Stabilization of the Minimum Latency Flow in Braess Graphs by State-Dependent Tax

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ABSTRACT

A selfish routing game is a simple model of selfish behaviors in networks. Braess's paradox is a well-known example of inefficiencies existing in the selfish routing games and it is an important problem to reduce such inefficiencies. To resolve such a problem, a notion of a marginal cost tax has been proposed. Although the marginal cost tax makes the minimum latency flow a Nash equilibrium, it also imposes an additional latency on the minimum latency flow. Thus, we apply replicator dynamics with a subsidy and a capitation tax to Braess graphs and extend the capitation tax to a state-dependent one. Using two simplest Braess graphs B^1 and B^2, we show that the minimum latency flow of the Braess graphs can be stabilized by our proposed state-dependent tax.

Keywords

Selfish routing, Braess's paradox, evolutionary game theory, tax and subsidy

1. INTRODUCTION

In large computer networks, there exist several inefficiencies due to selfish behaviors and it is an important problem to reduce them [7]. It is a main feature in ambient information networks that information is not only actively obtained by users but also autonomously provided to users by environment. So, the routing protocol in the networks may be selfish and the inefficiencies due to the protocol will be a more serious problem. To reduce the inefficiencies, an external agent to control packet flows will be needed and its design methodologies is an important issue. Recently, several game theoretical methodologies have been paid attention to model and resolve such a problem. A selfish routing game is a simple model of selfish behaviors in networks [5]. Its replicator dynamics has also been proposed [3]. In the selfish routing games, suppose that there is a fixed flow demand and it is routed from a source to a sink. Players are assumed to select paths in order to minimize the latency and such a selfish behavior causes several inefficiencies. Braess's paradox is a well-known example of inefficiencies existing in the selfish routing games. An equilibrium flow achieved by selfish behaviors of players is not the optimal minimum latency flow if there exists Braess's paradox. To reduce such inefficiencies in the networks, the notation of a marginal cost tax has been proposed [1]. It has been proved that we make the optimal latency flow a Nash equilibrium flow by assigning the marginal cost tax to each edge of networks as an additional fictitious latency. However, since the marginal cost tax is also imposed on the minimum latency flow, the average latency of the minimum latency flow may increase. Moreover, in some cases, the marginal cost tax is unnecessarily high for stabilizing the minimum latency flow.

On the other hand, inefficiencies due to players' selfish behaviors in social systems have also been studied. To model and resolve conflicts between a payoff of each player and the total payoff of players, replicator dynamics with a subsidy and a capitation tax has been introduced [4]. Unlike the notion of the marginal cost tax, this model does not only impose a capitation tax on players but also offers a state-dependent subsidy to them.

In this paper, we consider a stabilization problem of the minimum latency flow in Braess graphs. The Braess graphs are proposed to discuss Braess's paradox by Roughgarden [6] and Englert *et al.* [2]. We formulate replicator dynamics of the Braess graphs. We extend a capitation tax to a state-dependent one and apply replicator dynamics with a subsidy and a capitation tax to resolve Braess's paradox. Unlike the marginal cost tax, the latency of the target minimum latency flow is unaffected by the imposed tax and offered subsidy in our model. Moreover, we set the minimum latency flow to the target state and show a stabilization condition of the flow. Using two simplest Braess graphs B^1 and B^2, we show that the minimum latency flow of the Braess graphs can be stabilized by our proposed state-dependent tax.

