Two-Level Cooperation in Network Games

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Abstract. The problem of allocating a value in hierarchical cooperative structures is important in the game theoretic literature, and it often arises in practice. In this paper, we consider a two-level structure of players communication and propose a procedure allocating the value in two steps: first the value is allocated at the upper level among groups of players, and then each group allocates the designated value among its members. We demonstrate how to allocate the value in two steps using the Shapley value and show the difference with the classical one-step allocation procedure. We then adopt this approach for games with pairwise interactions and provide relations between several definitions of the characteristic function and the corresponding Shapley values.

Keywords: Network · Hierarchy · Cooperation · Two-level allocation · Shapley value

1 Introduction

Networks are very natural and convenient in describing and representing hierarchical communication structures. Such hierarchies model situations in which players (agents) are at different level of subordination. The simplest examples of the usage of hierarchies include leader-follower models, organization structures indicating communication and subordination between departments. When the hierarchical communication structure is specified, one can study the problem of optimal behavior of players. The game theoretic literature covers two types of behavior: equilibrium and cooperative. In this paper, we focus on a cooperative case as it can provide players with a better outcome in the game rather than an equilibrium outcome; however, the cooperative model will be built by using an initially given non-cooperative game. As it is common in the cooperative game theory, players jointly choose their actions to achieve the largest total payoff followed by its allocation among them. The hierarchical communication structure naturally dictates that this value has to be allocated step-by-step from

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the topmost level of hierarchy to its lowermost level. Specifically, in this paper we consider a two-level hierarchy, thus the value will first be allocated at the upper level, and then players at this level will allocate the designated values among the members of these groups at the lower level. Some potential areas of application and associated problems are discussed in the following sources. Two-level resource allocation models for LTE networks are studied in [6,9] as a bankruptcy game. A two-level cooperative model involving service providers is examined in [8] and includes coalition formation and coalition optimization steps for a revenue allocation problem. In [10,11] the authors propose a two-step cost allocation method for the transmission system among generators and loads in case of non-atomic players.

The structure of the paper is as follows. The model of a two-level network game is presented in Sect. 2. Section 3 is devoted to the allocation issues in the cooperative version of the game. There we show how to allocated a cooperative outcome in two steps and provide relations between several definitions of the characteristic function and the corresponding Shapley values. A special class of games so-called games with pairwise interactions is considered in Sect. 4. Conclusion is in Sect. 5.

2 The Model

Consider a situation in which a group of players make a decision in a constrained communication environment to achieve certain goals. We assume a two-level communication structure of players. The upper level of communication is represented by players from a given finite set A, $|A| \geqslant 2$, who are connected in a network g^1, which is a collection of undirected pairs $(i, j) \in A \times A$ called links. We exclude loops by supposing that $(i, i) \notin g^1$. At the lower level, there are $|A| = n$ finite groups of players $A_1, \dots, A_n$ such that $A_i \cap A_j = \varnothing$, $i \neq j$. We assume that each player $i \in A$ is associated with the set A_i. Further, players from set $B_i = \{i\} \cup A_i$ are connected in a network g^{2i}. Here network g^{2i} is a collection of undirected links (i, j), $j \in A_i$, i.e., in g^{2i} players from A_i are connected only with i. Thus we have a player set $N = A \cup A_1 \cup \dots \cup A_n = B_1 \cup \dots \cup B_n$ of $|A| + \sum_{i \in N} |A_i|$ players connected in a network $g = g^1 \cup g^{21} \cup \dots \cup g^{2n}$. Denote the neighbors of a player $i \in N$ in network g by $N_i(g) = \{j : (i, j) \in g\}$. An example of the network g is demonstrated in Fig. 1. It is a network of seven groups of players: players (nodes) from set A are filled black; other players compose six groups $A_1, \dots, A_6$.

