i,e} \bar{y}_i - \bar{u}_e \overset{(8)}{=} \sum_{i: e \in T_i} \rho_{i,e} \bar{y}_i \overset{(7)}{\leq} \sum_{i: e \in T_i} \rho_{i,e} \max_{f \in T_i \cap A} \left\{ \frac{P_{i,f}}{\rho_{i,f}} \right\} \overset{(1)}{\leq} c_e.$$

□

By Lemma 2, it follows that if $\bar{\mathbf{x}}, (\bar{\mathbf{y}}, \bar{\mathbf{u}})$ satisfy complementary slackness conditions, then $\bar{x}$ is optimal to $F_{LP}(\mathcal{B})$. Since also $\bar{x}$ is integral, that would imply that the integrality gap of $F_{LP}(\mathcal{B})$ is 1. Therefore, Lemma 1 follows by the lemma below.

Lemma 3. $\bar{\mathbf{x}}, (\bar{\mathbf{y}}, \bar{\mathbf{u}})$ *satisfy complementary slackness conditions* (3)*,* (4)*,* (5)*.*

Proof. First we examine (3) for an arbitrary agent i. If i is oversaturated, then by (7) we have $\bar{y}_i = 0$. If i is tight, then by the definition of tight agents, we have $d_i = \sum_{e \in T_i \cap A} \rho_{i,e} = \sum_{e \in T_i} \bar{\rho}_{i,e} \bar{x}_e$, as wanted.

Second, we study (4) for arbitrary $e \in E$. If $e \in A$, then by (6) we have $\bar{x}_e = 1$, while if $e \notin A$, then by (8) we have $\bar{u}_e = 0$. In any case, condition (4) is satisfied.

Third, we examine (5) for an arbitrary $e \in A$. If $e \notin A$, then by (6) we have $\bar{x}_e = 0$ and the condition is satisfied. Otherwise, $e \in A$. But then, note that by (8), $\bar{u}_e$ was chosen so as to make constraint $\sum_{i: e \in T_i} \rho_{i,e} \bar{y}_i - \bar{u}_e \leq c_e$ tight, independently of the valuations of $\bar{y}_i$ for agents $i \in e$. □

3.2 Stability from Integrality

In this section we prove the "if" claim of Theorem 1, that is we prove that

Lemma 4. *If for some covering-type problem $\mathcal{B}$ the integrality gap of $F_{LP}(\mathcal{B})$ is 1, then the underlying bargaining game $\mathcal{B}$ admits a stable outcome.*

For a fixed covering-type problem $\mathcal{B}$, consider an optimal integral solution $\bar{\mathbf{x}}$ to $F_{LP}(\mathcal{B})$. Next we propose an outcome to the bargaining game $\mathcal{B}$, and subsequently we show it is feasible (as per Definition 1) and stable (as per Definition 2). For this, we set

$$A := \{e \in E : \bar{x}_e = 1\}. \tag{9}$$

Notice that by construction, $\bar{\mathbf{x}}$ is feasible to $F_{IP}(\mathcal{B})$, and hence for each $i \in V$ we have $\sum_{e \in T_i \cap A} \rho_{i,e} = \sum_{e \in T_i} \bar{\rho}_{i,e}\bar{x}_e \geq d_i$, hence each agent meets her demand as per the (partial) requirement of feasible outcomes. In other words, A is a feasible solution to the covering-type problem $\mathcal{B}$.

In order to propose payments for players, we need a couple of observations. These arguments (Lemma 2 and in particular Lemma 5 below) require a much more delicate treatment than in the case of simple graphs with uniform demands and satisfactions.

Observation 2. *For each $e \in A$, there is at least one agent $i \in A$ that is tight.*

Proof. Since $\bar{\mathbf{x}}$ is optimal to $F_{LP}(\mathcal{B})$, it is also optimal to $F_{IP}(\mathcal{B})$, hence A is an optimal solution to the covering-type problem $\mathcal{B}$. For the sake of contradiction, assume that there is some $e_0 \in A$ for which

$$\sum_{e \in T_i} \rho_{i,e} x_e > d_i,$$

for all $i \in e_0$. Recall that for all $e \in E$ we have $\rho_{i,e} \geq 0$ and $c_e > 0$. As a result, there exists a small enough $\epsilon > 0$ so that by updating $\bar{x}_{e_0} \leftarrow \bar{x}_{e_0} - \epsilon$, vector $\bar{x}$ remains feasible and has cost strictly less than $\sum_{e \in A} c_e = optF_{LP}(\mathcal{B})$, a contradiction to the optimality of $\bar{\mathbf{x}}$. □

Next we recall that since $\bar{\mathbf{x}}$ is optimal to $F_{LP}^D(\mathcal{B})$ and by strong duality, $F_{LP}^D(\mathcal{B})$ admits an optimal solution, call it $(\bar{\mathbf{y}}, \bar{\mathbf{u}})$, and in particular, the primal-dual pair of feasible solutions satisfy complementary slackness conditions (3), (4), (5).

Lemma 5. *Given $(\bar{\mathbf{y}}, \bar{\mathbf{u}})$, and for every $e \in A$, there exists a distribution $\{\lambda_{i,e}\}_{i \in e}$ satisfying*

$$\rho_{i,e}\ \bar{y}_i - \lambda_{i,e}\bar{u}_e \geq 0, \forall i \in e, \tag{10}$$

and whose support lies only within the tight agents of e.

