CSRL Model Checking with Closed-Form Bounding Distributions *

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ABSTRACT

Continuous Stochastic Logic (CSL) which lets to express real time probabilistic properties on Continuous Time Markov Chains (CTMC) has been augmented by reward structures to check also performability measures. Thus Continuous Stochastic Reward Logic (CSRL) defined on Markov Reward Models (MRM) provides a framework to verify performance-related and as well as dependability-related measures. Probabilistic model checking can be provided through bounding transient, steady-state distributions of the underlying Markov chain, since models are checked to see if the considered measures are guaranteed or not. We propose to extend the model checking algorithm that we have proposed for CSL to the CSRL operators. This method is based on the construction of bounding models having closed-form transient and steady-state distributions by means of Stochastic Comparison technique. In the case when the model can be checked by this method we gain significantly in time and memory complexity. However in case when we can not conclude if the considered formula is satisfied or not, we may apply classical model checking algorithms.

Keywords

Stochastic comparison, Stochastic model checking, Markov Reward Model, class $\mathcal{C}$, CSRL, Stochastic model checking

1. INTRODUCTION

Model checking has been introduced as an automated technique to verify functional properties of systems expressed in a formal logic like Computational Tree Logic (CTL) [5]. This formalism has been extended with some probabilistic operators to Probabilistic CTL and Continuous Stochastic Logic (CSL)[1] [3]. Stochastic Model Checking is typically based on discrete time or continuous time Markov chains or Markov decision processes. For performance and/or dependability applications, stochastic model checking has been extended to models with some rewards on states and/or transitions in which logic formalisms PRCTL(Probabilistic Reward Computational Tree Logic) and CSRL(Stochastic Reward Logic)[8] are used.

Probabilistic model checking can be performed by numerical or statistical methods [20][4][3]. To perform numerical model checking, one needs to compute transient and steady-state distributions of the underlying Markovian model. This has been studied extensively and numerous algorithms have been devised and implemented in different model checkers [10] [9]. Despite the considerable works in the numerical Markovian analysis, the state space explosion still remains a problem. Bounding techniques have been largely applied to overcome the state space explosion problem of Markov chains and they are different according to the construction concepts and to the type of obtained bounds. We apply here stochastic comparison largely used in different areas of applied probability as well as in reliability, performance evaluation, dependability applications [14] [19]. We construct bounding chains in the sense of $\leq_{st}$ stochastic order belonging to class $\mathcal{C}$ Markov chains for which closed-form solutions of the transient and the steady-state distributions are given in [11].

Bounding methods are useful for model checking since we are interested in checking if the underlying formula meets the bounds or not. In [17], the bounds on the state reachability probabilities of Markov decision processes are computed by abstraction of the underlying model defined on smaller state spaces. In the case when the verification is not concluded the abstraction is refined. In [15], we have proposed to check PRCTL state formulas by stochastic bounding techniques by considering aggregated bounding Markov chains. In [13], we have proposed to apply the class $\mathcal{C}$ bounding models for CSL formulas. Contrary to other bounding aggregation methods, it is not possible to refine the bounding models when the verification can not be concluded but this method provides a significant gain on computation, memory complexity when the verification can be concluded. Thus it can be proposed as first step rapid model checking algorithm. We first apply the proposed method, and if the verification can not be concluded, we apply classical model checking algorithm.

In this paper we propose to extend this approach for CSRL. In the case of reward state formulas, the extension is straightforward since $\leq_{st}$ order is associated to the increasing functionals (rewards). In the case of time and reward bounded until formulas, we show that it is possible to provide closed-form lower or upper bounds for some cases.

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The paper is organized as follows: we first present CSRL model checking in section 2. Section 3 is devoted to a brief introduction of stochastic comparison technique and class $\mathcal{C}$ Markov chains. We present our bounding approach for CSL in section 4 and give a case study in section 5.

