Normalized Equilibrium in Tullock Rent Seeking Game

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Abstract. Games with Common Coupled Constraints represent many real-life situations. In these games, if one player fails to satisfy its constraints common to other players, then the other players are also penalized. Therefore these games can be viewed as being cooperative in goals related to meeting the common constraints, and non-cooperative in terms of the utilities. We study in this paper the Tullock rent-seeking game with additional common coupled constraints. We have succeeded in showing that the utilities satisfy the property of diagonal strict concavity (DSC), which can be viewed as an extension of concavity to a game setting. It not only guarantees the uniqueness of the Nash equilibrium but also of the normalized equilibrium.

Keywords: Normalized equilibrium · Common Coupled Constraints · Diagonal strict concavity

1 Introduction

Games with constraints have long been used for modeling and studying non-cooperative behavior in various areas. This includes road traffic [7,12] and telecommunications [9]. Various types of constraints may appear in everyday game situations; the simplest ones consisting of orthogonal constraints, where the strategies of the players are restricted independently of each other [15]. The second type of constraints are called Common Coupled Constraints (CCC) [3,14,15] in which all players have a common convex non-orthogonal multi-strategy space. This model can be viewed as constraints that are common to all users. A unilateral deviation of a player from some feasible multi-strategy (one that satisfies the constraints) to another strategy that is feasible for that player, does not result, therefore, in the violation of constraints of other users. CCC has often been used in telecommunications networking problems as well as in power transfer over a smart grid, where capacity constraints of links are naturally common. Games

K. Avrachenkov et al. (Eds.): GameNets 2019, LNICST 277, pp. 109–116, 2019.
https://doi.org/10.1007/978-3-030-16989-3_8

with this type of constraints are a special case of General Constrained Games (GCG) [6], see also [3–5, 8, 10, 16].

In this paper, we study the well known Tullock rent-seeking game with Common Coupled Constraints. This game describes contest over resources. Each player bids an amount that she is ready to pay. She then pays an amount proportional to her bid and receives, in turn, a payoff that is proportional to her bid divided by the sum of bids of all players.

The presence of a capacity constraint results in infinitely many equilibria and we are faced with a question of equilibrium selection. Using Kuhn Tucker conditions to the best response, we can solve a relaxed game instead of the original constrained game, which has, however, the same equilibria as the original game. The Lagrange multipliers can be interpreted as a shadow cost that a manager sets in order to guarantee that the equilibrium achieved satisfies the constraints. This approach may, however, be completely unscalable since KKT Theorem does not guarantee that the price per resource unit is the same for all players. In fact, since the Lagrange multipliers are obtained for the best response function, they could depend not only on the player but also on the policy of all other players, rending the approach even less scalable. We are interested in finding such shadow cost which is fixed per resource unit. Such an equilibrium along with a fixed shadow price is called a normalized equilibrium.

The Tullock rent-seeking game has been used recently to model and study several game phenomena in networking. It was used to model contests over timelines in social networks for maximizing visibility [17]. Each player i controls the rate $\lambda_i a_i$ of a Poisson process of posts that player i sends into a common timeline of length K. This rate is given by a basic popularity rate λ_i times the acceleration effort (e.g. through advertisement) given by a_i. Using basic queueing theory, the authors show that the stationary expected number of posts in the timeline originating from player i is given by

$$K \frac{\lambda_i a_i}{\sum_{j=1}^{N} \lambda_j a_j}$$

This visibility measure is the payoff in Tullock's model, while the cost for acceleration at a rate a_i is proportional to a_i as in Tullock's model.

Another application of the Tullock rent-seeking game is in the study of contests between miners in blockchain [2].

A few words on rent-seeking. According to Wikipedia, "In public choice theory and in economics, rent-seeking involves seeking to increase one's share of existing wealth without creating new wealth. Rent-seeking results in reduced economic efficiency through the poor allocation of resources, reduced actual wealth-creation, lost government revenue, increased income inequality, and (potentially) national decline."

Wikipedia further describes the origin of the idea: "The idea of rent-seeking was developed by Gordon Tullock in 1967, while the expression rent-seeking itself was coined in 1974 by Krueger [11]. The word "rent" does not refer specifically to payment on a lease but rather to Adam Smith's division of incomes into profit, wage, and rent. The origin of the term refers to gaining control of land or other natural resources."

Our first contribution is to show that the utilities satisfy a property that extends concavity to games, and is called Diagonally Strict Concavity. This is shown to imply the existence and uniqueness of a normalized equilibrium. We shall then show that this property further extends to the case of contests over several resources.

2 A Single Resource

Consider an N players game. Player m bids a quantity x^m. We have minimum constraints $x^m \geq \epsilon$ for all m.