Proof. Consider arbitrary $e \in A$ (and hence $\bar{x}_e = 1$). In what follows we construct non-negative $\{\lambda_{i,e}\}_{i \in e}$ with

$$\sum_{i \in e} \lambda_{i,e} = 1 \tag{11}$$

$$\lambda_{i,e} = 0, \forall i \in e \cap V_* \tag{12}$$

satisfying (10).

Since $\bar{x}_e = 1$, by complementary slackness condition (5), we know that

$$\sum_{i \in e} \rho_{i,e}\bar{y}_i - \bar{u}_e = c_e, \tag{13}$$

where $c_e > 0$. Recall that by Observation 2, service e contains at least one tight agent. Hence, if $\bar{u}_e = 0$ we put all weight to that agent and we are done.

In what follows, we assume $\bar{u}_e > 0$, and denote by e_* the set of agents within e that are oversaturated (i.e. not tight), and note that $e \setminus e_* \neq \emptyset$ (by Observation 2). For all $i \in e_*$ we set $\lambda_{i,e} = 0$, as required by (12). By complementary slackness condition (3) we conclude that $\bar{y}_i = 0$ whenever $i \in e_*$. Hence, for all oversaturated agents i, we have that (10) is satisfied tightly. Moreover, (13) can be rewritten as

$$\sum_{i \in e \setminus e_*} (\rho_{i,e}\bar{y}_i - \lambda_{i,e}\bar{u}_e) = c_e,$$

for arbitrary $\lambda_{i,e}$ satisfying (11). It remains to prove that $\{\lambda_{i,e}\}_{i \in e \setminus e_*}$ can be indeed chosen to be non-negative so as to also satisfy $\rho_{i,e}\ \bar{y}_i - \lambda_{i,e}\bar{u}_e \geq 0$, for all $i \in e \setminus e_*$.

To that end, notice that since $c_e > 0$ and by (13), we have $\sum_{i \in e \setminus e_*} \rho_{i,e}\bar{y}_i > \bar{u}_e \geq 0$. Then, set

$$\lambda_{i,e} = \frac{\rho_{i,e}\bar{y}_i}{\sum_{j \in e \setminus e_*} \rho_{j,e}\bar{y}_j}.$$

But then,

$$\rho_{i,e}\ \bar{y}_i - \lambda_{i,e}\bar{u}_e = \rho_{i,e}\ \bar{y}_i \left(1 - \frac{\bar{u}_e}{\sum_{j \in e \setminus e_*} \rho_{j,e}\bar{y}_j}\right) > \rho_{i,e}\ \bar{y}_i \geq 0,$$

where the last inequality follows by dual feasibility and that $\rho_{i,e} \geq 0$. □

Now, for any fixed collection of distributions $\{\{\lambda_{i,e}\}_{i \in e}\}_{e \in A}$ as per Lemma 5, we define payments

$$P_{i,e} := \begin{cases} \rho_{i,e}\ \bar{y}_i - \lambda_{i,e}u_e, & \text{if } e \in A \text{ and } i \in e \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

Altogether, (9) and (14) above determine outcome $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$. The following two lemmata verify that $\mathcal{F}$ is a feasible and stable, implying Lemma 4.

Lemma 6. *Outcome* $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$ *is feasible (as per Definition 1).*

Proof. Set $A \subseteq E$, as defined in (9) was already shown to satisfy Demand Satisfaction. Now, Lemma 5 implies, first, that all payments $P_{i,e}$ are indeed non-negative. In order to show that they also satisfy Cost Recovery, consider arbitrary

$e \in A$, and not that by construction, only tight agents within e may have positive payments. Therefore, we have

$$\sum_{i\in V} P_{i,e} = \sum_{i\in e\setminus V_*} P_{i,e} \overset{(14)}{=} \sum_{i\in e\setminus V_*} (\rho_{i,e}\ \bar{y}_i - \lambda_{i,e} u_e) \overset{(Lemma\ 5)}{=} \sum_{i\in e\setminus V_*} \rho_{i,e}\ \bar{y}_i - u_e. \tag{15}$$

By complementary slackness condition (3), oversaturated agents i in e have $\bar{y}_i = 0$, and hence (15) further equals,

$$\sum_{i\in e} \rho_{i,e}\ \bar{y}_i - u_e = c_e$$

where the last equality is due to dual feasibility and complementary slackness condition (5), since $\bar{x}_e = 1$. □

Lemma 7. *Outcome $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i\in V, e\in E})$ is stable (as per Definition 2).*

Proof. First we observe that by the definition of the payments (14), $P_{i,e} = 0$ if $i \notin e$ or if $e \notin A$. Now consider some $e \in A$ and some oversaturated agent $i \in e$. By Lemma 5 we have $\lambda_{i,e} = 0$, and by complementary slackness condition (3) we have $\bar{y}_i = 0$. Hence, by (14) we obtain $P_{i,e} = 0$, overall concluding property Greed.