2. PRELIMINARIES

2.1 Selfish Routing Game and its Replicator Dynamics

Consider a single-commodity network $G = (V, E)$ with a source vertex s and a sink vertex t, where V and E are sets of vertices and edges, respectively. Suppose that there is a fixed flow demand and it is routed from the source s to the sink t. Let P be the set of paths s-t. A flow $x = (x_{p_1}, \dots, x_{p_n})^T$ is

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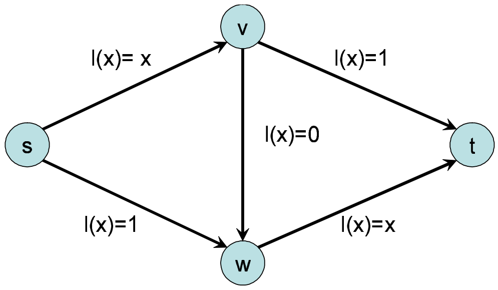

Figure 1: Example of Braess's paradox.

a nonnegative vector, where x_{p_i} is the amount of flow routed over the path $p_i \in P$ and n is the number of elements in P. For a flow vector x, we define the flow on an edge $e \in E$ by $x_e = \sum_{p_i \ni e} x_{p_i}$. Let $l_e(x_e)$ be the latency on the edge $e \in E$ and we assume that it is a nonnegative, continuous, and nondecreasing function. The latency $l_{p_i}(x)$ of $p_i \in P$ is given by $l_{p_i}(x) = \sum_{e \in p_i} l_e(x_e)$, and the average latency $\bar{l}(x)$ of a flow vector x is $\bar{l}(x) = \sum_{p_i \in P} x_{p_i} l_{p_i}(x)$. For simplicity, we assume the total amount of the fixed flow demand is equal to 1, that is, $\sum_{p_i \in P} x_{p_i} = 1$. A flow vector x is said to be a Nash equilibrium or a Nash flow if $l_{p_i}(x) \le l_{p_j}(x)$ holds for every pair of paths p_i and $p_j \in P$ with $x_{p_i} > 0$.

Suppose that players select paths in P in order to minimize the latency. A selfish routing game is a simple model of such a selfish behavior in networks [5]. Its replicator dynamics has been proposed as follows [3]: for all $p_i \in P$,

$$\dot{x}_{p_i} = f_{p_i}(x) := x_{p_i}(\bar{l}(x) - l_{p_i}(x)). \tag{1}$$

This equation means that the flow of a path with lower latency than the average increases, while one with higher latency than the average decreases.

2.2 Braess's Paradox

In the selfish routing games, there exist several inefficiencies due to selfish route selections of players. Braess's paradox is one of the well-known examples in such inefficiencies. Intuition may suggest that an additional edge with zero or sufficiently low latency reduces the average latency of the flow in the network, but it is incorrect in some cases and the average latency may increase due to players' selfishness. That is called Braess's paradox.

Consider the network in Fig. 1. To route all flow over the path *s*-*v*-*w*-*t* is a Nash equilibrium with the latency 2. However, if we omit the edge *v*-*w*, then to route a half of all flow over the path *s*-*v*-*t* and the rest over the path *s*-*w*-*t* becomes a Nash equilibrium with lower latency 3/2. This is a typical example of Braess's paradox.

2.3 Marginal Cost Tax

To reduce inefficiency as shown in Braess's paradox, the notion of a *marginal cost tax* has been proposed [1]. For the minimum latency flow x^*, the marginal cost tax for each edge $e \in E$ is defined as follows:

$$\tau_e^* = x_e^* \cdot l_e'(x_e^*), \tag{2}$$

where l_e' means the derivative of l_e with respect to x_e. The marginal cost tax τ_e^* is assigned to each edge $e \in E$ as an additional fictitious latency. The following proposition for the marginal cost tax has been proved [1]:

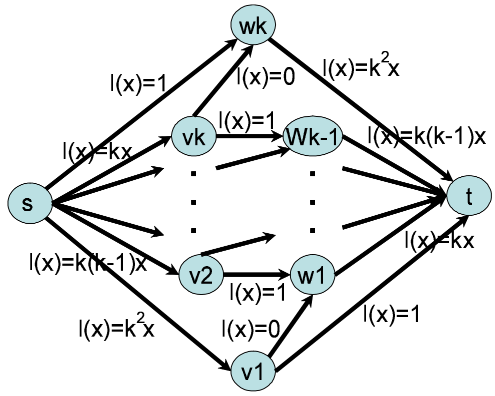

Figure 2: The kth Braess graph B^k.