In the network a player can remove (all or some) links if such links are not beneficial for her. Let $(d_i(g), u_i)$ denote a strategy of player $i \in N$ in the game. Here $d_i(g) = (d_{ij}(g))_{i \in N}$ whose components equal either 1 if i keeps links with corresponding players, or 0, otherwise. The second component $u_i \in U_i$ of player i's strategy reflects her action. In the game, players select strategies in two stages. First, players at the upper level (players from A) simultaneously announce their choices, and knowing them, players at the lower level (players from $N \setminus A$) simultaneously select theirs. Denote $(d(g), u) = ((d_i(g), u_i))_{i \in N}$

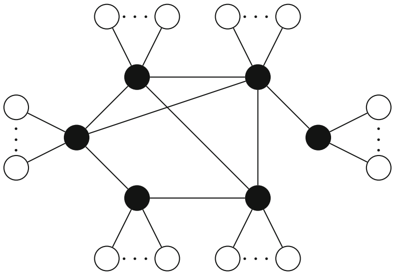

Fig. 1. A network of seven groups of players

and let $(d_S(g), u_S) = ((d_i(g), u_i))_{i \in S}$ for some subset $S \subset N$ called a *coalition*. Suppose that a player $i \in A$ selects $(d_i(g), u_i)$ and then a player $j \in A_i$ selects her strategy $(d_j(g), u_j)$ having the information about the choice of i. We note that profile $d(g)$ may change the network and thus we denote it by g^d.

Players are rewarded by payoffs depending upon their own strategies and the strategies of their neighbors in the given network g:

$$J_i(d(g), u) = h_i(u_i, u_{N_i(g^d)}) = \begin{cases} h_i(u_i, u_{N_i(g^{1,d})}, u_{N_i(g^{2i,d})}), & i \in A, \\ h_i(u_i, u_{N_i(g^{2j,d})}), & i \in A_j,\ j \in A. \end{cases}$$

This definition holds for all possible subnetworks g^d which can be realized when the strategy profile $(d(g), u)$ is played. The definition naturally comes from an option for players to remove some links from network g, and hence the payoff to the player is well-defined given any set of her neighbors (including the empty set). In the paper, we adopt the assumption from [13,14] that functions h_i, $i \in N$, satisfy the following property: for any player $i \in N$, any strategy $(d_i(g), u_i)$ and any coalition $S \subset N$, it holds true that $h_i(u_i, u_{N_i(g^d) \cap S'}) \leqslant h_i(u_i, u_{N_i(g^d) \cap S})$ for any $S' \subset S$. This property motivates players to keep all the links in the network. It also implies one useful result which will be discussed later in this paper.

3 Cooperation

Now suppose that in this two-stage decision process players coordinate their actions; the players seek to maximize the sum $\sum_{i \in N} J_i(d(g), u)$ followed by its allocation among them. Let

$$\begin{aligned}(\bar{d}(g), \bar{u}) &= \arg \max_{(d(g),u)} \sum_{i \in N} J_i(d_i(g), u_i) \\ &= \arg \max_{(d(g),u)} \sum_{i \in A} \left(h_i(u_i, u_{N_i(g^{1,d})}, u_{N_i(g^{2i,d})}) + \sum_{j \in A_i} h_j(u_j, u_{N_j(g^{2i,d})}) \right),\end{aligned}$$

and thus players should allocate $\sum_{i \in N} J_i(\bar{d}(g), \bar{u})$. Using the property of functions h_i, $i \in N$, we immediately conclude that under cooperation players do not have to revise the network by removing links, and hence the expression for the sum to be allocated can be simplified:

$$\sum_{i\in N} J_i(\bar{d}(g), \bar{u}) = \sum_{i\in A} \left(h_i(\bar{u}_i, \bar{u}_{N_i(g^1)}, \bar{u}_{A_i}) + \sum_{j\in A_i} h_j(\bar{u}_j, \bar{u}_i) \right) \tag{1}$$