Now we verify that $\mathcal{F}$ is also Envy-Free. For this, consider arbitrary $f = \{i_1, \dots, i_l\} \notin A$, and arbitrary $e_j \in A \cap T_{i_j}$ for $j = 1, \dots, l$. We have

$$\begin{aligned}
\sum_{j\in f} \frac{\rho_{j,f}}{\rho_{j,e_j}} P_{j,e_j} &= \sum_{j\in f\setminus V_*} \frac{\rho_{j,f}}{\rho_{j,e_j}} P_{j,e_j} && \text{(by Greed)}\\
&= \sum_{j\in f\setminus V_*} \frac{\rho_{j,f}}{\rho_{j,e_j}} \left(\rho_{j,e_j}\ \bar{y}_j - \lambda_{j,e_j} \bar{u}_{e_j}\right) && \text{(by (14))}\\
&\le \sum_{j\in f\setminus V_*} \rho_{j,f}\ \bar{y}_j && (\lambda_{j,e_j}, \bar{u}_{e_j} \ge 0)\\
&= \sum_{j\in f\setminus V_*} \rho_{j,f}\bar{y}_j - \bar{u}_f && (\bar{u}_f = 0, \text{since} f \notin A \text{ and by } (4))\\
&= \sum_{j\in f} \rho_{j,f}\bar{y}_j - \bar{u}_f && (\bar{y}_j = 0 \text{ for } j \in V_*,\ \text{by } (3))\\
&\le c_f. && ((\bar{\mathbf{y}}, \bar{\mathbf{u}}) \text{ feasible to } F_{LP}^D(\mathcal{B}))
\end{aligned}$$

□

4 Stability Notions Based on Socially-Aware Agents

4.1 Critical Constraints

Consider a covering-type problem $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$, along with the underlying bargaining game. The following definition is the starting point toward deriving a new notion of stability.

Definition 3 (Critical Constraint). *Let $R \subseteq E$, $t \in \mathbb{Q}_{++}$ and $\sigma : R \mapsto \mathbb{Q}_{++}$. Triplet (R, t, σ) is called a critical constraint for $\mathcal{B}$ if for every $A \subseteq E$ satisfying demand satisfaction (as per Definition 1), we have*

$$\sum_{e \in A \cap R} \sigma_e \geq t.$$

A critical constraint for $\mathcal{B}$ models a property that any feasible solution to a bargaining game satisfies, at least when it comes to demand satisfaction. In the combinatorial optimization world, (R, t, σ) corresponds exactly to the *redundant constraint*

$$\sum_{e \in R} \sigma_e x_e \geq t. \tag{16}$$

of $F_{IP}(\mathcal{B})$, i.e. a constraint that is derivable by the remaining constraints of the integer program. In other words, it is a constraint that is always satisfied by any integral solution to the IP formulation, and that, on one hand, it can be syntactically derived by the demands' requirements in the integral lattice, but which, on the other hand, might be independent of the demands' requirements in the realm of LPs (when variables assume non-integral values). Indeed, constraint (16) might not be redundant for the relaxation $F_{LP}(\mathcal{B})$ (i.e. it might be a cutting-plane). Even more, the integrality of $F_{LP}(\mathcal{B})$ could be less than 1, while the addition of constraint (16) could result in a Linear Program with integrality gap 1.

Example 3. Continuing from Example 2, consider graph G_0. It was already observed that any solution to the covering-type problem chooses at least 2 services among $\{1, 2, 3\}$. In particular, $(\{1, 2, 3\}, 2, \mathbf{1})$ is a critical constraint for the bargaining game. Constraint $x_1 + x_2 + x_3 \geq 2$ is a redundant constraint for the IP formulation of the problem, nevertheless, the addition of the constraint to the LP relaxation improves the integrality gap from 3/4 to 1.

For a covering-type problem $\mathcal{B}$, fix now a family of k critical constraints $\mathcal{R} = \left\{ \left(R_i, t_i, \sigma^{(i)} \right) \right\}_{i \in [k]}$. The following is a linear program relaxation to $F_{IP}(\mathcal{B})$

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e \, x_e & & (F_{LP}(\mathcal{B}, \mathcal{R})) \\
\text{s.t.} \quad & \sum_{e \in T_i} \rho_{i,e} x_e \geq d_i, & & \forall i \in V \\
& \sum_{e \in R_j} \sigma_e^{(j)} x_e \geq t_j, & & \forall j = 1, \ldots, k \\
& \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}$$

As discussed earlier, the already established characterization of stable outcomes is tailored to the syntactic formulation of the covering-type problem as Integer Program $F_{IP}(\mathcal{B})$. Specific to the formulation is that feasible collections of services are determined by a number of constraints associated exactly with each of the agents. It seems a fortunate coincidence that a natural property of IP formulations of combinatorial optimization problems matches an intuitive notion of fair (stable) outcomes in bargaining games. At the same time, the bargaining game of Example 3 admits no stable solution, even though its structure is simple, and none of the agents seems to be in an advantageous position in the network. It is reasonable to assume that agents would prefer to compromise so as to be able to arrive at a (new-type of) stable solution, rather than not agreeing at all. Put it differently, $F_{LP}(\mathcal{B}, \mathcal{R})$ might have integrality gap 1, when $F_{LP}(\mathcal{B})$ does not. At the same time our findings (as well as all previously known results) pertaining to the existence of stable solutions do not apply to formulation $F_{LP}(\mathcal{B}, \mathcal{R})$, since it involves constraints that are not associated with agents. In the next section we investigate new relaxed notions of stability that are captured exactly by the integrality of $F_{LP}(\mathcal{B}, \mathcal{R})$.

On the practical side, these new notions of stability admit an interpretation according to which agents would prefer to compromise as otherwise no "fair/stable" bargaining outcome would be agreed among the players. Critical constraints describe exactly necessary conditions of feasible solutions to the combinatorial optimization problem. When such critical constraints are identified by the players, they may choose to relax their bargaining power (based on their contribution in the critical constraints) for the sake of arriving at a fair/stable bargaining outcome. This notion of fair/stable outcome is explored in the next section, and is it derived by polyhedral mechanics (same way our extended notion of stability was derived in the previous sections).