2. CSRL MODEL CHECKING

In this section we briefly introduce MRMs [2] and CSRL [6]. Then we present the model checking procedure based on the computation of steady-state, transient and joint distributions to verify CSRL formulas [8].

2.1 Preliminaries

A (labelled) MRM [2] $\mathcal{M}$ is a 4-tuple $(S, \mathbf{R}, L, \rho)$ where S is a finite set of *states*, $\mathbf{R} : S \times S \to \mathcal{R}^+$ is the *rate matrix* and $L : S \to 2^{AP}$ is the *labelling* function which assigns to each state $s \in S$ the set $L(s)$ of atomic propositions $a \in AP$ that are valid in s (AP denotes the finite set of atomic propositions) and $\rho : S \to \mathcal{R}^+$ is a *reward structure* that assigns to each state $s \in S$ a reward $\rho(s)$.
Remark that the infinitesimal generator $\mathbf{Q}$ can be easily deduced as $\mathbf{Q}(s,s') = \mathbf{R}(s,s')$ if $s \neq s'$ and $\mathbf{Q}(s,s) = -\sum_{s' \in S} \mathbf{R}(s,s')$.

In the sequel, we denote by $\mathbf{P}_\lambda$ the uniformized matrix defined as $\mathbf{P}_\lambda = \mathbf{I} + \frac{1}{\lambda}\mathbf{Q}$, where λ is the *uniformization rate* and $\lambda \geq sup_i |q_{i,i}|$.

With *state reward*, MRM can be seen as two dimensional stochastic process $\{(X(t), Y(t)), t \geq 0\}$ on $S \times \mathcal{R}^+$. $X(t)$ takes values in discrete set S and describes the transition behavior of $\mathcal{M}$ while $Y(t)$ takes real values and describes the accumulated reward gained over time. The stochastic process $\{Y(t), t \geq 0\}$ is not Markovian and it represents the accumulated reward from time 0 to t and it is determined by $X(t)$ and the reward structure ρ :

$$Y(t) = \int_0^t \rho(X(x))dx$$

We can distinguish three basic measures over MRMs: transient distribution where the system is considered at time t, steady-state distribution when the system reaches an equilibrium (if it exists) and joint distribution with respect to time and the accumulated reward. In the sequel, we denote by $\mathbf{\Pi}_s^{\mathcal{M}}(t)$ the transient distribution at time t of Markov chain $\mathcal{M}$ starting from the initial state s. The probability to be in state s' at time t starting from the initial state s will be denoted by $\mathbf{\Pi}_s^{\mathcal{M}}(s',t)$. $\mathbf{\Pi}_s^{\mathcal{M}}(s') = \lim_{t\to\infty} \mathbf{\Pi}_s^{\mathcal{M}}(s',t)$ is the steady-state probability to be in state s'. If $\mathcal{M}$ is ergodic, $\mathbf{\Pi}_s^{\mathcal{M}}(s')$ exists and it is independent of the initial distribution that is denoted by $\mathbf{\Pi}^{\mathcal{M}}(s')$ and $\mathbf{\Pi}^{\mathcal{M}}$ is the steady-state probability vector. The joint distribution of state and reward is used to check CSRL path formulas. Let $\mathbf{\Upsilon}_s^{\mathcal{M}}(s',t,r)$ (resp. $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(s',t,r)$) denote the probability that at time t, MRM $\mathcal{M}$ is in state s' and has accumulated a reward lower or equal (resp. higher) than r having started in state s. $\mathbf{\Upsilon}_s^{\mathcal{M}}(t,r)$ and $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(t,r)$ denote the corresponding joint probability vectors. We can remark that:

$$\begin{aligned}\mathbf{\Upsilon}_s^{\mathcal{M}}(s',t,r) &= Pr\{Y(t) \leq r, X(t) = s' \mid X(0) = s\} \\ &= Pr\{X(t) = s' \mid X(0) = s\} \\ &\quad -Pr\{Y(t) > r, X(t) = s' \mid X(0) = s\} \\ &= \mathbf{\Pi}_s^{\mathcal{M}}(s',t) - \overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(s',t,r)\end{aligned}$$

