



Games and Random Search

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Abstract. We use dynamics of measures, i.e. iteration of the operators from measurable space to space of probabilistic measures on this space, to model and prove properties of random search algorithms. Specifically using this technique in the context of Game Theory we show that stochastic better response dynamics, where players in the potential game perform their moves independently choosing the random strategy improving their outcome, converges in stochastic sense to playing strategies near equilibrium.

Keywords: Game Theory · Random search · Measure dynamics

1 Introduction

Main aim of the paper is to study an effective method of mathematical analysis of random searching as dynamics in the space of probabilistic measures. This method allows to relatively easily prove the convergence of the random search algorithms, also those realized in multi-agent scenarios. Although work was conducted with applications to wireless networks in mind, discussion is abstract making its outcomes applicable in the broad context. Specifically, we will show correspondence between “classical” random search and random search, as may be realized in Game Theoretic problem of finding equilibria. Formalism used to capture stochasticity here is inspired by [1] and applications of measure dynamics for proving convergence of the random search algorithms in [4] and [6] (where it is applied to much more sophisticated algorithms than considered in this work).

The paper has a following structure. First, we introduce the method of measure dynamics by presenting some basic facts and a simple example of random search in optimization. Results in this section largely overlap with results in [5], but the formal approach is different. In the next section, we pose the problem of finding equilibrium of a game as an optimization problem, introducing some auxiliary function associated with a game and proving some of its basic properties. Main result, i.e. convergence of the stochastic best response dynamics in potential games to equilibria as anticipated in [3], is proven in the next section. In the last section, we comment on the main result, discuss assumptions under which it is proven and outline further development and applications. The appendix

contains some important facts and proofs required for the completeness of the proof of the main theorem but are not essential from the point of view of its exposition. We will use following notation and conventions in the text:

- $U_\epsilon = f^{-1}((\epsilon, \infty))$ for $f : S \rightarrow \mathbb{R}$, where S and f is known from context. In text care will be taken to clearly specify this context.
- For $s \in \prod_{j \in P} S_j$ and $x \in S_i$ we have $(x, s_{-i}) = y \in \prod_{j \in P} S_j$, such that $s_j = y_j$ for $j \neq i$ and $x = y_i$.
- For $s \in \prod_{j \in P} S_j = S$ we will denote by $\iota_{s,i} : S_i \rightarrow S$ embedding defined as $\iota_{s,i}(x) = (x, s_{-i})$.
- For a measurable space (X, σ) by $\mathcal{P}(X)$ we will denote the set of all probabilistic measures on X .

2 Stochastic Search

Random search is the well established, generic method of solving optimization problem or finding an object satisfying some special criteria. The latter is to some extent a special case of the former, since one can think of finding objects as of finding maxima of a characteristic function. This random search method is also known as stochastic optimization or stochastic search. It is an important component of methods known in Operational Research as Metaheuristics [2].

There are numerous variants of the method. In this section, we will only consider one, which is the easiest for formal analysis. Other will be only mentioned. For more sophisticated examples one can refer e.g. to [4] or [6] and to literature referred by these works.

Let (X, σ) be a measurable space (σ is a σ -algebra on X). Let f be a function $f : X \rightarrow \mathbb{R}$ Let μ_0 be some probabilistic measure on (X, σ) being the “reference” measure in further discussion. Let $\mathcal{P}(X)$ be the set of probabilistic measures on (X, σ) General problem we are trying to solve is to find $x^* \in X$ such that certain maximality condition is satisfied. Exact form of this condition depends on the quality of the solution we want to achieve. Possible forms of such a condition are:

$$\begin{aligned}
 f(x^*) &= \max_{x \in X} f(x) \text{ (if } x^* \text{ exists, e.g. } X \text{ is finite or } X \text{ is compact and } f \text{ is} \\
 &\text{continuous)} \\
 f(x^*) &> \sup_{x \in X} f(x) - \epsilon \text{ for some } \epsilon > 0 \\
 \mu_0(\{x : f(x) > f(x^*)\}) &< \epsilon \text{ for some } \epsilon > 0
 \end{aligned}$$

Last condition, which amounts to finding the solution difficult to beat, is not vastly discussed in the literature, however it seems to be potentially important in certain applications like games, risk management, security etc.

Essentials of the method lie in finding stochastic (Markov) operator

$$A : X \rightarrow \mathcal{P}(X)$$

such that condition similar to:

input : terminating condition: $C : \mathbb{R} \times \mathbb{N} \rightarrow \{True, False\}$
input : objective function: $f : X \rightarrow \mathbb{R}$
input : Initial point: $x_0 \in X$
output: $x^* \in X$
initialize variables: $x = x_0$;
 $n = 0$;
while $C(f(x), n) = False$ **do**
 | $x = sample(A(x))$;
 | $n = n + 1$;
end
return x

Algorithm 1. Basic random search

$$\mathbb{E}_{A(x)}[f] = \int_X f dA(x) > f(x)$$

is satisfied sufficiently often. For reference we will call this condition “martingal” condition. In Operational Research setting, one can think of A as about formal representation of heuristics. Further assumptions clarifying meaning of vague term “often” and introducing conditions stronger than simple inequality depend on the particular method and the context.

Assuming that the operator A is given, and for $x \in X$ there is a method of sampling the space X according to probability measure $A(x)$ one can give computational procedure as Algorithm 1. In more precise formulation this means, that given $A(x)$ one has also $S(x)$ which is a random variable with values in the space X and distribution $A(x)$. In practice, it is most often the case that sampling method for $A(x)$ is derived from the given sampling method for measure μ_0 - see e.g. [5]¹.

It is worth to note, that non-stochastic search can be treated as a special case of the stochastic search, where $A(x) = \delta_{D(x)}$ where $D : X \rightarrow X$ is a deterministic operator given by the search method and δ_x is a proper Dirac’s measure, (i.e. $\int_X g d\delta_x = g(x)$ for each $g : X \rightarrow \mathbb{R}$).²

¹ Opposite direction, where one has simple method of sampling for $A(x)$ but sampling X is difficult, is also used in practice. When μ_0 is stationary distribution of Markov process given by A , famous Metropolis algorithm is such an example.