Lastly, an orthogonal question to consider, which is outside the scope of this paper, is how these critical constraints are identified. Since these constraints are exactly valid constraints for the integral hull of linear programs, an answer is given by numerous techniques proposed by well-known combinatorial optimization methods, including generic cutting planes methods, e.g. Gomory-Chvatal cuts [10,17] (see also [26]).

4.2 Stability Based on Critical Constraints

Our main contribution in this section is the introduction of a relaxed, still natural and intuitive, notion of stability that can be characterized by the integrality of linear programs. Notably, we still study outcomes $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$ of a bargaining game $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$. More specifically, we will again propose a refinement of feasible outcomes (as per Definition 1), i.e. the notion of feasibility stays invariant. The new refinement, i.e. new notion of stability, is associated with a family of k critical constraints $\mathcal{R} = \left\{ \left(R_i, t_i, \sigma^{(i)} \right) \right\}_{i \in [k]}$ for $\mathcal{B}$, which will be provably a relaxed notion of the stability of Definition 2. In what follows, and for the j-th critical constraint, we denote by $V(R_j)$ the set of agents that are served by some service in R_j, i.e. $V(R_j) = \cup_{e \in R_j} e$.

Definition 4 ($\mathcal{R}$-stable Outcome). *Given a bargaining game* $\mathcal{B} = (\mathcal{H} = (V, E), d, c, \rho)$ *consider family of critical constraints* $\mathcal{R} = \left\{ \left(R_i, t_i, \sigma^{(i)} \right) \right\}_{i \in [k]}$.

A feasible outcome $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$ *is called* $\mathcal{R}$*-stable if there exist non-negative* $P_{i,e}^{ind}, P_{i,e}^{R_j}$ *such that*

$$P_{i,e} = P_{i,e}^{ind} + \sum_{j \in [k]} P_{i,e}^{R_j}$$

satisfying:

- *(Individual Greed)* $P_{i,e}^{ind} > 0$*, implies that agent* i *is tight,* $i \in e$*, and service* e *is chosen.*
- *(Collective Greed)* $P_{i,e}^{R_j} > 0$*, implies that demand* t_j *of the* j*-th critical constraint is satisfied tightly,* $e \in R_j$ *and* $i \in V(R_j)$*.*
- *(Collective Envy-Free) For every* $f \notin A$*, suppose that* f *contains* $\{i_1, \ldots, i_l\}$ *of the agents, as well as critical constraints* $\{j_1, \ldots, j_m\}$ *are serviced by* f*, i.e.* $f \in R_{j_t}, t = 1, \ldots, m$*. Then, for all* $e_j \in A \cap T_{i_j}$ *and for all* $e_t \in A \cap R_{j_t}$

$$\sum_{j=1}^{l} \frac{\rho_{i_j,f}}{\rho_{i_j,e_j}} P_{i_j,e_j}^{ind} + \sum_{t=1}^{m} \frac{\sigma_f^{(j_t)}}{\sigma_{e_t}^{(j_t)}} \sum_{s \in V(R_{j_t})} P_{s,e_t}^{R_{j_t}} \leq c_f. \tag{17}$$

$\mathcal{R}$-stable outcomes admit an intuitive and natural interpretation. Indeed, consider outcome $(A \subseteq E, \{P_{i,e}\}_{i \in V, e \in E})$. Each agent, as per the feasibility requirement, is still expected to make some contribution for each service $e \in A$, and the cost of the chosen services is covered by the agents. Now each agent has a relaxed notion of stability which is associated with the critical constraints. In particular, the payment of each agent i for $e \in A$ has two components, the *individual* contribution and the *collective* contribution.

More specifically, critical constraints $\mathcal{R}$ identify coalitions $V(R_j)$, $j = 1, \ldots, m$ of agents. Members within each coalition $V(r_j)$ are guaranteed to meet demand t_j with respect to satisfaction function $\sigma^{(j)}$, for every collection of services satisfying agents' demands (due to the definition of critical constraints). As such, *socially-aware* agents in $V(R_j)$ are asked to make some *collective* contribution $P_{i,e}^{R_j}$ for being members of the coalition.

On top of that, each agent makes another *individual* contribution $P_{i,e}^{ind}$ which can be thought as the only contribution agent would make in a stable outcome. Still, a notion of *fairness* is required for both types of payments to be appealing to the agents. As before, no agent should have a positive individual payment for a service e that is either not chosen in the solution or if agent is not served by e. Similarly, each agent i will have a positive collective contribution toward coalition R_j only if $i \in V(R_j)$ (i.e. only if i lies within a service required by R_j) and only when coalition R_j is satisfied tightly with respect to demand t_j and satisfaction $\sigma^{(j)}$.

Lastly, agents V and coalitions $V(R_j)$ can be thought as individual players. In particular, coalition $V(R_j)$ can be thought to make payment $\sum_{s \in V(R_j)} P_{s,e}^{R_j}$ for each $e \in E$. A natural notion of fairness then requires that each of these players makes no payment which is more than her outside option in a naturally defined "augmented" bargaining game. At a high level, each of the agents or coalitions could deviate from the current proposed selection of services, if that would result to a smaller payment, either for the agents or the coalitions. The fact that coalition-payments are distributed, in our definition of $\mathcal{R}$-stability, arbitrarily among a carefully chosen collection of agents is a natural consequence of the definition of critical constraints. This is explored in detail in the next section.