The payoff from this contest to player m is

$$P^m = \frac{x^m}{\sum_{j=1}^{M} x^j}.$$

This comes at a cost of $x^m\gamma$ to player m where γ is a constant. The utility for player m is thus

$$U^m(x) = \frac{x^m}{\sum_{j=1}^{M} x^j} - x^m\gamma.$$

Theorem 1. *(i) The utility of player m is concave in its action and is continuous in the actions of other players.*
(ii) For any strictly positive value of γ, the above game has a unique Nash equilibrium in pure policies.

Proof. Direct calculation leads to (i). The existence then directly follows from [15]. Uniqueness is established in [1], see also [18]. Other related uniqueness results in the asymmetric case can be found in [17,19].

3 Normalized Equilibrium

The games we have seen so far involved orthogonal constraints. By that, we mean that the actions that a player can use do not depend on the actions of other players. We next introduce capacity constraint. We require that the following holds for some constant V:

$$\sum_{j=1}^{M} x^j \leq V \tag{1}$$

Capacity constraints may represent physical bounds on resources, such as bounded power, or resources that are bounded by regulation. For example, legislation may impose bounds on the power used or on the emission of CO_2 by cars. With the additional capacity constraint, the Nash equilibrium is no more unique and there may, in fact, be an infinite number of equilibria. We call this the game with capacity constraint.

Let y be an equilibrium in the above game and let $y_{[-m]}$ denote the action vectors of all players other than m. By KKT Theorem, since for each m, U^m is concave in x^m, there is a Lagrange multiplier $\lambda^m(y_{[-m]})$ such that y^m maximizes the Lagrangian

$$L^m(x^m) = U^m(x, y_{[-m]}) - \lambda^m(y_{[-m]}) \left(V_k - \sum_{j=1}^{M} x^j \right)$$

and

$$\lambda^m(y_{[-m]}) \left(V - \sum_{j=1}^{M} x^j \right) = 0$$

(complementarity property). We call the game with the Lagrangian L^m replacing the utilities U^m the relaxed game.

The Lagrange multipliers can be interpreted as shadow prices: if a price is set on player m such that when other players are at equilibrium, the player pays $x^m \lambda^m(y_{[-m]})$ for its use of the capacity, then y is an equilibrium in the game with capacity constraints. Yet this pricing is not scalable since for the same use of the resources it may vary from user to user and it further depends on the chosen equilibrium. For billing purposes, one would prefer λ^m not to depend on y nor on m, but to be a constant.

Does there exist a constant Lagrange multiplier λ independent of strategies of the payers and of the identity m of the player, along with an associated equilibrium y for the corresponding relaxed game? If the answer is positive then y is called a *normalized equilibrium* [15].

Our goal is to establish the existence and uniqueness of the normalized equilibrium.

4 Diagonal Strict Concavity

For a vector of real non-negative numbers r, define

$$\sigma(x, r) = \sum_{m=1}^{N} r_m U^m(x)$$

$$g(x, r) = \begin{bmatrix} r_1 \frac{\partial}{\partial x_1} U^1(x_1, x_{-1}) \\ r_2 \frac{\partial}{\partial x_2} U^2(x_2, x_{-2}) \\ \vdots \\ r_N \frac{\partial}{\partial x_N} U^N(x_N, x_{-N}) \end{bmatrix} \tag{2}$$

σ is called diagonally strict concave (DSC) for a given r if for every distinct x^0 and x^1,

$$(x^1 - x^0)' g(x^0, r) + (x^0 - x^1)' g(x^1, r) > 0$$

Let $G(x,r)$ be the Jacobian of $g(x,r)$ with respect to x and let $G_{i,j}$ be i^{th} row and j^{th} column of $G(x,r)$. Then a sufficient condition for σ to be diagonally strict concave for some r is that for all x, $[G(x,r)+G'(x,r)]$ is negative definite.

Our interest in diagonally strict concave utility functions is due to the following properties of games possessing such utilities.

Theorem 2 *(Theorem 4 from [15]). Let σ be diagonally strict concave for some r. Then there exists a unique normalized equilibrium.*

5 Proof of DSC

In this section we establish that the Tullock game with capacity constraint has a DSC structure and thus has a unique normalized equilibrium.