We consider transferable payoffs. Then to allocate (1), we need to determine corresponding TU games and find their solutions. We will do it in two steps: first we allocate (1) among the subgroups $B_1, \ldots, B_n$ and then the assigned values to subgroups B_i, $i = 1, \ldots, n$, will be allocated among their members. Following this idea, we first define a TU game (B, v_1) for the upper level, where $B = \{B_1, \ldots, B_n\}$ denote the set of players-groups and v_1 is a *characteristic function* assigning to any coalition $S \subseteq B$ its worth. In [14], the authors propose the definition of a characteristic function for a network game with an exogenously given network. The idea of that definition is very simple and intuitive: the worth of a coalition is only determined by the strength of its members as if the members of the complement would have no impact on the coalition. This definition also goes exactly in line with the approach of von Neumann and Morgenstern [15] defining the worth of a coalition as the maxmin value of a corresponding zero-sum game between this coalition and its complement. It has been shown that under the assumptions on payoff functions, the maxmin optimization problem is reduced to a maximization problem [14]. Hence adopting this approach to the model under consideration, we come to the following:

$$v_1(S) = \begin{cases} \sum\limits_{i\in N} J_i(\bar{d}(g), \bar{u}), & S = B, \\ \max\limits_{u_S} \sum\limits_{i\in A\cap S} \left(h_i(u_i, u_{N_i(g^1)\cap S}, u_{A_i}) + \sum\limits_{j\in A_i} h_j(u_j, u_i) \right), & S \subset B, \\ 0, & S = \varnothing. \end{cases}$$

From the above, we obverse that the worth of the coalition of all players is just their largest total payoff, whereas to find the worth of a smaller coalition, we take into account only the strategies of its members. Indeed, according to the approach of von Neumann and Morgenstern, the coalition will not use an existing link with its complement if it is beneficial to it (as the complement will remove the link itself). Finally, the empty coalition gains nothing.

An imputation $\xi[v_1]$ in TU game (B, v_1) is a profile from $\mathbb{R}^n$ (as we have n players-groups) such that $\sum_{B_i\in B} \xi_{B_i}[v_1] = v_1(B)$ and $\xi_{B_i}[v_1] \geqslant v_1(B_i)$ for each $B_i \in B$. The set of all imputations called the *imputation set* will be denoted by $\mathcal{I}[v_1]$. A *cooperative solution* to the TU game (B, v_1) is a map assigning a subset $\mathcal{M}[v_1] \subseteq \mathcal{I}[v_1]$ to the game. Thus at the upper level, the value $v_1(B)$ is allocated among the players-groups by an imputation $\xi[v_1] \in \mathcal{M}[v_1]$. Under such the allocation a group $B_i \in B$ receives $\xi_{B_i}[v_1]$. Next, this value is allocated among the members of this group. For each B_i this is done with the use of a TU game (B_i, v_{2i}) in which the characteristic function is given by

$$v_{2i}(S) = \begin{cases} \xi_{B_i}[v_1], & S = B_i, \\ \max\limits_{u_S} \left(h_i(u_i, u_{N_i(g^{2i})}) + \sum\limits_{j \in A_i \cap S} h_j(u_j, u_i) \right), & S \subset B_i,\ i \in S, \\ \max\limits_{u_S} \sum\limits_{j \in S} h_j(u_j), & S \subset B_i,\ i \notin S, \\ 0, & S = \varnothing. \end{cases}$$

In a similar way, one can define the imputation set $\mathcal{I}[v_{2i}]$. To allocate $\xi_{B_i}[v_1]$ among the members of coalition $B_i = \{i\} \cup A_i$ we use the same cooperative solution $\mathcal{M}[v_{2i}] \subseteq \mathcal{I}[v_{2i}]$ which differs only in the characteristic function (v_{2i} replaces v_1). Thus selecting an imputation $\xi[v_{2i}] \in \mathcal{M}[v_{2i}]$ we solve the problem of allocating $\xi_{B_i}[v_1]$, $B_i \in B$.

When the cooperative solution $\mathcal{M}$ is the Shapley value and therefore consists of a single imputation, the corresponding allocations at both levels will have the following form:

$$\mathrm{Sh}_{B_i}[v_1] = \sum_{S \subseteq B,\, B_i \in S} \frac{(|B| - |S|)!(|S| - 1)!}{|B|!} \left(v_1(S) - v_1(S \setminus B_i) \right),$$

for any player-group $B_i \in B$ at the upper level, and

$$\begin{aligned} \mathrm{Sh}_j[v_{2i}] &= \sum_{S \subseteq B_i,\, j \in S} \frac{(|B_i| - |S|)!(|S| - 1)!}{|B_i|!} \left(v_{2i}(S) - v_{2i}(S \setminus \{j\}) \right) \\ &= \sum_{S \subseteq B_i,\, j \in S} \frac{(|A_i| - |S| + 1)!(|S| - 1)!}{(|A_i| + 1)!} \left(v_{2i}(S) - v_{2i}(S \setminus \{j\}) \right), \end{aligned}$$

for any player j at the lower level from group $B_i = \{i\} \cup A_i \in B$, and player i at the upper level.