4.3 $\mathcal{R}$-Stability from $\emptyset$-Stability

In this section we justify how $\mathcal{R}$-stability is derived naturally from the notion of stability of Definition 2. First, it is not difficult to see that $\emptyset$-stable outcomes as per Definition 4 are exactly the stable outcomes of Definition 2. Next we argue that $\mathcal{R}$-stability can be derived naturally from $\emptyset$-stability for a carefully defined bargaining game.

Indeed, consider a bargaining game $\mathcal{B} = (\mathcal{H}, d, c, \rho)$, where $\mathcal{H} = (V, E), V = \{1, 2, \dots, n\}$, along with a family of k critical constraints $\mathcal{R} = \left\{ \left(R_i, t_i, \sigma^{(i)} \right) \right\}_{i \in [k]}$. We introduce the $\mathcal{R}$-augmented bargaining game $\mathcal{B}' = (\mathcal{H}' = (V', E'), d', c', \rho')$. At a high level $\mathcal{B}'$ contains one auxiliary new agent $n+j$ for every critical constraint R_j, $j = 1, \dots, k$. The set of services is updated so that each service contains also all of the critical constraint-players it serves. The new services have the same cost as the services in $\mathcal{B}$. Finally, the demands of the new players and their satisfactions are given by t_i and $\sigma^{(i)}$, respectively.

Formally, the $\mathcal{R}$-augmented bargaining game $\mathcal{B}'$ is defined as: $V' := V \cup \{n + 1, \dots, n+k\}$; E' contains all $e' \subseteq V'$ obtained from $e \in E$ as $e' := e \cup \{n+j : e \in R_j \text{ for some } j \in [k]\}$; $d'_i = d_i$ whenever $i \in V$, and $d'_{n+j} = t_j$ for all $j = 1, \dots, k$; $c'_{e'} = c_e$ for all $e \in E$ (recall the two games have the same number of services); $\rho'_{i,e'} = \rho_{i,e}$ whenever $i \in V$, and $\rho'_{n+j,e'} = \sigma^{(j)}_e$ for all $j = 1, \dots, k$. Note that by contruction, there is a bijection between E and E', and so for each $e \in E$ we denote below by $e' \in E'$ the corresponding edge in the $\mathcal{R}$-augmented bargaining game $\mathcal{B}'$.

Since $F_{LP}(\mathcal{B}, \mathcal{R})$ is a relaxation to the exact formulation $F_{IP}(\mathcal{B})$, the following is immediate from Theorem 1.

Corollary 1. *$\mathcal{R}$-augmented bargaining game $\mathcal{B}'$ admits an $\emptyset$-stable (stable) outcome if and only if the integrality gap of $F_{LP}(\mathcal{B}, \mathcal{R})$ is 1.*

Notably, we were able to obtain Corollary 1 only because Theorem 1 accounted for bargaining games over networks induced by hypergraphs (the augmented bargaining game involves services that are hyperedges even if the underlying graph of the original bargaining game is simple). We use the same property to obtain the main contribution of this section which reads as follows.

Theorem 3. *Let $\mathcal{R}$ be a family of critical constraints for the bargaining game $\mathcal{B}$, and let $\mathcal{B}'$ be the associated $\mathcal{R}$-augmented bargaining game. Then, $\mathcal{B}$ admits an $\mathcal{R}$-stable (and feasible) solution iff $\mathcal{B}'$ admits an $\emptyset$-stable (and feasible) solution.*

We prove Theorem 3 in Lemmata 8, 9 below.

Lemma 8. *If $\mathcal{B}$ admits an $\mathcal{R}$-stable (and feasible) solution then $\mathcal{B}'$ admits an $\emptyset$-stable (and feasible) solution.*

Proof. Consider an $\mathcal{R}$-stable (and feasible) solution $\mathcal{F} = (A \subseteq E, \{P_{i,e}\}_{i\in V, e\in E})$ for $\mathcal{B}$. By Definition 4, there exist non-negative $P_{i,e}^{ind}, P_{i,e}^{R_j}$ such that $P_{i,e} = P_{i,e}^{ind} + \sum_{j\in[k]} P_{i,e}^{R_j}$.

Now we define the outcome for $\mathcal{B}'$. First recall that there is a bijection between E and E', and in particular services e' in E are obtained from services in E containing (as edges) all critical constraints they serve. We set $A' = A$. For each $i \in V$ (set of original players), we set $P'_{i,e'} = P_{i,e}^{ind}$. For each critical constraint $j \in [k]$, we set $P'_{n+j,e'} = \sum_{i\in V(R_j)} P_{i,e}^{R_j}$. Next we prove that $\mathcal{F}' = \left(A' \subseteq E', \{P'_{i,e'}\}_{i\in V', e'\in E'}\right)$ is feasible and $\emptyset$-stable for the $\mathcal{R}$-augmented game $\mathcal{B}'$.

Indeed, A satisfies Demand Satisfaction for $\mathcal{B}$. At the same time, we know that any collection of services with this property also satisfies any critical constraint (see Definition 3). Hence, A' satisfies Demand Satisfaction for $\mathcal{B}'$.

Now consider $e' \in A'$ (and hence $e \in A$). We verify Cost Recovery for $\mathcal{B}'$. Indeed,

$$\sum_{i\in V'} P'_{i,e'} = \sum_{i\in V} P'_{i,e'} + \sum_{j\in[k]} P'_{n+j,e'} = \sum_{i\in V} P_{i,e}^{ind} + \sum_{j\in[k]} \sum_{i\in V(R_j)} P_{i,e}^{R_j}$$

By the definition of $\mathcal{R}$-stability, $P_{i,e}^{R_j} = 0$, whenever $i \notin V(R_j)$. Hence, expression above equals

$$\sum_{i\in V} \left(P_{i,e}^{ind} + \sum_{j\in[k]} P_{i,e}^{R_j} \right) = \sum_{i\in V} P_{i,e} = c_e$$

as wanted (where the last equality is due to Cost recovery of $\mathcal{B}$). This concludes that outcome $\mathcal{F}'$ is feasible for $\mathcal{B}'$.