Proposition 1. Consider a network $G = (V, E)$ with a differentiable latency functions $l_e(x_e)$ for each $e \in E$. Let x^* be the minimum latency flow of G with l_e and τ_e^* be the marginal cost tax for each edge e. Then x^* is a Nash equilibrium for G with a latency function $l_e(x_e) + \tau_e^*$.

The marginal cost tax makes the minimum latency flow a Nash equilibrium.

3. SELFISH ROUTING WITH STATE-DEPENDENT TAX

3.1 The kth Braess Graph

To discuss Braess's paradox, the generalized Braess graphs are defined by Roughgarden [6] and Englert *et al.* [2]. Their definitions are similar in the structures of the graphs but are different in the formulations of their latency functions. In this paper, we employ the definition of Englert *et al.* [2] as follows:

Definition 1. For a given natural number k, a graph $B^k = (V_k, E_k)$ with $V_k = \{s, v_1, \ldots, v_k, w_1, \ldots, w_k, t\}$ and $E_k = \{(s, v_i), (v_i, w_i), (w_i, t) : 1 \le i \le k\} \cup \{(v_i, w_{i-1}) : 2 \le i \le k\} \cup \{(s, w_k)\} \cup \{(v_1, t)\}$ is said to be the kth Braess graph B^k if it has the latency functions satisfying the following conditions:

- $l^k_{v_i,w_i}(x_{v_i,w_i}) = 0$ and $l^k_{s,v_{k-i+1}}(x_{s,v_{k-i+1}}) = l^k_{w_i,t}(x_{w_i,t}) = ikx_{w_i,t}$ for $1 \le i \le k$;
- $l^k_{v_i,w_{i-1}}(x_{v_i,w_{i-1}}) = 1$ for $2 \le i \le k$; and
- $l^k_{s,w_k}(x_{s,w_k}) = l^k_{v_1,t}(x_{s,w_k}) = 1$,

where $x_{v,w}$ and $l^k_{v,w}(x_{v,w})$ are the flow on the edge v-w and the latency of v-w in B^k, respectively.

Let $(p_1, \ldots, p_{2k+1}) = (p_{s,w_k,t}, p_{s,v_k,w_k,t}, p_{s,v_k,w_{k-1},t}, \ldots, p_{s,v_1,t})$ be the corresponding path of the graph B^k. By Definition 1, the latency of each path of B^k is given by

$$l_{p_i}(x) = \begin{cases} k\left[\dfrac{i-1}{2}(x_{p_{i-1}} - x_{p_{i+1}}) + k(x_{p_i} + x_{p_{i+1}})\right] + 1, \\ \qquad\qquad \text{for } i = 1, 3, \ldots, 2k+1, \\ k\left[\dfrac{i}{2}(x_{p_{i+1}} - x_{p_{i-1}}) + (k+1)(x_{p_{i-1}} + x_{p_i})\right], \\ \qquad\qquad \text{for } i = 2, 4, \ldots, 2k, \end{cases} \tag{3}$$

where $x_{p_0} = x_{p_{2k+2}} = 0$. Figure 2 shows the structure of the graph B^k.

The Nash flow of B^k is

$$x_{p_i} = \begin{cases} 0 & \text{for } i = 1, 3, \ldots, 2k+1, \\ 1/k & \text{for } i = 2, 4, \ldots, 2k. \end{cases} \quad (4)$$

In the graph B^k, the minimum latency flow x^* is

$$x^*_{p_i} = \begin{cases} 1/(1+k) & \text{for } i = 1, 3, \ldots, 2k+1, \\ 0 & \text{for } i = 2, 4, \ldots, 2k. \end{cases} \quad (5)$$

Therefore, the average latency of the minimum latency flow is $\bar{l}(x^*) = 1 + \frac{k^2}{k+1}$.