Remark 1. It worth mentioning that characteristic functions v_1 and v_{2i} are consistent in terms of their definitions except for coalition B_i. Indeed, B_i can guarantee for itself the value of

$$v_1(B_i) = \max_{u_{B_i}} \left(h_i(u_i, u_{A_i}) + \sum_{j \in A_i} h_j(u_j, u_i) \right)$$

since $u_{N_i(g^1) \cap B_i} = \varnothing$ and player i becomes disconnected from other players from A in g^1. However, selecting an imputation $\xi[v_1]$ at the upper level, coalition B_i will gain ξ_{B_i} according to it. Noting that the imputation satisfies individual rationality, it follows that $\xi_{B_i}[v_1] \geqslant v_1(B_i)$. Thus if we proceed to the lower level for allocating $\xi_{B_i}[v_1] \equiv v_{2i}(B_i)$, we get that $v_{2i}(B_i) - v_1(B_i) \geqslant 0$. This makes some inconsistency in the definition of the characteristic functions. For all other coalitions, functions v_1 and v_{2i} are associated with maximization problems of the same type and hence are consistent in their definitions. One can deal with the inconsistent but not crucial definition of the characteristic functions in two ways. In the first scenario, group B_i allocates what it has been allocated at the upper level, $\xi_{B_i}[v_1]$. For this reason, we do not define $v_{2i}(B_i)$ as the solution of a corresponding maximization problem, but just set $v_{2i}(B_i) \equiv \xi_{B_i}[v_1]$ and define v_{2i} only for subsets of B_i. This scenario has been used above: v_{2i} has been defined according to the mentioned procedure. In the second scenario, we can make a small change in the definition of the characteristic function at the lower level: instead of values $v_{2i}(S)$, $S \subseteq B_i$, we may introduce the adjusted values: the difference $\xi_{B_i}[v_1] - v_1(B_i)$ (a surplus for coalition B_i) can be allocated only to player i, and then the value $v_1(B_i)$, which this coalition can guarantee for itself, is allocated among the members. In this case $v_{2i}(S)$ remains unchanged for coalitions not containing player i, but for a coalition containing i, its worth is $v_{2i}(S) + \xi_{B_i}[v_1] - v_1(B_i)$, i.e.,

$$\tilde{v}_{2i}(S) = \begin{cases} \xi_{B_i}[v_1] \equiv v_1(S) + (\xi_{B_i}[v_1] - v_1(B_i)), & S = B_i, \\ v_{2i}(S) + (\xi_{B_i}[v_1] - v_1(B_i)), & S \subset B_i,\ i \in S, \\ v_{2i}(S), & S \subset B_i,\ i \notin S, \\ 0, & S = \varnothing. \end{cases}$$

Alternatively, the v_{2i} can be determined differently:

$$\tilde{\tilde{v}}_{2i}(S) = \begin{cases} \xi_{B_i}[v_1] \equiv v_1(S) + (\xi_{B_i}[v_1] - v_1(B_i))|S|/|B_i|, & S = B_i, \\ v_{2i}(S) + (\xi_{B_i}[v_1] - v_1(B_i))|S|/|B_i|, & S \subset B_i, \\ 0, & S = \varnothing. \end{cases}$$

for any $S \subseteq B_i$ by letting $v_{2i}(B_i) = v_1(B_i)$ and supposing that each player from B_i gets an additional and the same surplus of $(\xi_{B_i}[v_1] - v_1(B_i))/|B_i|$ from the allocation of $v_1(B_i)$. Here for all coalitions S, the definition of v_1 and v_{2i} becomes consistent (recall that $v_1(S)$ and $v_{2i}(S)$ are defined in the same manner, and the subscripts refer only to the level of the hierarchy).