Now we show that $\mathcal{F}'$ is $\emptyset$-stable for $\mathcal{B}'$. First we verify property Greed. Consider $i \in V'$ and $e' \in E'$ such that $P'_{i,e'} > 0$. We consider two cases. If $i \in V$, then $P_{i,e}^{ind} > 0$. But then, Individual Greed for $\mathcal{F}$ implies that agent i is tight and $i \in e$ as wanted. If $i \in V' \setminus V$, then $i = n + j$ for some $j \in [k]$ and $P'_{n+j,e'} = \sum_{t\in V(R_j)} P_{t,e}^{R_j} > 0$. Since all payments are non-negative, there exists $t \in V(R_j)$ such that $P_{t,e}^{R_j} > 0$. Due to Collective Greed, critical constraint must be satisfied tightly, and $e' \in R_j$ (in other words, e' serves player j), as wanted.

Finally, we verify Envy-Free. Consider some $f' \notin A'$. Suppose that f' contains original agents $U \subseteq V$ and critical constraints $W \subseteq [k]$. For each $i \in U$, consider $e'_i \in A' \cap T_{i_s}$, and for each $j \in W$ consider $e'_j \in A' \cap R_j$. Taking into consideration the definition of $\mathcal{B}'$ (and the satisfaction rates of players corresponding to critical constraints), we have

$$\sum_{i\in f}\frac{\rho_{i,f'}}{\rho_{i,e'_i}}P'_{i,e'_i}=\sum_{i\in U}\frac{\rho_{i,f'}}{\rho_{i,e'_i}}P'_{i,e'_i}+\sum_{j\in W}\frac{\sigma^{(j)}_{f'}}{\sigma^{(j)}_{e'_j}}P'_{n+j,e'_j}$$
$$=\sum_{i\in U}\frac{\rho_{i,f}}{\rho_{i,e_i}}P^{ind}_{i,e_i}+\sum_{j\in W}\frac{\sigma^{(j)}_{f}}{\sigma^{(j)}_{e_j}}\sum_{k\in V(R_j)}P^{R_j}_{k,e_j}$$

which is at most c_f (due to that $\mathcal{F}$ satisfies the Collective Envy-Free property), as wanted. ☐

Lemma 9. *If $\mathcal{B}'$ admits a stable (and feasible) solution then $\mathcal{B}$ admits an $\mathcal{R}$-stable (and feasible) solution.*

Proof. Consider outcome $\mathcal{F}'=\left(A'\subseteq E',\{P'_{i,e'}\}_{i\in V',e'\in E'}\right)$ for bargaining game $\mathcal{B}'$, which is feasible and stable.

For each critical constraint $j\in[k]$, we consider an arbitrary distribution $\{\tau^{(j)}_i\}_{i\in V(R_j)}$ over agents in $V(R_j)$, i.e. $\sum_{i\in V(R_j)}\tau^{(j)}_i=1$, with all $\tau^{(j)}_i\geq 0$. For completeness, we also set $\tau^{(j)}_i=0$ whenever $i\notin R_j$. Next we define outcome $\mathcal{F}=(A\subseteq E,\{P_{i,e}\}_{i\in V,e\in E})$ for bargaining game $\mathcal{B}$ as follows; $A=A'$, and for every $i\in V$ and $e\in E$, we set $P^{ind}_{i,e}=P'_{i,e'}$ and for each critical constraint $j\in[k]$ we set $P^{R_j}_{i,e}=\tau^{(j)}_i P'_{n+j,e'}$, so that

$$P_{i,e}=P'_{i,e'}+\sum_{j\in[k]}\tau^{(j)}_i P'_{n+j,e'}.$$

Next we show that $\mathcal{F}$ is feasible and $\mathcal{R}$-stable.

First note that A' by definition satisfies all agents' demands in $\mathcal{B}'$ (even the demands of the critical constraints), hence A satisfies demand satisfaction for $\mathcal{B}$. Next we study Cost Recovery, so we consider some arbitrary $e\in A$ (hence $e'\in A'$). We have

$$\sum_{i\in V}P_{i,e}=\sum_{i\in V}\left(P'_{i,e'}+\sum_{j\in[k]}\tau^{(j)}_i P'_{n+j,e'}\right)$$
$$=\sum_{i\in V}P'_{i,e'}+\sum_{j\in[k]}P'_{n+j,e'}\sum_{i\in V}\tau^{(j)}_i$$
$$=\sum_{i\in V}P'_{i,e'}+\sum_{j\in[k]}P'_{n+j,e'}$$
$$=c_e$$

where the last equality is due to that $\mathcal{F}'$ satisfies Cost recovery for $\mathcal{B}'$.

In what follows we show that $\mathcal{F}$ is $\mathcal{R}$-stable. First we study Individual Greed. Consider $i\in V,e\in E$ so that $P^{ind}_{i,e}>0$. But then, $P'_{i,e'}>0$ in $\mathcal{B}'$, and since $\mathcal{F}'$ is stable, we conclude that $i\in e$ and i is satisfied tightly. as wanted.