The average latency of a Nash flow in the kth Braess graph B^k can be improved by removing edges v_i-w_i for all $i = 1, 2, \ldots, k$ with the latency 0 from B^k.

3.2 State-Dependent Tax

The marginal cost tax can make the minimum latency flow a Nash equilibrium as shown in Sect. 2.3. However, since the marginal cost tax is also imposed on the minimum latency flow, the average latency of the minimum latency flow may increase. Moreover, in some cases, the marginal cost tax is unnecessarily high for stabilizing the minimum latency flow. Thus, in this section, we consider that a stabilization problem of the minimum latency flow of the Braess graph B^k by imposing a state-dependent (flow-dependent) tax on each path of B^k.

On the other hand, in social systems, to reduce inefficiencies due to selfish behaviors of players, the government tries to control the players by imposing tax on and offering subsidies to them depending on how desirable their behaviors are. Fukumoto *et al.* have proposed a replicator dynamics with a subsidy and a capitation tax [4]. We apply this model to the selfish routing game.

Suppose that the government imposes a capitation tax c on each player and offering a subsidy $\alpha x_i^*/x_i$ to each player with strategy i, where $x = (x_1, \ldots, x_n)^T$ is a population state, x_i is a proportion of players with strategy i, and $x^* = (x_1^*, \ldots, x_n^*)^T$ is a target state. In the selfish routing game, they correspond to a flow vector x, a flow routed over a path p_i, and the minimum latency flow x^*, respectively. In this paper, since we consider the state-dependent tax, we extend the capitation tax c to depend on a state x, that is, $c = c(x)$. Thus, the latency of each path $p_i \in P$ with the subsidy and the capitation tax is given by

$$l_{p_i}(x) + c(x) - \alpha \frac{x^*_{p_i}}{x_{p_i}}. \quad (6)$$

Suppose that every path is used at least in the initial state, that is, $x_{p_i}(0) > 0$ is assumed to hold for any path $p_i \in P$. By this assumption, within any finite-time interval, $x_{p_i} > 0$ holds for all $p_i \in P$ and Eq. (6) is well-defined.

We can regard the capitation tax c as additional fictitious delay to the flow of each path $p_i \in P$ in routing games. However, we cannot apply the subsidies to the selfish routing since we cannot reduce the latency in the network by the control. Therefore, we set $c(x) = \max_{p_i \in P}\{\alpha x^*_{p_i}/x_{p_i}\}$. Equation (6) is modified as follows:

$$l_{p_i}(x) + \max_{p_i \in P} \left\{ \alpha \frac{x^*_{p_i}}{x_{p_i}} \right\} - \alpha \frac{x^*_{p_i}}{x_{p_i}}. \quad (7)$$

Since Eq. (7) is always greater than or equal to $l_{p_i}(x)$ for all $p_i \in P$ with equality if and only if $x = x^*$, it is well-defined as a latency function in the selfish routing game and the latency of the target minimum latency flow is unaffected by the additional latency.

Thus, replicator dynamics of the selfish routing with a state-dependent tax is given by

$$\dot{x}_{p_i} = x_{p_i}(\bar{l}(x) - l_{p_i}(x)) + \alpha(x^*_{p_i} - x_{p_i}). \quad (8)$$

In spite of some modifications, Eq. (8) is identical with the replicator dynamics with a subsidy and a capitation tax proposed in [4].

We have the replicator dynamics of the selfish routing in the graph B^k with a state-dependent tax as follows:

$$\dot{x}_{p_i} = \begin{cases} x_{p_i}(\bar{l}(x) - l_{p_i}(x)) + \alpha\left(\dfrac{1}{k+1} - x_{p_i}\right), \\ \qquad\qquad \text{for } i = 1, 3, \ldots, 2k-1, \\ x_{p_i}(\bar{l}(x) - l_{p_i}(x)) - \alpha x_{p_i}, \\ \qquad\qquad \text{for } i = 2, 4, \ldots, 2k. \end{cases} \quad (9)$$

Since the all amount of the flow is always 1, we can eliminate the equation for $\dot{x}_{p_{2k+1}}$.