We now demonstrate the relationship between the Shapley values for the above characteristic functions. Given the network g, let for any coalition $S \subseteq N$

$$v(S) = \max_{u_i,\, i \in S} \sum_{i \in S} h_i(u_i, u_{N_i(g) \cap S}).$$

For $T \subset N$ we define a restriction $v|_T$ of v: $v|_T(S) \equiv v(S)$ for all $S \subseteq T$. It is clear that $v_1(B) = v(N)$, $v_1(B_i) = v(B_i) = v|_{B_i}(B_i)$ and $v_{2i}(S) = v(S)$ for any $S \subset B_i$ and $i = 1, \ldots, n$.

Proposition 1. *For any group B_i and any player $j \in B_i$, the following relations hold true:*

$$\mathrm{Sh}_j[v_{2i}] = \mathrm{Sh}_j[v|_{B_i}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))/|B_i|,$$

$$\mathrm{Sh}_j[\tilde{v}_{2i}] = \begin{cases} \mathrm{Sh}_j[v|_{B_i}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i)), & j = i, \\ \mathrm{Sh}_j[v|_{B_i}], & j \neq i, \end{cases}$$

$$\mathrm{Sh}_j[\tilde{\tilde{v}}_{2i}] = \mathrm{Sh}_j[v_{2i}].$$

Proof. To prove the first equality, we note:

$$\begin{aligned}
\mathrm{Sh}_j[v_{2i}] &= \sum_{S \subseteq B_i,\, j \in S} \frac{(|B_i| - |S|)!(|S|-1)!}{|B_i|!} \left(v_{2i}(S) - v_{2i}(S \setminus \{j\})\right) \\
&= \frac{v_{2i}(B_i) - v_{2i}(B_i \setminus \{j\})}{|B_i|} \\
&\quad + \sum_{S \subset B_i,\, j \in S} \frac{(|B_i| - |S|)!(|S|-1)!}{|B_i|!} \left(v_{2i}(S) - v_{2i}(S \setminus \{j\})\right) \\
&= \frac{v_{2i}(B_i) - v_1(B_i) + v_1(B_i) - v_{2i}(B_i \setminus \{j\})}{|B_i|} \\
&\quad + \sum_{S \subset B_i,\, j \in S} \frac{(|B_i| - |S|)!(|S|-1)!}{|B_i|!} \left(v|_{B_i}(S) - v|_{B_i}(S \setminus \{j\})\right) \\
&= \frac{v_{2i}(B_i) - v_1(B_i) + v|_{B_i}(B_i) - v|_{B_i}(B_i \setminus \{j\})}{|B_i|} \\
&\quad + \sum_{S \subset B_i,\, j \in S} \frac{(|B_i| - |S|)!(|S|-1)!}{|B_i|!} \left(v|_{B_i}(S) - v|_{B_i}(S \setminus \{j\})\right) \\
&= \frac{\mathrm{Sh}_j[v_1] - v_1(B_i)}{|B_i|} + \mathrm{Sh}_j[v|_{B_i}],
\end{aligned}$$

where $v_{2i}(B_i) = \mathrm{Sh}_j[v_1]$.

Now we show the fulfillment of the second equality. Since only coalitions $S \subseteq B_i$, $S \ni i$, gain a constant value of $\mathrm{Sh}_{B_i}[v_1] - v_1(B_i)$, then by the properties of the Shapley value player i gets this value as her additional payoff. Further, as $v_1(B_i) = v|_{B_i}(B_i)$ and $v_{2i}(S) = v|_{B_i}(S)$ for all $S \subset B_i$, we get that $\mathrm{Sh}_j[\tilde{v}_{2i}] = \mathrm{Sh}_j[v|_{B_i}]$ for $j \neq i$ and $\mathrm{Sh}_i[\tilde{v}_{2i}] = \mathrm{Sh}_i[v|_{B_i}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))$.