Next we study Collective Greed. Consider $i\in V,e\in E,j\in[k]$ so that $P^{R_j}_{i,e}>0$. But then, $\tau^{(j)}_i P'_{n+j,e'}>0$ in $\mathcal{B}'$. Recall that distribution $\{\tau^{(j)}_i\}_i$ has its support within $V(R_j)$, hence $i \in V(R_j)$. Moreover, $P'_{n+j,e'} > 0$ implies that $e' \in A' \cap R_j$ and critical constraint R_j has its demand satisfied tightly, since $\mathcal{F}'$ is stable. Hence $\mathcal{F}$ satisfies Collective Greed.

Finally, we show that $\mathcal{F}$ is Collective Envy-Free. Indeed, consider $f \notin A$ (hence $f' \notin A'$). Suppose that f contains $\{i_1, \ldots, i_l\}$ of the agents, as well as critical constraints $\{j_1, \ldots, j_m\}$ are serviced by f. Consider also arbitrary $e_j \in A \cap T_{i_j}$ and $e_t \in A \cap R_{j_t}$ (hence $e'_j, e'_t \in A'$). Then,

$$\begin{aligned}
&\sum_{j=1}^{l} \frac{\rho_{i_j,f}}{\rho_{i_j,e_j}} P^{ind}_{i_j,e_j} + \sum_{t=1}^{m} \frac{\sigma_f^{(j_t)}}{\sigma_{e_t}^{(j_t)}} \sum_{s \in V(R_{j_t})} P^{R_{j_t}}_{s,e_t} \\
&= \sum_{j=1}^{l} \frac{\rho_{i_j,f}}{\rho_{i_j,e_j}} P'_{i_j,e'_j} + \sum_{t=1}^{m} \frac{\sigma_f^{(j_t)}}{\sigma_{e_t}^{(j_t)}} \sum_{s \in V(R_{j_t})} \tau_s^{(j_t)} P'_{n+j_t,e'_t} \\
&= \sum_{j=1}^{l} \frac{\rho_{i_j,f}}{\rho_{i_j,e_j}} P'_{i_j,e'_j} + \sum_{t=1}^{m} \frac{\sigma_f^{(j_t)}}{\sigma_{e_t}^{(j_t)}} P'_{n+j_t,e'_t} \sum_{s \in V(R_{j_t})} \tau_s^{(j_t)}
\end{aligned}$$

Recalling that $\{\tau_s^{(j_t)}\}_s$ is a distribution over $s \in V(R_{j_t})$, the last expression is less than c_f, since $\mathcal{F}'$ is Envy-Free for $\mathcal{B}'$. □

Corollary 1 together with Theorem 3 characterize the existence of $\mathcal{R}$-stable outcomes.

Corollary 2. *Bargaining game $\mathcal{B}$ admits an $\mathcal{R}$-stable outcome if and only if the integrality gap of $F_{LP}(\mathcal{B}, \mathcal{R})$ is 1.*

We conclude this section by demonstrating an application of our findings pertaining to $\mathcal{R}$-stable outcomes.

Example 4. Consider the covering-type problem of Example 1, and fix graph $G = (V_0, E_0)$. A number of cutting planes are known for the problem, including the so-called odd-cycle constraints; for every odd cycle $C \subseteq V$, and for every vertex cover A of G, we have $|A \cap C| \geq (|C| + 1)/2$. Effectively, cutting planes $\sum_{i \in C} x_i \geq (|C| + 1)/2$, for all odd cycle $C \subseteq V$ can be added to formulation (2). Moreover each odd cycle C is a critical constraint. The specific graph of bargaining game of Example 2 did not admit a stable solution. However, together with the odd-cycle (critical) constraint $R = (\{1, 2, 3\}, 2, \mathbf{1})$, and as explained in Example 3, the new LP relaxation has integrality gap 1. Hence it admits an R-stable solution, in which each *socially-aware* agent not only makes an individual payment, but also contributes towards covering the cost of the global solution as a member of a coalition (the odd cycle) which always receives at least 2 services.

5 Discussion and Open Problems

We studied covering-type problems and their underlying bargaining games over networks. Our work generalized previously known results in two directions. First, our networks are induced by hypergraphs, and hence bargaining is associated with generic subsets of players (not only of size 2). Second, we assumed non-uniformity for the "value" of the objects to be bargained over, in contrast to previous results in which all objects were worth the same to all agents. These generalizations allowed us to further extend previous results by introducing and characterizing new and relaxed notions of stability based on combinatorial properties of the underlying optimization problems, which admit intuitive interpretations based on socially-aware players.

Our work is a just starting point toward introducing natural notions of stability based on properties of the underlying combinatorial optimization problems. Indeed, we introduced intuitive notions of stability based on linear programs strengthened by constraints valid for the integral hull of the relaxations, and we only considered linear tightenings. Is there a natural stability notion associated with semidefinite programming tightenings (or in general non-linear tightenings)? What if the starting linear program relaxation is tightened by any of the well-studied lift-and-project systems, e.g. Lovasz-Schrijver [24], Sherali-Adams [29], or Lasserre [23]? Similarly, is there a natural stability notion associated with convex relaxations in which the integrality gap is not 1 but bounded?