It is easy to show that the minimum latency flow of the graph B^k is an equilibrium point of Eq. (1). It has been proved that an equilibrium point of Eq. (1) is also an equilibrium point of Eq. (8), and a stabilization condition of the target state which is an equilibrium point of Eq. (1) has been derived as follows [4]:

Theorem 1. Let the linearization system of Eq. (1) at the target state $x = x^*$ be $\dot{x} = J_0 x$, and the eigenvalues of J_0 be λ_{0i} $(i = 1, \ldots, n)$. Then, the linearization system of Eq. (8) at the target state $x = x^*$ is given by

$$\dot{x} = (J_0 - \alpha I_n)x, \quad (10)$$

where I_n is the $n \times n$ unit matrix. The origin is asymptotically stable in Eq. (10) if and only if $\alpha > \max_i \{\Re(\lambda_{0i})\}$, where $\Re(\lambda_{0i})$ is the real part of λ_{0i}.

Note that since n means the number of strategies, it corresponds to the number of paths in the graph B^k, that is, $n = 2k+1$ for B^k. If the origin of the linearization system Eq. (10) is asymptotically stable, then the target state $x = x^*$ of Eq. (8) is locally asymptotically stable. As a result, we can stabilize the minimum latency flow by setting the minimum latency flow to a target state, and selecting the parameter $\alpha > \max_i\{\mathrm{Re}(\lambda_{0i})\}$, where λ_{0i} is an eigenvalue of the Jacobian matrix at the target state.

4. EXAMPLES

In this section, we show that the minimum latency flow of the Braess graphs can be stabilized by our proposed state-dependent tax. We consider two simplest cases that $k = 1$ and 2. The Jacobian matrices of the two graphs and their eigenvalues are obtained, and the parameter α is selected for each of the graphs B^1 and B^2 based on them. We show that the minimum latency flow is stabilized by selected α in each case.

4.1 The Case $k = 1$

Consider the 1st Braess graph B^1 shown in Fig. 1. Let paths 1, 2, and 3 correspond to *s*-*v*-*t*, *s*-*v*-*w*-*t*, and *s*-*w*-*t*, respectively. We set the minimum latency flow $(1/2, 0, 1/2)^T$ to the target state x^*. A Nash equilibrium flow is $x =$

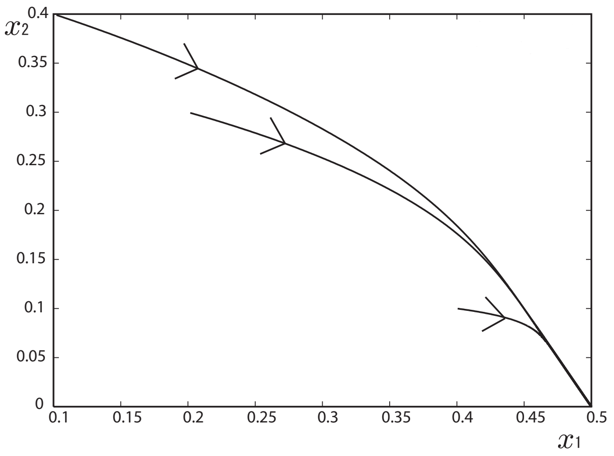

Figure 3: Orbits of Eq. (13) with $\alpha = 0.51$.

$(0, 1, 0)^T$ and it is a unique asymptotically stable equilibrium point without the state-dependent tax.