Finally, the third equality also follows by the properties of the Shapley value. Since each coalition $S \subseteq B_i$ gains a value of $(\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))|S|/|B_i|$, then each player just gets an additional value of $(\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))/|B_i|$. Further, as $v_1(B_i) = v|_{B_i}(B_i)$ and $v_{2i}(S) = v|_{B_i}(S)$ for all $S \subset B_i$, we get that $\mathrm{Sh}_j[\tilde{\tilde{v}}_{2i}] = \mathrm{Sh}_j[v|_{B_i}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))/|B_i|$ and therefore, $\mathrm{Sh}_j[\tilde{\tilde{v}}_{2i}] = \mathrm{Sh}_j[v_{2i}]$. This concludes the proof. □

The proposed two-level allocation procedure differs from the standard one-level scheme used in the classical cooperative game theory. We illustrate this difference for the Shapley value. To compute the one-level Shapley value, one has to use all subsets of the player set N, i.e. $2^{|N|}$ sets. However for the two-level Shapley value, we first use only subsets of N consisting of all possible unions of groups $B_1, \ldots, B_n$, and then at the lower level we use all subsets of the groups, i.e. get only $2^{|B|} + 2^{|A_1|} + \ldots + 2^{|A_n|}$ subcoalitions ($|B| + |A_1| + \ldots + |A_n| = |N|$). Thus in the two-level allocation scheme we do not list all subsets of N. To point out the difference between the classical (one-level) allocation procedure and the two-level procedure, we consider the next example.

Example 1. Consider a five-person game. Let the player set $N = \{1, 2, 3, 4, 5\}$ be decomposed into three groups: $N = A \cup A_1 \cup A_2$ where $A = \{1, 4\}$ is the set of players at the upper level, $A_1 = \{2, 3\}$ is the set of players at the lower level subordinated to player 1, and $A_2 = \{5\}$ is the set of players at the lower level subordinated to player 4. The network g is demonstrated in Fig. 2. Let $B_1 = \{1\} \cup A_1$, $B_2 = \{4\} \cup A_2$.

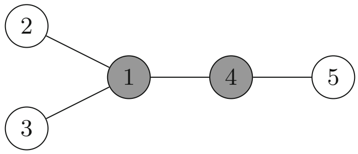

Fig. 2. A network g of three groups of players for Example 1

Let $u_i \in U_i = [0, \infty)$ be an action for player $i \in N$, and this player be rewarded with her payoff function $h_i(u_i, u_{N_i(g)}) = \ln(1 + u_i + \sum_{j \in N_i(g)} u_j) - cu_i$, for $c > 0$. This is an example of a logarithmic utility function for a model of public goods provision [2]. Here we suppose that $c = 0.1$.

One can show that under cooperation, the total players payoff equals $v(N) = v(\{B_1, B_2\}) = 12.810$ (we round off all numbers to third decimal place). For a two-level model, we first have to allocate $v(\{B_1, B_2\})$ at the upper level between groups B_1 and B_2. To do this find $v_1(B_1) = 7.303$ and $v_1(B_2) = 4.091$ and then get the Shapley value $\text{Sh}[v_1] = (\text{Sh}_{B_1}[v_1], \text{Sh}_{B_2}[v_1])$ with $\text{Sh}_{B_1}[v_1] = 8.011$ and $\text{Sh}_{B_2}[v_1] = 4.799$. Next the corresponding values are allocated at the lower level. For this reason for group B_1 we find $v_{21}(B_1) = \text{Sh}_{B_1}[v_1]$, $v_{21}(\{1, 2\}) = v_{21}(\{1, 3\}) = 4.091$, $v_{21}(\{2, 3\}) = 2.805$, $v_{21}(\{1\}) = v_{21}(\{2\}) = v_{21}(\{3\}) = 1.403$ and therefore the Shapley value $\text{Sh}[v_{21}] = (\text{Sh}_1[v_{21}], \text{Sh}_2[v_{21}], \text{Sh}_3[v_{21}])$ consists of the components: $\text{Sh}_1[v_{21}] = 3.099$, $\text{Sh}_2[v_{21}] = \text{Sh}_3[v_{21}] = 2.456$. For group B_2 we find $v_{22}(B_2) = \text{Sh}_{B_2}[v_1]$, $v_{22}(\{4\}) = v_{22}(\{5\}) = 1.403$ and therefore the Shapley value $\text{Sh}[v_{22}] = (\text{Sh}_4[v_{22}], \text{Sh}_5[v_{22}])$ consists of the components: $\text{Sh}_4[v_{22}] = \text{Sh}_5[v_{22}] = 2.400$ (the components do not sum up to 4.799 due to rounding).