² To some extent one can look at the stochastic search method as having significant advantage over the deterministic one, because deterministic method relies on some internal structure of the X . If function we are maximizing is incompatible with this internal structure, deterministic search may be completely lost. A method of sampling assumed is in this case a way to escape from this pitfall. On the other hand, ability to sample requires some source of randomness - using of pseudo-random variables for sampling makes the method de facto deterministic. Intuition when such a method cannot work efficiently comes from Algorithmic Information Theory. In purely computable setting there is no randomness. This is an interesting philosophical aspect of random search.

Example 1 (Random search). For $v \in \mathbb{R}$ let's denote $U_v = \{x \in X : f(x) > v\}$ and define operator A (remembering, that $A(x)$ is a measure):

$$A(x)(B) = \begin{cases} \mu_0(B \cap U_{f(x)}) + \mu_0(X - U_{f(x)}) & x \in B \\ \mu_0(B \cap U_{f(x)}) & \text{otherwise} \end{cases}$$

Please note, that this operator corresponds to $A(x)$ sampling procedure that uses sampling for the μ_0 . We can, with some small abuse of the notation, express it as:

$$\text{sample}(A(x)) = \begin{cases} y & y = \text{sample}(\mu_0) \text{ and } f(y) > f(x) \\ x & y = \text{sample}(\mu_0) \text{ and } f(y) \leq f(x) \end{cases}$$

Less formally we can say, that at each step we are drawing an element according to distribution μ_0 and return this element if the value of f on this element is greater than $f(x)$ or return x otherwise.

We can check in this case:

$$\begin{aligned} \mathbb{E}_{A(x)}[f] &= \int_X f dA(x) = \int_{X - U_{f(x)}} f dA(x) + \int_{U_{f(x)}} f dA(x) = \\ &= f(x)\mu_0(X - U_{f(x)}) + \int_{U_{f(x)}} f dA(x) \geq f(x)\mu_0(X - U_{f(x)}) + f(x)\mu_0(U_{f(x)}) = f(x) \end{aligned}$$

If additionally $\mu_0(U_{f(x)}) > 0$ aforementioned weak inequality becomes strong. So to have “martingal” condition satisfied one has to assume something about f (to satisfy this “sufficiently often” part). E.g. assumption that $\mu_0(\{x \in X : \mu_0(U_{f(x)}) > 0\}) = 1$ may be a good in this case.

As the extension of Example 1 we can assume stronger conditions on X postulating additional structure e.g. that X is topological or metric space and require from f to be compatible with this structure: e.g. to be continuous.

Example 2 (Simple local search). Let's assume, that X is sufficiently regular topological space (e.g. metrical), where τ denotes topology and let σ be a Borrel's σ -algebra (i.e. the smallest σ -algebra containing all open sets). Let $N : X \rightarrow \tau$ be such, that $x \in N(x)$. We can define operator $A : X \rightarrow \mathcal{P}(X)$:

$$A(x)(B) = \begin{cases} \frac{1}{\mu_0(N(x))} \mu_0(B \cap U_{f(x)} \cap N(x)) + \mu_0((X - U_{f(x)}) \cap B \cap N(x)) & x \in B \\ \frac{1}{\mu_0(N(x))} \mu_0(B \cap U_{f(x)} \cap N(x)) & \text{otherwise} \end{cases}$$

This operator corresponds to “local search” where sampling is performed over some neighborhood $N(x)$ of x . Additional assumptions about $N(x)$ should be made to allow Algorithm 1 to escape from local maxima. Particularly bad choice of X and N may result in not all the space being reachable (so, in the language of the Markov chains, process is not irreducible).

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input : terminating condition:  $C : \mathbb{R} \times \mathbb{N} \rightarrow \{True, False\}$ 
input : objective function:  $f : X \rightarrow \mathbb{R}$ 
input : Initial point:  $x_0 \in X$ 
output:  $x^* \in X$ 
initialize variables:  $x = x_0$ ;
 $n = 0$ ;
 $s = s_0$  while  $C(f(x), n) = False$  do
    |  $x = \text{sample}(\pi_1(A(x, s)))$ ;
    |  $s = \pi_2(A(x, s))$ ;
    |  $n = n + 1$ ;
end
return  $x$ 

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Algorithm 2. State random search

In both examples, the choice of the operators A prevented sampling process from decreasing value of the $f(x)$ along the trajectory constituted by selection of subsequent x s in the Algorithm 1. This may not necessarily be the case in random searching. E.g it is not the case in the famous simulated annealing algorithm, where there are possible some moves (choices of the points) decreasing optimized function. Their probability however as a function of number of the step in which the choice is made is decreasing. This means also, that in this kind of algorithm we do not have a fixed A operator, but we use different operators in subsequent steps.

To properly explore this extended setting, one, following e.g. [2], may extend definition by introducing state space. We take (X, σ) as previously, but introduce space S which we can think of as about some “memory” or “learning ability”. We take operator $A : X \times S \rightarrow \mathcal{P}(X) \times S$. We can now state Algorithm 2.

In the Algorithm 2 π_i means projection on the i -th component in the Cartesian product. Let’s observe, that previous, stateless operators A can be treated as special cases of these new A when we consider dummy, single element, state space $\{*\}$ and define $A(x, *) = (A(x), *)$ We can also fit into this model case, where A changes as a function of step as in the simulated annealing. In this case operator is of the form $A(x, n) = (A_n(x), n + 1)$.

Now, let’s analyze issues of convergence of the algorithms. Let’s define measures μ_i for $i > 0$ inductively by defining for $B \in \sigma$ $\mu_{n+1}(B) = \int_X A_n(x)(B) d\mu_n$. Under some technical conditions, that we will not discuss for this case, by Fubini’s theorem μ_i is well defined and is indeed a measure. We have:

Theorem 1. *If for each i , $A_i : X \rightarrow \mathcal{P}(X)$ is such, that $\mathbb{E}_{A_i(x)}[f] > f(x)$ then for each i $\mathbb{E}_{\mu_{i+1}}[f] \geq \mathbb{E}_{\mu_i}[f]$.*