From Eq. (1), we have the replicator dynamics of the graph B^1 as follows:

$$\dot{x}_1 = f_1(x) := x_1\{2x_1^2 + x_2^2 + 2x_1x_2 - 3x_1 - 2x_2 + 1\}, \quad (11)$$
$$\dot{x}_2 = f_2(x) := x_2\{2x_1^2 + x_2^2 + 2x_1x_2 - 2x_1 - 2x_2 + 1\}. \quad (12)$$

From Eq. (9), the replicator dynamics of B^1 with the state-dependent tax is given by

$$\dot{x}_1 = f_1(x) + \alpha\left(\frac{1}{2} - x_1\right), \quad \dot{x}_2 = f_2(x) - \alpha x_2. \qquad (13)$$

The Jacobian matrix J_0 of Eqs. (11) and (12) is

$$J_0 = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=x^*} = \begin{bmatrix} -1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \qquad (14)$$

where

$$\frac{\partial f_1}{\partial x_1} = 6x_1^2 + x_2^2 + 4x_1x_2 - 6x_1 - 2x_2 + 1, \qquad (15)$$
$$\frac{\partial f_1}{\partial x_2} = 2x_1x_2 + 2x_1^2 - 2x_1, \qquad (16)$$
$$\frac{\partial f_2}{\partial x_1} = 4x_1x_2 + 2x_2^2 - 2x_2, \qquad (17)$$
$$\frac{\partial f_2}{\partial x_2} = 2x_1^2 + 3x_2^2 + 4x_1x_2 - 2x_1 - 4x_2 + 1. \qquad (18)$$

In this case, the eigenvalues of J_0 are $1/2$ and $-1/2$. By Theorem 1, the target minimum latency flow is a locally asymptotically stable equilibrium point of Eq. (13) if $\alpha > 1/2$. Figure 3 shows orbits of Eq. (13) for $\alpha = 0.51 > 1/2$, where the horizontal axis is a flow routed over the path *s-v-t* (x_1), and the vertical axis is that routed over *s-v-w-t* (x_2).

4.2 The Case $k = 2$

Consider the 2nd Braess graph B^2 shown in Fig. 4. Let paths $1, 2, \ldots, 5$ correspond to paths s-w_2-t, s-v_2-w_2-t, $\ldots$, s-v_1-t, respectively. We set the minimum latency flow $(1/3, 0, 1/3, 0, 1/3)^T$ to the target state x^*. A Nash equilibrium flow is $x = (0, 1/2, 0, 1/2, 0)^T$ and it is a unique asymptotically stable equilibrium point without the state-dependent tax.

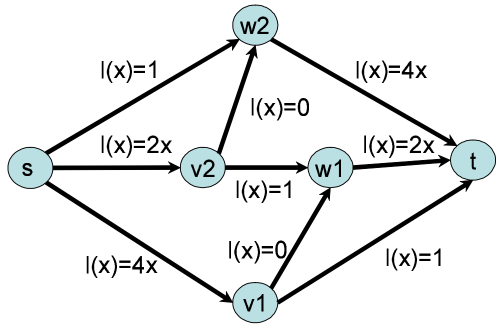

Figure 4: The 2nd Braess graph B^2.

From Eq. (1), we have the replicator dynamics of the graph B^1 as follows:

$$\dot{x}_1 = f_1(x) := x_1\{\bar{l}(x) - (1 + 4x_1 + 4x_2)\}, \qquad (19)$$
$$\dot{x}_2 = f_2(x) := x_2\{\bar{l}(x) - (4x_1 + 6x_2 + 2x_3)\}, \qquad (20)$$
$$\dot{x}_3 = f_3(x) := x_3\{\bar{l}(x) - (1 + 2x_2 + 4x_3 + 2x_4)\}, \qquad (21)$$
$$\dot{x}_4 = f_4(x) := x_4\{\bar{l}(x) - (4 - 4x_1 - 4x_2 - 2x_3 + 2x_4)\}, \quad (22)$$

where

$$\bar{l}(x) = x_1(-8 + 8x_1 + 16x_2 + 8x_3) + x_2(-9 + 10x_2 + 12x_3) + x_3(-8 + 8x_3 + 4x_4) + x_4(-1 + 2x_4). \qquad (23)$$