If the players had allocated the value of $v_1(N) = v_1(\{B_1, B_2\}) = 12.810$ only in one step under the classical allocation procedure, we would have had the following one-level characteristic function v: $v(\{N\}) = 12.810$, $v(\{1, 2, 3, 4\}) = 10.856$, $v(\{1, 2, 3, 5\}) = v(\{1, 2, 4, 5\}) = 9.519$, $v(\{2, 3, 4, 5\}) = 6.897$, $v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{1, 4, 5\}) = 7.304$, $v(\{1, 2, 5\}) = v(\{1, 3, 5\}) = v(\{2, 4, 5\}) = v(\{3, 4, 5\}) = 5.494$, $v(\{2, 3, 4\}) = v(\{2, 3, 5\}) = 4.208$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 4\}) = v(\{4, 5\}) = 4.091$, $v(\{1, 5\}) = v(\{2, 3\}) = v(\{2, 4\}) = v(\{2, 5\}) = v(\{3, 4\}) = v(\{3, 5\}) = 2.805$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\{5\}) = 1.403$, and the one-level Shapley value $\text{Sh}[v] = (\text{Sh}_1[v], \text{Sh}_2[v], \text{Sh}_3[v], \text{Sh}_4[v], \text{Sh}_5[v])$ whose components equal $\text{Sh}_1[v] = 3.633$, $\text{Sh}_2[v] = \text{Sh}_3[v] = 2.247$, $\text{Sh}_4[v] = 2.812$, $\text{Sh}_5[v] = 1.869$ (the components do not sum up to 12.810 due to rounding).

Comparing the two values we observe that players 2, 3 and 5 (lower-level players) benefit from the two-level allocation procedure whereas players 1 and 4 (upper-level players) benefit from the classical one-level scheme. Moreover, following the two-level scheme players 4 and 5 (players of different levels) are rewarded equally which is not the case when the one-level allocation procedure is applied. The example demonstrates that there should be a trade-off between the schemes as players benefit unequally from them. We conclude this example by noting that $\mathrm{Sh}_1[v|_{B_1}] = 2.863$, $\mathrm{Sh}_2[v|_{B_1}] = \mathrm{Sh}_3[v|_{B_1}] = 2.220$, $\mathrm{Sh}_4[v|_{B_2}] = \mathrm{Sh}_5[v|_{B_2}] = 2.046$. And finally, $\mathrm{Sh}_1[\tilde{v}_{21}] = 3.571$, $\mathrm{Sh}_2[\tilde{v}_{21}] = \mathrm{Sh}_3[\tilde{v}_{21}] = 2.220$, $\mathrm{Sh}_4[\tilde{v}_{22}] = 2.753$, $\mathrm{Sh}_5[\tilde{v}_{22}] = 2.046$.

4 A Case of Pairwise Interactions

This section deals with a case of pairwise interactions introduced in [4] and then developed in [12] for cooperative network games. Pairwise interaction means that a player gains from each of her neighbors by choosing actions not necessarily the same for each of the neighbors. An interaction between two connected players in the network is generally represented by a bimatrix game in which each of the players has a finite number of actions. For example, a network game with a 2×2 coordination game played between neighbors was studied in [7] and later in [5]. A coordination stag-hunt game with repeated rounds was considered in [3]. In [16], the authors developed a network game model based on prisoner's dilemma.