Proof. Let $f_k : X \rightarrow \mathbb{R}$ be a function such that $f_k|_{B_k} \equiv c_k$ and $f_k|_{X-B_k} \equiv 0$ for certain $B_k \in \sigma$, $c_k \in \mathbb{R}$. For any ϵ there is countable K and family $\{B_k\}_{k \in K}$ such that for each $k \in K$ $B_k \in \sigma$ and sets are pairwise mutually exclusive, such that step function $F_\epsilon = \sum_{k \in K} f_k \leq f$ and $\int_X f d\mu_i - \int_X F_\epsilon d\mu_i \leq \epsilon$ and $\int_X f d\mu_{i+1} - \int_X F_\epsilon d\mu_{i+1} \leq \epsilon$ Now, taking some fixed ϵ we have:

$$\begin{aligned}
\int_X f d\mu_{i+1} &> \int_X F_\epsilon d\mu_{i+1} = \int_X \left(\sum_{k \in K} f_k \right) d\mu_{i+1} = \sum_{k \in K} \int_X f_k d\mu_{i+1} = \\
&\sum_{k \in K} c_k \mu_{i+1}(B_k) = \sum_{k \in K} c_k \int_X A_i(x)(B_k) d\mu_i = \\
&\int_X \left(\sum_{k \in K} c_k A_i(x)(B_k) \right) d\mu_i = \int_X \left(\int_X \left(\sum_{k \in K} f_k \right) dA_i(x) \right) d\mu_i
\end{aligned}$$

Taking subsequent approximation f by step functions $\sum_{k \in K} f_k$ we have the last expression converging to

$$\int_X \int_X f dA_i(x) d\mu_i = \int_X \mathbb{E}_{A_i(x)}[f] d\mu_i \geq \int_X f d\mu_i = \mathbb{E}_{\mu_i}[f]$$

We also have following form of convergence of measures to maximal solution:

Theorem 2. *Let's fix $\gamma > 0$. If for each i and each x $A_i(x)(U_{max-\epsilon}) > \gamma$ and $A_i(x)(X - U_{f(x)}) = A(x)(x)$ then $\mu_i(U_{max-\epsilon}) \xrightarrow{i \rightarrow \infty} 1$.*

Proof.

$$\begin{aligned}
\mu_{i+1}(U_{max-\epsilon}) &= \int_X A_i(x)(U_{max-\epsilon}) d\mu_i = \\
&\int_{U_{max-\epsilon}} A_i(x)(U_{max-\epsilon}) d\mu_i + \int_{X-U_{max-\epsilon}} A_i(x)(U_{max-\epsilon}) d\mu_i \geq \\
&\int_{U_{max-\epsilon}} 1 d\mu_i + \int_{X-U_{max-\epsilon}} \gamma d\mu_i = \mu_i(U_{max-\epsilon}) + \gamma(1 - \mu_i(U_{max-\epsilon})) = \\
&(1 - \gamma)\mu_i(U_{max-\epsilon}) + \gamma
\end{aligned}$$

Now we can inductively obtain:

$$\mu_i(U_{max-\epsilon}) \geq (1 - \gamma)^i \mu_0(U_{max-\epsilon}) + \gamma \left(\sum_{k=0}^{i-1} (1 - \gamma)^k \right) = 1 + (1 - \gamma)^i (\mu_0(U_{max-\epsilon}) - 1)$$

Thus $\mu_i(U_{max-\epsilon}) \xrightarrow{i \rightarrow \infty} 1$

3 Finding Equilibrium as Optimization Problem

Let's consider, as in the Game Theory, a game in extensive form:

Definition 1 (Game). A game Γ in extensive form is triple $(P, \{S_i\}_{i \in P}, \{u_i\}_{i \in P})$ where P is a (finite) set of players, for each $i \in P$, S_i is a set of strategies (or actions) available to player i and

$$u_i : \prod_{i \in P} S_i \rightarrow \mathbb{R}$$

is a function called "payoff" of player i . For convenience we will denote $S = \prod_{i \in P} S_i$.

Important notion in the Game Theory is the Nash Equilibrium (NE). A point $s \in S$ is NE when no player wants to deviate from it individually. Formally

Definition 2 (Nash Equilibrium). Having game Γ , under the notation of Definition 1, an element $s \in S$ is Nash Equilibrium if for each $i \in P$ and for each $s'_i \in S_i$ $u_i(s'_i, s_{-i}) \leq u_i(s)$.

As it is vastly discussed in the Game Theory literature, NE models situation which will be stable in the game assuming that players are perfectly rational, which means they want to secure highest possible payoff for themselves. We will denote:

$$N_\Gamma = \{x \in S : x \text{ is NE for } \Gamma\}$$

For game Γ let's define function $\chi_\Gamma \rightarrow \mathbb{R} \cup \{-\infty\}$:

$$\chi_\Gamma(s) = \min_{i \in P} (\min_{x_i \in S_i} (u_i(s) - u_i(x_i, s_{-i})))$$

In general χ_Γ may be not defined for all points (or rather take infinite values as some points). In practice however we often assume more about structure of the strategies and payoff function, which makes χ_Γ behaving more regularly and having its range in the reals.

Theorem 3. Let game $\Gamma = (P, \{S_i\}_{i \in P}, \{u_i\}_{i \in P})$ be such that for each $i \in P$ (S_i, d_i) is compact metric space. Let for each i u_i be continuous in topology on S given by the metric equivalent to product metric on S i.e. metric d such that $d(s, s') = \sum_{i \in P} d_i(s_i, s'_i)$. Under these assumptions $\chi_\Gamma(S) \subset \mathbb{R}$ and χ_Γ is continuous.

Proof. For each $i \in P$ and $s \in S$ function $\phi_{i,s}(x) = u_i(x, s_i) - u_i(s)$ is continuous, so because S_i is compact it attains minimum. Thus $\chi_\Gamma(s) = \min_{i \in P} \min \phi_{i,s} \in \mathbb{R}$.

Now, let's take $s \in S$ and $\epsilon > 0$. Since S is compact each u_i and ϕ_i is uniformly continuous. There is δ_i such that $d(x, y) < \delta_i$ implies $|u_i(x) - u_i(y)| < \epsilon$ Now if $d(x, y) < \delta_i$ we have for each t $|\phi_{i,x}(t) - \phi_{i,y}(t)| < \epsilon$ (since $d((t, x_{-i}), (t, y_i)) \leq$

$d(x, y)$). This implies that $|\min \phi_{i,x} - \min \phi_{i,y}| < \epsilon$ So, taking $\delta < \min_{i \in P} \delta_i$ we have, that for each s' such that $d(s, s') < \delta$ $|\chi_\Gamma(s) - \chi_\Gamma(s')| = |\min_{i \in P} \min \phi_{i,S} - \min_{i \in P} \min \phi_{i,S'}| < \epsilon$

Similarly we have following:

Theorem 4. *Let game $\Gamma = (P, \{S_i\}_{i \in P}, \{u_i\}_{i \in P})$ be such that for each $i \in P$ (S_i, d_i) is compact metric space. Let for each i u_i be Lipschitz with respect to metric equivalent to product metric on S . Under these assumptions $\chi_\Gamma(S) \subset \mathbb{R}$ and χ_Γ is Lipschitz.*

Proof. For each $i \in P$ let L_i be a Lipschitz constant for the u_i . We will give proof by contradiction. Let's assume, that χ_Γ is not Lipschitz. So for each $M > 0$ there exists $s \in S$ and $s' \in S$ such that:

$$|\chi_\Gamma(s) - \chi_\Gamma(s')| > Md(s, s') = M \left(\sum_{i \in P} d_i(s_i, s'_i) \right)$$

Let's take $M > 2 \times \max_{i \in P} L_i$ and s and s' such this inequality is fulfilled. Without loss of the generality we will assume, that $\chi_\Gamma(s) > \chi_\Gamma(s')$. From inequality above we have:

$$\chi_\Gamma(s') < \chi_\Gamma(s) - Md(s, s')$$

Since each u_i is continuous we can find:

$$j \in P, y \in S_j \text{ such that } \chi_\Gamma(s') = u_j(s') - u_j(y, s_{-j})$$

Now, we have:

$$u_j(s) - u_j(y, s_{-j}) = u_j(s') + (u_j(s) - u_j(s')) - u_j(y, s'_{-j}) + (u_j(y, s_{-j}) - u_j(y, s'_{-j}))$$

so:

$$\begin{aligned} u_j(s) - u_j(y, s_{-j}) &\leq \chi_\Gamma(s') + |u_j(s) - u_j(s')| + |u_j(y, s_{-j}) - u_j(y, s'_{-j})| \\ &\leq \chi_\Gamma(s') + 2L_j d(s', s) \end{aligned}$$

So, from definition of χ_Γ and assumptions about s, s' and M one has:

$$\begin{aligned} \chi_\Gamma(s) &\leq u_j(s) - u_j(y, s_{-j}) \leq \chi_\Gamma(s') + 2L_j d(s', s) < \\ &\chi_\Gamma(s) - (M - 2L_j) d(s', s) < \chi_\Gamma(s) \end{aligned}$$

what is a contradiction.

Obviously, assumption about the metric d being the exact product metric in previous theorems is purely technical and simplifies proofs. In fact, any metric that is equivalent to product metric will work, but inequalities to consider will be somewhat more complicated.

We have a following observation:

Theorem 5. *Let Γ be a game. $s \in \prod_{p \in P} S_i$ is NE of Γ if and only if s is maximum of χ_Γ and $\chi_\Gamma(s) = 0$.*

Proof. Let s be a NE. Let's take $i \in P$. By the Definition 2 for each $x_i \in S_i$ we have

$$u_i(x_i, s_{-i}) \leq u_i(s).$$

This implies

$$\min_{x_i \in S_i} (u_i(s) - u_i(x_i, s_{-i})) \geq 0$$

so in fact

$$\min_{x_i \in S_i} (u_i(s) - u_i(x_i, s_{-i})) = 0$$

because equality is reached at $x_i = s_i$. Since we can follow this for each $i \in P$ we have for s

$$\min_{i \in P} (\min_{x_i \in S_i} (u_i(s) - u_i(x_i, s_{-i}))) = 0$$

For s which is not NE, for at least one i we have $u_i(x_i, s_{-i}) \leq u_i(s)$ so $\min_{x_i \in S_i} (u_i(s) - u_i(x_i, s_{-i})) < 0$ and in consequence $\chi_\Gamma(s) < 0$. Thus if s is NE, s is maximum of χ_Γ . Proof in opposite direction basically goes along the same lines and will be omitted.

Let's observe, that although under assumptions about game Γ as in the Theorem 3 the χ_Γ always has maximum. Value of this maximum may be different than 0. This corresponds to situation when game has no Nash Equilibrium.

Now, let's define, often more practical then NE, notion of ϵ -Nash Equilibrium:

Definition 3 (ϵ -Nash Equilibrium). *Given $\epsilon > 0$ and game Γ as in Definition 1, point $s \in S$ is ϵ -Nash Equilibrium (ϵ -NE in short) if for each $i \in P$ and each $x \in S_i$ we have $u_i(x, s_{-i}) \leq u_i(s) + \epsilon$*

We may denote set of ϵ -NE of the game: $NE_{\Gamma, \epsilon} = \{s : s \text{ is } \epsilon\text{-NE}\}$ We have simple theorem:

Theorem 6. *$NE_{\Gamma, \epsilon}$ has following properties:*

1. $NE_{\Gamma, \epsilon} = \chi_\Gamma^{-1}((-\epsilon, \infty]) = \chi_\Gamma^{-1}((-\epsilon, 0])$
2. $NE_\Gamma \subset NE_{\Gamma, \epsilon'} \subset NE_{\Gamma, \epsilon}$ for each ϵ, ϵ' such that $0 < \epsilon' \leq \epsilon$
3. under assumption of Theorem 3 $NE_{\Gamma, \epsilon}$ is closed and if $\epsilon > \epsilon' > 0$ and $NE_{\Gamma, \epsilon'} \neq \emptyset$ then $\text{int}(NE_{\Gamma, \epsilon}) \neq \emptyset$ where int denotes operation of taking interior of the set.

Proof. First two items are direct consequence of definition. Last item is consequence of first two and fact, that χ_Γ is continuous.

4 Better Response Dynamics in Potential Game

It is well known that following class of games is important from the point of view of applications (see e.g. [3]):

Definition 4 (Potential Game). *Let Γ be a game as in Definition 1. We call it potential game if there exists function $V : S \rightarrow \mathbb{R}$ called potential, such that for each $i \in P$, for each $x \in S_i$ and for each $s \in S$ we have:*

$$u_i(x, s_{-i}) - u_i(s) = V(x, s_{-i}) - V(s)$$

We call Γ ordered potential game if there exists $V : S \rightarrow \mathbb{R}$, called potential, such that for each $i \in P$ for each $x \in S_i$ and for each $s \in S$ we have:

$$u_i(x, s_{-i}) - u_i(s) > 0 \iff V(x, s_{-i}) - V(s) > 0$$

We have following theorem that we will leave without a proof (and we will not use it further in the paper).