From Eq. (9), the replicator dynamics of B^1 with the state-dependent tax given by

$$\dot{x}_1 = f_1(x) + \alpha\left(\frac{1}{3} - x_1\right), \quad \dot{x}_2 = f_2(x) - \alpha x_2, \quad (24)$$
$$\dot{x}_3 = f_3(x) + \alpha\left(\frac{1}{3} - x_3\right), \quad \dot{x}_4 = f_4(x) - \alpha x_4. \quad (25)$$

Similar to the case $k = 1$, we calculates the eigenvalues of the Jacobian matrix J_0 of Eqs. (19)–(22). In this case, the eigenvalues of J_0 are $1/3$, $1/3$, $-4/3$, and $-4/3$, and the target minimum latency flow is a locally asymptotically stable equilibrium point of Eqs. (24) and (25) if $\alpha > 1/3$. Figure 5 shows a transient behavior of Eqs. (24) and (25) with $\alpha = 0.4 > 1/3$.

5. CONCLUSIONS

In this paper, we have considered a stabilization problem of the minimum latency flow in Braess graphs. Replicator dynamics of the Braess graphs have been formulated. We have extended a capitation tax to a state-dependent one and applied replicator dynamics with a subsidy and a state-dependent capitation tax to resolve Braess's paradox. We have also shown a stabilization condition of the minimum latency flow. Using two simplest Braess graphs B^1 and B^2, we have demonstrated that the minimum latency flow of the Braess graphs can be stabilized by our proposed state-dependent tax.

In this paper, we only consider a single-commodity flow which routed from a single source to a single sink. However, several flows which routed from different sources to different sinks coexist in real computer networks. So, it is an important problem to extend our work to multi-commodity flows. We applied a model of social systems with a subsidy and a capitation tax proposed by Fukumoto *et al.* [4], where we

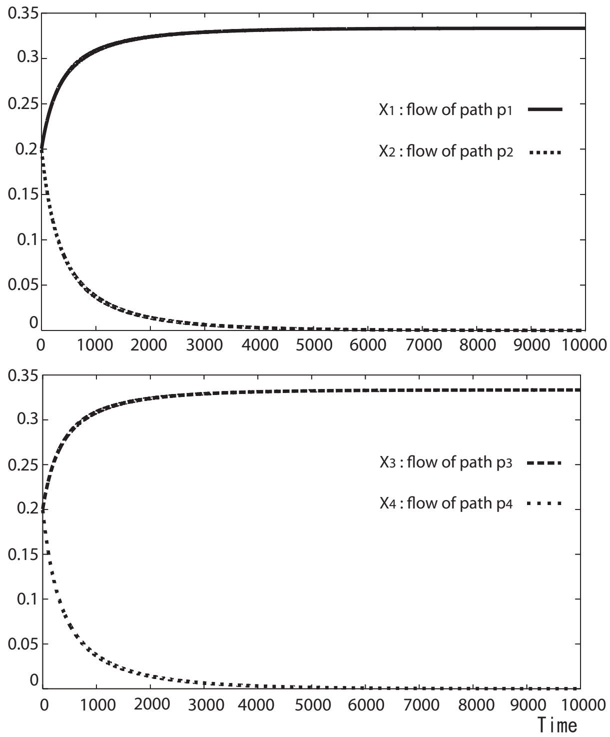

Figure 5: Transient behavior of Eqs. (24) and (25) with $\alpha = 0.4 > 1/3$.

consider interactions among players in a single population. To apply the model to multi-commodity networks, we generalize the model to a system with multi-populations. It is future work to consider taxations and subsidizations to the system with multi-populations, and apply them to reduce inefficiencies in multi-commodity networks. Moreover, we focus on networks modeled by the simple Braess graphs in this paper. It is also future work to generalize the proposed method to more complex and more general networks.

6. ACKNOWLEDGMENTS

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