Specifically, let $i \in N$, $j \in N_i(g)$, then player i plays with her neighbor j a bimatrix game with payoff matrices $\mathrm{A}_{ij} = \{a^{ij}_{p\ell}\}$ and $\mathrm{B}_{ij} = \{b^{ij}_{p\ell}\}$ for players i and j, respectively. Given a profile of players' strategies, the player's payoff represents the sum of her payoffs in all bimatrix games played with her neighbors in the network. Let

$$m_{ij} = \begin{cases} \max\limits_{p,\ell}(a^{ij}_{p\ell} + b^{ji}_{p\ell}), & \text{if } i \text{ and } j \text{ are neighbors,} \\ 0, & \text{otherwise.} \end{cases}$$

Then characteristic functions v_1 and v_{2i}, $i \in A$ will have the following form:

$$v_1(S) = \begin{cases} \dfrac{1}{2} \sum\limits_{i \in A} \left(\sum\limits_{j \in N_i(g^1)} m_{ij} + 2 \sum\limits_{j \in A_i} m_{ij} \right), & S = B, \\ \dfrac{1}{2} \sum\limits_{i \in A \cap S} \left(\sum\limits_{j \in N_i(g^1) \cap S} m_{ij} + 2 \sum\limits_{j \in A_i} m_{ij} \right), & S \subset B, \\ 0, & S = \varnothing, \end{cases}$$

and for any $B_i \in B$,

$$v_{2i}(S) = \begin{cases} \xi_{B_i}[v_1], & S = B_i, \\ \sum\limits_{j \in A_i \cap S} m_{ij}, & S \subset B_i,\ i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

By the definition of v_1, we get $v_1(B_i) = \sum_{j \in A_i} m_{ij}$. Consider the characteristic function

$$v_{2i}^{\mathrm{PI}}(S) = \begin{cases} \sum\limits_{j \in A_i \cap S} m_{ij}, & S \subseteq B_i,\ i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

At the lower level, network g^{2i} is a star-network whose hub is player i. For a star network, [12] provides the formula for the Shapley value $\mathrm{Sh}[v_{2i}^{\mathrm{PI}}]$. Adopting this formula for $v_{2i}^{\mathrm{PI}}(S)$, we obtain:

$$\mathrm{Sh}_j[v_{2i}^{\mathrm{PI}}] = \begin{cases} \dfrac{1}{2} \sum\limits_{k \in A_i} m_{ik}, & j = i, \\ \dfrac{m_{ij}}{2}, & j \neq i. \end{cases}$$

Corollary 1. *In the case of pairwise interactions for any group B_i and any player $j \in B_i$, the following relations hold true:*

$$\mathrm{Sh}_j[v_{2i}] = \mathrm{Sh}_j[v_{2i}^{\mathrm{PI}}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i))/|B_i|,$$

$$\mathrm{Sh}_j[\tilde{v}_{2i}] = \begin{cases} \mathrm{Sh}_j[v_{2i}^{\mathrm{PI}}] + (\mathrm{Sh}_{B_i}[v_1] - v_1(B_i)), & j = i, \\ \mathrm{Sh}_j[v_{2i}^{\mathrm{PI}}], & j \neq i, \end{cases}$$

$$\mathrm{Sh}_j[\tilde{\tilde{v}}_{2i}] = \mathrm{Sh}_j[v_{2i}].$$

Proof. This statement directly follows from Proposition 1 by noting that $v_{2i}^{\mathrm{PI}} = v|_{B_i}$, $i = 1, \ldots, n$.

5 Conclusion

In the paper, we have proposed an allocation procedure for a two-level communication structure. At the upper level, we allocate the total payoff among the groups of players, and then the designated values are allocated within these groups. This approach can find its application in hierarchical structures or organizations. In the paper, we illustrated the results for the Shapley value chosen as an cooperative solution (allocation rule) at both levels. Definitely, the model is flexible with respect to the selection of cooperative solutions and is not limited to the Shapley value only. Moreover, in the model we can adopt two different allocation rules at the upper and lower levels of communication (even solutions from the non-transferable utility theory). In some instances, the solutions for games with non-transferable utility, for example, the Nash bargaining solution, could fit the model more precisely [1], in particular, when at the upper level groups have payoffs of different types. Once the total payoff has been allocated, players can use a cooperative solution for transferable payoffs within the group as its members can have payoffs of the same type. In this case, one should use proper allocations at both levels of communication. The model may be extended to the case of multiple levels of hierarchy, and we hope that the proposed in the paper technique can be adopted to it. This is left for future research.

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