Theorem 7. *If game Γ is like in the assumption of Theorem 3 and is a potential game, then potential V is continuous.*

Main aim of this section is to prove that certain way of playing (which we call stochastic better response) multiple times the same potential game leads to playing almost surely strategy that is ϵ -NE

We will assume that we have a game Γ that is potential, compact, payoffs are continuous and that potential of the game is also continuous. By stochastic better response dynamics we will call a process with discrete time T , such that at each moment in time one of the player performs a move (so unilaterally chooses strategy) in such a way that this strategy gives the same or better outcome than the strategy chosen recently assuming that adversaries play their recent strategies. We assume that a strategy is random - so “chooses” means chooses randomly from the set of strategies according to some probability measure. We assume that which player is making move at time t is random variable, that for different times variable are independent and probability of choosing any player is non-zero³.

Let's put it in the following, more formal, framework. Let P be finite. Let (S_i, d_i) be a compact metric space for each $i \in P$. Let σ_i be a Borel σ -algebra on S_i . Let $S = \prod_{i \in P} S_i$. Obviously, there is a natural product metrics on S and σ -algebra of Borel sets for S is $\sigma = \bigotimes_{i \in P} \sigma_i$ (product σ -algebra). Let each S_i be

³ To some extent it can model situation where players are continuously playing their strategies and making decisions about change of the strategy in the moments that are distributed according to a Poisson process. Probability of choice of the same moments for some players is then 0, so we can concentrate only on the order of changes.

equipped with measure $\mu_0^i \in \mathcal{P}(S_i)$ and we take measure $\bigotimes_{i \in P} \mu_0^i = \mu_0 \in \mathcal{P}(S)$. For $s \in S$ and $i \in P$ let's define embedding function $\iota_{s,i} : S_i \rightarrow S$ by

$$\iota_{s,i}(x) = (x, s_{-i})$$

This definition induces operators of type $\tilde{\mathcal{V}}_i : S \rightarrow \mathcal{P}(S)$. For Borel sets B we define:

$$\tilde{\mathcal{V}}_i(s)(B) = \mu_0^i(\iota_{s,i}^{-1}(B))$$

Let's note, that, obviously, $\mathcal{V}_i(s)$ is no absolutely continuous relative to μ_0 . Let $\alpha \in \prod_{i \in P} (0, 1)$ and such that $\sum_{i \in P} \alpha_i = 1$. Note, that for each i $\alpha_i > 0$. We define $\tilde{A} : S \rightarrow \mathcal{P}(S)$ as

$$\tilde{A}(s) = \sum_{i \in P} \alpha_i \tilde{\mathcal{V}}_i(s)$$

By analogy to examples discussed in the Sect. 2, this operator models situation when at some position s one of the players (chosen according to distribution α) choses unilaterally strategy, randomly according to measure μ_0^i . By modeling here it is understood, that random choice, is lifted to the space of measures.

Now we will introduce operator A that models stochastic better response dynamics, so procedure where at each step the single random player chooses better strategy randomly. Choice for each player is performed according to the distribution which corresponds to the measure μ_0^i restricted to the set of better strategies and normalized.

For $i \in P$ and $s \in S$ let's denote:

$$U_{i,s} = \iota_{i,s}^{-1}([u_i(s), \infty)) = \{x \in S_i : u_i(x, s_{-i}) \geq u_i(s)\}$$

We take $A : S \rightarrow \mathcal{P}(S)$ defined as

$$A(s) = \sum_{i \in P} \alpha_i \mathcal{V}_i(s)$$

where

$$\mathcal{V}_i(s)(B) = \begin{cases} \frac{\tilde{\mathcal{V}}_i(s)(B \cap U_{i,s})}{\tilde{\mathcal{V}}_i(s)(U_{i,s})} & \tilde{\mathcal{V}}_i(s)(U_{i,s}) > 0 \\ 1 & \tilde{\mathcal{V}}_i(s)(U_{i,s}) = 0, s \in B \\ 0 & otherwise \end{cases}$$

Let us, as in the Sect. 2, introduce sequence of measures starting of μ_0 inductively:

$$\mu_{n+1}(B) = \int_X A_n(x)(B) d\mu_n$$

for B being the Borel set. μ_n is distribution of n -th points on the trajectory of stochastic better response dynamics, when the initial choice of starting point was made according to distribution μ_0 and subsequent choices are governed by distributions given by A .

Now we will formulate theorem, being an variant of theorem postulated in [3].

Theorem 8. *Let Γ be a compact ordinal potential game with the Lipschitz potential V and let each S_i is endowed with σ -algebra of Borel sets and probabilistic measure μ_0^i satisfying additional assumption (*): for each $\delta > 0$, $\inf_{x \in S_i} \mu_0^i(B_i(x, \delta)) > 0$.*

Let $\epsilon > 0$. We further assume that S is endowed with product metrics, product σ -algebra and product measure. Let's consider better response dynamics described by the operator A above. We have: $\mu_n \xrightarrow{n \rightarrow \infty} 1$. $n \rightarrow 1$

Proof. For the simplification of the argument we will assume that $\min_{x \in S} V(x) = 0$. We do not loose generality by making this assumption since it is not value of potential but difference of the potential in different points that matters in definition of potential game (by adding constant to potential one still obtains valid potential for the game)

Let's take $\epsilon' = \frac{\epsilon}{2}$. We define:

$$f(s) = A(s)(U_{V(s)+\epsilon'})$$

We claim, that:

1. f is measurable
2. there is $\delta > 0$ such that $f(s) > \delta$ for all s in compact set $\chi_\Gamma^{-1}((-\infty, \epsilon])$.

We will postpone technical proof of first property to Appendix. Last property follows from the fact that, as we have already observed if the $\chi_\Gamma(s) < \epsilon$ there is i and $x \in S_i$ such that $V(x, s_{-i}) - V(s) = u_i(x, s_{-i}) - u_i(s) > \epsilon$. So $V(x, s_{-i}) > V(s) + \epsilon$. From Lipschitzity of V , for $y \in B((x, s_{-i}), \frac{\epsilon'}{L})$ we have

$$V(y) > V(x, s_{-i}) - \epsilon' > V(s) + \epsilon - \epsilon' = V(s) + \epsilon'$$

We see, that $B((x, s_{-i}), \frac{\epsilon'}{L}) \subset U_{V(s)+\epsilon'}$

This gives:

$$B_i(x, \frac{\epsilon'}{L}) \subset \{x \in S_i : (x, s_{-i}) \in B((x, s_{-i}), \frac{\epsilon'}{L})\}$$

and

$$B_i(x, \frac{\epsilon'}{L}) \subset U_{i,s}$$

Since we have assumed (*) we have lower bound:

$$\mathcal{V}_i(s)(U_{V(s)+\epsilon'}) > \inf_{x \in S_i} \mu_0^i(B_i(x, \frac{\epsilon'}{L})) = \delta_i$$

Thus, we can take $\delta = \min_{i \in P} \alpha_i \delta_i$ and we have required lower bound for f .