In our case we have

$$g(x,r) = \begin{bmatrix} \frac{r_1 x_{-1}}{X} \\ \frac{r_2 x_{-2}}{X} \\ \vdots \\ \frac{r_N x_{-N}}{X} \end{bmatrix} \tag{3}$$

where $X = \sum_{i=1}^{N} x_i$ and $x_{-m} = \sum_{i=1, i\neq m}^{N} x_i$

$$G_{i,j} = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\frac{r_i x_i}{X}\right) \tag{4}$$

$$r_i \frac{\partial}{\partial x_j}\left(\frac{x_{-i}}{X^2}\right) = \begin{cases} r_i \frac{-2x_i}{X^3} & \text{if } i = j \\ r_i \frac{x_i - x_{-i}}{X^3} & \text{if } i \neq j \end{cases} \tag{5}$$

For $[G+G']$ consider

$$G_{i,j} + G_{j,i} = \begin{cases} \frac{-4 r_i x_{-i}}{X^3} & \text{if } i = j \\ \frac{r_i(x_i - x_{-i}) + r_j(x_j - x_{-j})}{X^3} & \text{if } i \neq j \end{cases} \tag{6}$$

$[G+G']$ is negative definite if $A'\,[G+G']\,A < 0, \forall A, A \neq 0$ where A is the column vector

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$$

$$A'\,[G+G']\,A$$
$$= \sum_{i=1}^{N} \left[\sum_{j=1, j\neq i}^{N} a_i a_j \frac{r_i(x_i - x_{-i}) + r_j(x_j - x_{-j})}{X^3}\right] - a_i^2 \frac{4 r_i x_{-i}}{X^3} \tag{7}$$

We choose $r_i = 1$ for all i. Then (7) equals $-Z/X^3$ where Z is given by

$$\sum_{i=1}^{N} a_i^2 4x_{-i} + \left[\sum_{j=1, j\neq i}^{N} a_i a_j \left((x_{-i} - x_i) + (x_{-j} - x_j) \right) \right] \tag{8}$$

$$= \sum_{i=1}^{N} a_i^2 4 \left(X - x_i \right) + \left[\sum_{j>i}^{N} 4 a_i a_j \left(X - x_i - x_j \right) \right] \tag{9}$$

$$= 4 \sum_{i=1}^{N} a_i^2 \left(X - x_i \right) + \left[\sum_{j>i}^{N} a_i a_j \left(X - x_i - x_j \right) \right] \tag{10}$$

$$= 4 \sum_{i=1}^{N} \left[a_i^2 \sum_{j=1, j\neq i}^{N} x_i + \sum_{j>i}^{N} a_i a_j \sum_{k=1, k\neq j, k\neq i}^{N} x_k \right] \tag{11}$$

$$= \sum_{i=1}^{N} 4x_i \left[\sum_{j=1, j\neq i}^{N} a_j^2 + a_j \sum_{k>j, k\neq i}^{N} a_k \right] \tag{12}$$

Now (12) is positive for any positive value of x and hence [G'+G] matrix is negative definite.

6 Several Resources

We consider next the following extension to the case of K resources. Each player m of the M players has a budget $B(m)$ that he can invest by bidding x_k^m of resource k. The following (orthogonal) constraint should hold:

$$\sum_{k=1}^{K} x_k^m \leq B(m).$$

The payoff for player m is the sum of payoffs in all K contests, i.e.

$$P^m(x) = \sum_{k=1}^{K} P_k^m(x_k)$$

where x_k is the vector $x_k^1, \ldots, x_k^M$ and where

$$P_k^m(x_k) = \frac{x_k^m}{\sum_{j=1}^{M} x_k^j}.$$

and the cost of a contest k to player m is $\gamma(k) x_k^m$. Player m' s utility is thus

$$U^m(x) = \sum_{k=1}^{K} \left(\frac{x_k^m}{\sum_{j=1}^{M} x_k^j} - \gamma_k x_k^m \right)$$

For the study of such games, see [17].

We next define capacity constraint on each of the K resources. Let V be the column vector with the k^{th} entry being a constant V_k. We then require for each k that

$$\sum_{m=1}^{N} x_k^m \leq V_k$$

Note that when applying KKT conditions to the best response at equilibrium, we shall have K Lagrange multipliers. We wish to find a vector of K Lagrange multipliers that do not depend on the player nor on the policy of other players, such that the Nash equilibrium for the relaxed game will be an equilibrium for the original constrained game and in particular the constraints would be met and would satisfy the complementarity conditions. This is the vector version of a normalized equilibrium.

According to Theorem 4 of Rosen [15], we have to show that the set of utilities is diagonally strict concave in order to have existence and uniqueness of the normalized equilibrium. The latter follows from the fact that DSC holds for each resource separately and then apply the proof of Corollary 2 in [13].

7 Conclusions and Future Work

We have shown that the utilities in the Tullock game are strict diagonal concave. This allows to conclude using Rosen's result that in absence of common correlated constraints, the Nash equilibrium exists and is unique, while in presence of such constraints, the normalized equilibrium exists in pure strategies and is unique. Note that while the statements on the Nash equilibrium have already been available in [1] which proposed an extension to the DSC property, that reference does provide tools to handle the normalized equilibrium.

Another advantage from the derivation of the DSC structure is that one can use dynamic distributed algorithms to converge to the normalized equilibrium and convergence is guaranteed under DSC, see [15].

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