References

1. Azar, Y., Birnbaum, B., Celis, L.E., Devanur, N.R., Peres, Y.: Convergence of local dynamics to balanced outcomes in exchange networks. In: 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, Atlanta, Georgia, USA, 25–27 October 2009, pp. 293–302. IEEE Computer Society (2009)
2. Bateni, M.H., Hajiaghayi, M.T., Immorlica, N., Mahini, H.: The cooperative game theory foundations of network bargaining games. In: Abramsky, S., Gavoille, C., Kirchner, C., Meyer auf der Heide, F., Spirakis, P.G. (eds.) ICALP 2010. LNCS, vol. 6198, pp. 67–78. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-14165-2_7
3. Bayati, M., Borgs, C., Chayes, J., Kanoria, Y., Montanari, A.: Bargaining dynamics in exchange networks. J. Econ. Theory **156**, 417–454 (2015)
4. Bock, A., Chandrasekaran, K., Könemann, J., Peis, B., Sanità, L.: Finding small stabilizers for unstable graphs. Math. Program. **154**(1–2), 173–196 (2015)
5. Celis, L.E., Devanur, N.R., Peres, Y.: Local dynamics in bargaining networks via random-turn games. In: Saberi, A. (ed.) WINE 2010. LNCS, vol. 6484, pp. 133–144. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-17572-5_11
6. Chakraborty, T., Kearns, M.: Bargaining solutions in a social network. In: Papadimitriou, C., Zhang, S. (eds.) WINE 2008. LNCS, vol. 5385, pp. 548–555. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-92185-1_61
7. Chakraborty, T., Kearns, M., Khanna, S.: Network bargaining: algorithms and structural results. In: Proceedings 10th ACM Conference on Electronic Commerce, EC 2009, Stanford, California, USA, 6–10 July 2009, pp. 159–168 (2009)
8. Chalkiadakis, G., Elkind, E., Wooldridge, M.: Computational Aspects of Cooperative Game Theory. Morgan & Claypool Publishers, San Rafael (2011)

9. Chan, T.-H.H., Chen, F., Ning, L.: Optimizing social welfare for network bargaining games in the face of unstability, greed and spite. In: Epstein, L., Ferragina, P. (eds.) ESA 2012. LNCS, vol. 7501, pp. 265–276. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-33090-2_24
10. Chvátal, V.: Edmonds polytopes and a hierarchy of combinatorial problems. Discrete Math. **306**(10–11), 886–904 (2006)
11. Cook, K.S., Emerson, R.M., Gillmore, M.R., Yamagishi, T.: Distribution of power in exchange networks: theory and experimental results. Am. J. Sociol. **89**, 275–305 (1983)
12. Deng, X., Fang, Q.: Algorithmic cooperative game theory. In: Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L. (eds.) Pareto Optimality, Game Theory and Equilibria. SOIA, vol. 17, pp. 159–185. Springer, New York (2008). https://doi.org/10.1007/978-0-387-77247-9_7
13. Deng, X., Ibaraki, T., Nagamochi, H.: Algorithmic aspects of the core of combinatorial optimization games. Math. Oper. Res. **24**(3), 751–766 (1999)
14. Draief, M., Vojnovic, M.: Bargaining dynamics in exchange networks. CoRR, abs/1202.1089 (2012)
15. Farczadi, L., Georgiou, K., Könemann, J.: Network bargaining with general capacities. In: Bodlaender, H.L., Italiano, G.F. (eds.) ESA 2013. LNCS, vol. 8125, pp. 433–444. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-40450-4_37
16. Georgiou, K., Karakostas, G., Könemann, J., Stamirowska, Z.: Social exchange networks with distant bargaining. Theoret. Comput. Sci. **554**, 263–274 (2014)
17. Gomory, R.E.: Solving linear programming problems in integers. In: Bellman, R., Hall Jr., M. (eds.) Combinatorial Analysis, pp. 211–215, Providence, RI. Symposia in Applied Mathematics X. American Mathematical Society (1960)
18. Granot, D., Granot, F.: On some network flow games. Math. Oper. Res. **17**(4), 792–841 (1992)
19. Hajiaghayi, M., Mahini, H.: Bargaining networks. In: Kao, M.Y. (ed.) Encyclopedia of Algorithms, pp. 1–5. Springer, Boston (2014). https://doi.org/10.1007/978-3-642-27848-8
20. Ito, T., Kakimura, N., Kamiyama, N., Kobayashi, Y., Okamoto, Y.: Efficient stabilization of cooperative matching games. Theoret. Comput. Sci. **677**, 69–82 (2017)
21. Kanoria, Y., Bayati, M., Borgs, C., Chayes, J.T., Montanari, A.: A natural dynamics for bargaining on exchange networks. CoRR, abs/0911.1767 (2009)
22. Kleinberg, J.M., Tardos, É.: Balanced outcomes in social exchange networks. In: Proceedings of the ACM Symposium on Theory of Computing, pp. 295–304 (2008)
23. Lasserre, J.B.: An explicit exact SDP relaxation for nonlinear 0-1 programs. In: Aardal, K., Gerards, B. (eds.) IPCO 2001. LNCS, vol. 2081, pp. 293–303. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-45535-3_23
24. Lovász, L., Schrijver, A.: Cones of matrices and set-functions and 0-1 optimization. SIAM J. Optim. **1**(2), 166–190 (1991)
25. Nash, J.: The bargaining problem. Econometrica **18**, 155–162 (1950)
26. Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization. Wiley, Hoboken (1988)
27. Rochford, S.C.: Symmetrically pairwise-bargained allocations in an assignment market. J. Econ. Theory **34**(2), 262–281 (1984)

28. Shapley, L.S., Shubik, M.: The assignment game: the core. Int. J. Game Theory **1**(1), 111–130 (1971)
29. Sherali, H.D., Adams, W.P.: A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM J. Discrete Math. **3**(3), 411–430 (1990)
30. Willer, D.: Network Exchange Theory. Praeger Publishers, Westport (1999)