Let $\Delta_k = V^{-1}([(k\epsilon', (k+1)\epsilon')])$ for $k \in \mathbb{N}$. Let's further denote:

$$\Delta_{E,k} = \Delta_k \cap \chi_{\Gamma}^{-1}((-\epsilon, 0])$$

$$\Delta_{N,k} = \Delta_k \cap \chi_{\Gamma}^{-1}((-\infty, -\epsilon])$$

We obviously have following properties:

$$\Delta_k = \Delta_{E,k} \cup \Delta_{N,k}$$

There exists $M \in \mathbb{N}$ s.t.

$$\bigcup_{0 \leq k < M} \Delta_k = S$$

and

$$\bigcup_{0 \leq k < M} \Delta_{E,k} = \chi_{\Gamma}^{-1}((-\epsilon, 0])$$

$$\bigcup_{0 \leq k < M} \Delta_{N,k} = \chi_{\Gamma}^{-1}((-\infty, -\epsilon])$$

By the definition of the operator A , we have for $j < i$ and $s \in \Delta_i$

$$A(s)(\Delta_j) = 0 \quad (**)$$

For each i we will show:

1. there exists a_k such that $\mu_n(\Delta_k) \xrightarrow{n \rightarrow \infty} a_k \geq 0$
2. $\mu_n(\Delta_{N,k}) \xrightarrow{n \rightarrow \infty} 0$

Now we will follow inductive reasoning. First we will prove, that $\mu_n(\Delta_{N,0}) \xrightarrow{n \rightarrow \infty} 0$ Let's observe, that, because of the property (**):

$$\begin{aligned} \mu_{n+1}(\Delta_0) &= \int_S A(s)(\Delta_0) d\mu_n = \\ &= \int_{\Delta_0} A(s)(\Delta_0) d\mu_n + \int_{S-\Delta_0} A(s)(\Delta_0) d\mu_n = \int_{\Delta_0} A(s)(\Delta_0) d\mu_n \leq \mu_n(\Delta_0) \end{aligned}$$

So, $\mu_n(\Delta_0)$ is decreasing sequence, thus there is limit $\mu_n(\Delta_0) \xrightarrow{n \rightarrow \infty} a_0 \geq 0$.

Now, let's assume, that $\mu_n(\Delta_{N,0})$ does not tend to 0 i.e. that there is λ such that for each $M \in \mathbb{N}$ there is $n_M > M$ such that $\mu_{n_M}(\Delta_{N,0}) > \lambda$. Let's take M such that for all $n > M$

$$0 < |\mu_n(\Delta_0) - a_0| = (\mu_n(\Delta_0) - a_0) < \frac{\lambda\delta}{2}$$

Now, let's take $n = n_M > M$ as above. We have:

$$\begin{aligned}
\mu_{n+1}(\Delta_0) &= \int_{\Delta_{E,0}} A(s)(\Delta_0) d\mu_n + \int_{\Delta_{N,0}} A(s)(\Delta_0) d\mu_n = \\
&\int_{\Delta_{E,0}} A(s)(\Delta_0) d\mu_n + \int_{\Delta_{N,0}} A(s)(S - (S - \Delta_0)) d\mu_n = \\
&\int_{\Delta_{E,0}} A(s)(\Delta_0) d\mu_n + \int_{\Delta_{N,0}} A(s)(S) d\mu_n - \int_{\Delta_{N,0}} A(s)(S - \Delta_0) d\mu_n \leq \\
&\int_{\Delta_{E,0}} A(s)(\Delta_0) d\mu_n + \int_{\Delta_{N,0}} A(s)(S) d\mu_n - \int_{\Delta_{N,0}} A(s)(U_{V(s)+\epsilon'}) d\mu_n = \\
&\int_{\Delta_{E,0}} A(s)(\Delta_0) d\mu_n + \int_{\Delta_{N,0}} A(s)(S) d\mu_n - \int_{\Delta_{N,0}} f d\mu_n < \\
&\mu_n(\Delta_{E,0}) + \mu_n(\Delta_{N,0}) - \mu_n(\Delta_{N,0})\delta \leq \mu_n(\Delta_0) - \delta\lambda \leq \\
&a_0 + \frac{\lambda\delta}{2} - \lambda\delta = a_0 - \frac{\lambda\delta}{2}
\end{aligned}$$

First weak inequality above comes from the fact, that for $s \in \Delta_0$ we have $U_{V(s)+\epsilon'} \subset S - \Delta_0$. This gives contradiction, because

$$\mu_{n+1}(\Delta_0) - a_0 \leq a_0 - \frac{\lambda\delta}{2} - a_0 = -\frac{\lambda\delta}{2}$$

so

$$|\mu_{n+1}(\Delta_0) - a_0| \geq \frac{\lambda\delta}{2}$$

what contradicts our choice of n .

Now, let's assume that assumptions are true for $l \in \{0, \dots, k-1\}$ and we will show, that they are true also for k . First, let's notice, that as in the case of $k=0$, because of condition (***) for each n one has:

$$\begin{aligned}
\mu_{n+1}\left(\bigcup_{l \leq k} \Delta_l\right) &= \int_S A(s)\left(\bigcup_{l \leq k} \Delta_l\right) d\mu_n = \\
&\int_{\bigcup_{l \leq k} \Delta_l} A(s)\left(\bigcup_{l \leq k} \Delta_l\right) d\mu_n + \int_{\bigcup_{l > k} \Delta_l} A(s)\left(\bigcup_{l \leq k} \Delta_l\right) d\mu_n = \\
&\int_{\bigcup_{l \leq k} \Delta_l} A(s)\left(\bigcup_{l \leq k} \Delta_l\right) d\mu_n \leq \mu_n\left(\bigcup_{l \leq k} \Delta_l\right)
\end{aligned}$$

Thus, $\mu_n(\bigcup_{l \leq k} \Delta_l)$ is decreasing so $\mu_n(\bigcup_{l \leq k} \Delta_l) \xrightarrow{n \rightarrow \infty} \tilde{a}_0$ for certain \tilde{a}_k . Now, since $\mu_n(\bigcup_{l \leq k} \Delta_l) = \sum_{l \leq k} \mu_n(\Delta_l)$ and from induction assumption $\mu_n(\Delta_l)$ converges for $l < k$, so we have that $\mu_n(\Delta_k)$ converges.

Now we can repeat the same reasoning as in the case $k = 0$ to show, that $\mu_n(\Delta_{N,k}) \xrightarrow{n \rightarrow \infty} 0$.

So, remembering that there is only finite k such that $\Delta_k \neq \emptyset$ we have:

$$1 - \mu_n(NE_{\Gamma,\epsilon}) = \mu_n(S - NE_{\Gamma,\epsilon}) = \mu_n\left(\bigcup_{l \in \mathbb{N}} \Delta_{N,l}\right) = \sum_{l \in \mathbb{N}} \mu_n(\Delta_{N,l}) \xrightarrow{n \rightarrow \infty} 0$$

so in consequence:

$$\mu_n(NE_{\Gamma,\epsilon}) \xrightarrow{n \rightarrow \infty} 1$$

We can observe also, that choice of the ϵ above is arbitrary so we can make ϵ arbitrarily small.

5 Comments and Future Work

Theorem 8 makes some relatively strong assumption about metric and measure theoretic properties of the strategy sets S_i and payoff functions u_i . Some of them are essential. E.g. without assumption about lower limit of measures of open balls with constant radius, theorem is not true. It is relatively easy to come up with counterexample taking $S_i = ([0, 1][0, 1]) \sqcup [0, 1]$ being the disjoint union of 2-dimensional cube and interval, where measure as restricted to cube is 2-dimensional Lebesgue (or it's restriction to Borel) and measure of the interval is 0 and preparing V to be small on product of all cubes and high on the product of all intervals. Anyway, in many natural situations assumption is satisfied. E.g. in reasonable case of S_i being the n -dimensional compact convex polytope with n -dimensional Lebesgue measure they are satisfied.

Special choice of the operator A in the Theorem 8 reflects specific random better response dynamics: assumption, that randomly chosen player samples strategy from the set of strategies being better than last strategy responses to strategies played by opponents. Additionally it assumes that the distribution according to which strategy is sampled is “uniform” with respect to measure on the space S_i . This assumes the knowledge of the better responses.

More realistic scenario would be following: random player choses strategy sampled from all available to her and verify its outcome. If the outcome is better, she stays with this new strategy. Otherwise she comes back to previously played. In this case we must assume, that no other player will move in the trial play where choice of the strategy is verified, so protocol becomes more complex. Leaving aside this complexity issue, better response dynamics under such trial and error scheme may be still modeled using our formalism. It just requires changing the \mathcal{V}_i definitions used to define operator A to following one:

$$\mathcal{V}_i(s)(B) = \begin{cases} \tilde{\mathcal{V}}_i(s)(B \cap U_{i,s}) + \tilde{\mathcal{V}}_i(S_i - U_{i,s}) & s \in B \\ \tilde{\mathcal{V}}_i(s)(B \cap U_{i,s}) & otherwise \end{cases}$$

Theorem 8 for such stochastic better response dynamics should still be valid with essentially the same proof. Also for other distributions on the set of better

strategies, as long as conditions similar to (*) are valid for them and operator A remains well defined, should lead to convergence in measure to $NE_{\Gamma, \epsilon}$. However the exact form of such a more general theorem is not yet formulated.

In general, when thinking about stochastic optimization as discussed in the Sect. 2 we treat the cost of every trial in terms of increasing number of steps executed, so a constant. In game-theoretic approach, when searching is done by playing actual game, cost of searching for equilibrium depends on the actual payoff in the game so can be potentially very high and not constant. Analysis of algorithms of reaching equilibrium from this perspective must then consider “cost of searching”.

Assumption about lack of collision between players when choosing strategy is essential in the proof of the Theorem 8, since it warranties non-decreasing of the potential V along the trajectories in the space of game profiles. It is an interesting question under what assumption about game and probability of collision one can still assure convergence to equilibrium. In such a case, some special steps have to be built in protocol executed by players to assure, that they are able to distinguish (at least with high probability) between situations when change in the outcome of payoff function is caused by their choice of the strategy and the situation when someone else moved. In such cases communication between players or implementing some “backoff” scheme may play a role.

Comparing to version postulated in [3] the Theorem 8 assumes potential game, not an ordered potential game. This stronger assumption we make seems to be rather technical and it looks like Theorem 8 remains true for ordered potential game, but the proof, even if can follow the lines of the one presented here, would be much more technical.

It is of great interest how one can go beyond assumption that game is potential. It leads to the realm of questions similar to these addressed by Young in [7] under the name of “learning by trial and errors”. A part of the planned future studies is a question if function χ_{Γ} we introduced may be used as a tool in the proofs of theorems and designing protocols of reaching equilibrium in a general game case, reducing it to searching maximum of this function.

A Appendix

Following is the proof of measurability of the function f defined in the proof of Theorem 8. We will infer it from two lemmas.

Lemma 1. *Let P be the finite set, and for each $i \in P$ S_i be a compact metric space with metric d_i . For each i by σ_i we will denote σ -algebra of Borel sets and μ_i be a regular probabilistic measure on S_i . Let $S = \prod_{i \in P}$ be a metric space with metric d :*

$$d(x, y) = \sum_{i \in P} d_i(x_i, y_i).$$

which gives a product topology in S . We denote by σ the σ -algebra of Borel set in S , and take measure $\mu = \bigotimes_{i \in P} \mu_i$. Let $V : S \rightarrow \mathbb{R}$ be continuous, $\alpha \geq 0$ and

function $g_i : S \rightarrow \mathbb{R}$ be defined as follows:

$$g_i(s) = \mu_i(\iota_{i,s}^{-1}(U_{V(s)+\alpha}))$$

where $U_t = V^{-1}((t, \infty))$. Under this assumptions we have: g_i is lower semi-continuous

Proof. We will show, that $g_i^{-1}((\rho, \infty))$ for $\rho \in \mathbb{R}$ is open. Let's assume that $g_i^{-1}((\rho, \infty)) \neq \emptyset$ (if it is empty, it is open trivially) and take $s \in g_i^{-1}((\rho, \infty))$. From definition we have:

$$\mu_i(\iota_{i,s}^{-1}(U_{V(s)+\alpha})) > \rho$$

Since μ_i is regular, there is a compact set K such that $K \subset \iota_{i,s}^{-1}(U_{V(s)+\alpha})$ and $\mu_i > \rho$ Obviously, $\iota_{i,s}(K)$ is compact so V takes it's extrema on K . Thus, there is $\epsilon > 0$ such that, for each $x \in \iota_{i,s}(K)$ we have $V(x) > V(s) + \alpha + \epsilon$ For each $x \in \iota_{i,s}(K)$ then, there is open neighborhood O_x of x such that: $O_x = \prod_{j \in P} O_x^j$, where O_x^j is neighborhood of x_j in S_j and for each $y \in O_x$ we have $V(y) > V(s) + \alpha + \frac{\epsilon}{2}$ From the compactness, we can cover $\iota_{i,s}(K)$ by finite number of these neighborhoods:

$$\iota_{i,s}(K) \subset \bigcup_{j \in \{0, \dots, n\}} O_{x_j} = \bigcup_{j \in \{0, \dots, n\}} O_j$$

where last equality is introduced to simplify further referencing these sets. Now let $O_s = V^{-1}((-\infty, V(s) + \frac{\epsilon}{2}))$. It is open neighborhood of s in S . Finally let's construct another open neighborhood of s following way:

$$O = O_s \cap ((\bigcap_{j=0, \dots, n} O_j^{-i}) \times S_i)$$

where $O_j^{-i} = \prod_{l \in (P - \{i\})} O_j^l$

We will show, that $O \subset g_i^{-1}((\rho, \infty))$.

Let's take $s' \in O$. For each $x \in K$ we have: $\iota_{i,s'}(x) = (x, s'_{-i})$ where $s'_{-i} \in \bigcap_{j=0, \dots, n} O_j^{-i}$, thus belongs to all O_j^{-i} . Since O_j is cover of $\iota_{i,s}(K)$, there is specific j , such that $(x, s_{-i}) \in O_j$. So, also $(x, s'_{-i}) \in O_j$. Now, we have:

$$V(\iota_{i,s'}(x)) > V(s) + \alpha + \frac{\epsilon}{2} > V(s') + \alpha$$

thus $K \subset \iota_{i,s'}^{-1}(U_{V(s')+\alpha})$ and since $\mu_i(K) > \rho$ we have $O \subset g_i^{-1}((\rho, \infty))$. This completes the proof.

Lemma 2. Let (X, d) be a metric space, $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ be two lower semi-continuous functions such that, for each $x \in M$ we have $0 \leq g(x) \leq f(x) \leq M$. Let $\phi : X \rightarrow \mathbb{R}$ be defined as

$$\phi(x) = \begin{cases} 1 & g(x) = f(x) \\ \frac{g(x)}{f(x)} & \text{otherwise} \end{cases}$$

then ϕ is measurable in the sense of the Borel σ -algebra

Proof. We will show, that ϕ is Baire class 2 function (so measurable with regard to Borel's σ -algebra. Let's take sequence $(\epsilon_n)_{n \in \mathbb{N}}$, such that: $\epsilon_n \xrightarrow[n \rightarrow \infty]{} 0$ for each $n \in \mathbb{N}$ we have $\epsilon_n > \epsilon_{n+1}$.

Obviously functions $f_n = f + \epsilon_n$ and $g_n = g + \epsilon_n$ are also lower semi-continuous, and what's more strictly positive ($> \epsilon_n$). From Baire theorem on lower semi-continuous functions [8] there are pointwise increasing sequence of continuous functions $(f_{n,i})_{i \in \mathbb{N}}$ and $(g_{n,i})_{i \in \mathbb{N}}$ such that pointwise $f_{n,i} \xrightarrow[i \rightarrow \infty]{} f_n$ and $g_{n,i} \xrightarrow[i \rightarrow \infty]{} g_n$. By taking

$$\tilde{f}_{n,i}(x) = \max\{f_{n,i}(x), \frac{\epsilon}{2}\}$$

and

$$\tilde{g}_{n,i}(x) = \max\{g_{n,i}(x), \frac{\epsilon}{2}\}$$

we can assure, that sequences are bounded below from 0 and taking

$$\tilde{\tilde{f}}_{n,i}(x) = \max\{\tilde{f}_{n,i}(x), \tilde{g}_{n,i}(x)\}$$

and

$$\tilde{\tilde{g}}_{n,i}(x) = \min\{\tilde{f}_{n,i}(x), \tilde{g}_{n,i}(x)\}$$

we can assure that $\tilde{\tilde{g}}_{n,i} \leq \tilde{\tilde{f}}_{n,i}$. Now, we have $\frac{\tilde{\tilde{g}}_{n,i}}{\tilde{\tilde{f}}_{n,i}} \xrightarrow[i \rightarrow \infty]{} \frac{g_n}{f_n}$ so $\frac{g_n}{f_n}$ is of first category as a pointwise limit of continuous functions.

We can also check (by easy calculation), that for each $x \in X$ $\frac{g_n(x)}{f_n(x)} \geq \frac{g_{n+1}(x)}{f_{n+1}(x)}$. Thus $\frac{g_n}{f_n}$ is pointwise convergent equal to 1 whenever $f(x) = g(x)$, so convergent to ϕ . Thus ϕ is of second Baire category.

Let us now recall definition of function f from Theorem 8 (we will use notation and assumptions from Sect. 4 and proof of Theorem 8):

$$f(s) = A(s)(U_{V(s)+\epsilon})$$

We have for each $i \in P$ that functions $f_i(s) = \tilde{\mathcal{V}}_i(s)(U_{i,s}) = \mu_0^i(\iota_{i,s}^{-1}(U_{V(s)}))$ and $g_i(s) = \tilde{\mathcal{V}}_i(s)(U_{V(s)+\epsilon'}) = \mu_0^i(s)(\iota_{i,s}^{-1}(U_{V(s)+\epsilon'}))$ both are lower semi-continuous by Lemma 1, and what's more satisfy conditions of Lemma 2. So, $\mathcal{V}_i(U_{V(s)+\epsilon'})$ is measurable for each i and $f(s) = A(s)(U_{V(s)+\epsilon'}) = \sum_{i \in P} \alpha_i \mathcal{V}_i(U_{V(s)+\epsilon'})$ is measurable as a linear combination of the measurable functions.

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