For $S' \subseteq S$ and time t, we denote by $\rho_s^{\mathcal{M}}(S',t)$ the *instantaneous reward* that denotes the rate at which reward is earned in states of S' at time t. It is defined as:

$$\rho_s^{\mathcal{M}}(S',t) = \sum_{s' \in S'} \mathbf{\Pi}_s^{\mathcal{M}}(s',t) \cdot \rho(s')$$

$\rho^{\mathcal{M}}(S')$ is the *expected (or long run) reward rate* that is equal to:

$$\rho^{\mathcal{M}}(S') = \sum_{s' \in S'} \mathbf{\Pi}^{\mathcal{M}}(s') \cdot \rho(s')$$

We denote by $\mathbf{\Upsilon}_s^{\mathcal{M}}(S',t,r)$ (resp. $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(S',t,r)$) the probability that at time t, MRM $\mathcal{M}$ is in subset S' and has accumulated a reward lower or equal (resp. higher) than r having started from an initial state s.

$$\mathbf{\Upsilon}_s^{\mathcal{M}}(S',t,r) = \sum_{s' \in S'} \mathbf{\Upsilon}_s^{\mathcal{M}}(s',t,r)$$

$$\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(S',t,r) = \sum_{s' \in S'} \overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(s',t,r)$$

2.2 Continuous Stochastic Reward Logic

Continuous Stochastic Reward Logic (CSRL)[8] is an extension of Continuous Stochastic Logic (CSL) [1] [3] by adding constraints over rewards. In [2], CSRL is extended to support further reward-based measures by allowing more state operators.

Syntax.

Let p be a probability threshold, $\triangleleft$ be a comparison operator such as $\triangleleft \in \{\leq, \geq, <, >\}$ and I (resp. J) be an interval of real number which represents a timing constraint (resp. bound for the cumulative reward). In the sequel, we denote by S_ϕ or ϕ-states the set of states that satisfy ϕ and by $\models$ the satisfaction relation. The syntax of CSRL is as follows:

$$\phi ::= true \mid a \mid \phi \wedge \phi \mid \neg\phi \mid \mathcal{P}_{\triangleleft p}(\phi\ \mathcal{U}_J^I \phi) \mid \mathcal{E}_J(\phi) \mid \mathcal{E}_J^t(\phi) \mid \mathcal{C}_J^I(\phi)$$

In this paper, for the sake of simplicity we do not consider the next operator X_J^I. We do not consider the other boolean connectives (false, $\vee$, $\Rightarrow$) that are derived in the usual way [2]. The steady-state operator and the transient operator of CSL logic are also omitted since we have already considered these operators with bounding approach in [13]. The path formula $\phi_1\ \mathcal{U}_J^I \phi_2$ asserts that ϕ_2 will be satisfied at some time $t \in I$ and that at all previous times ϕ_1 holds and the earned cumulative reward up to time t lies in J. However the formula $\mathcal{P}_{\triangleleft p}(\phi_1\ \mathcal{U}_J^I \phi_2)$ asserts that the probability measure of paths satisfying $\phi_1\ \mathcal{U}_J^I \phi_2$ meets the bound given by $\triangleleft p$. The state operator $\mathcal{E}_J(\phi)$ asserts that the expected (long run) reward rate for ϕ-states lies in J. $\mathcal{E}_J^t(\phi)$ asserts that the expected instantaneous reward rate at time t for ϕ-states lies in J. $\mathcal{C}_J^I(\phi)$ states that the expected amount of reward accumulated in ϕ-states during interval I lies in J.

Semantics .

Let us present briefly the semantics of these formulae [1],

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[2]:

$$s \models true \qquad \text{for all} \quad s \in S$$

$$s \models a \qquad \text{iff} \quad a \in L(s)$$

$$s \models \neg\phi \qquad \text{iff} \quad s \not\models \phi$$

$$s \models \mathcal{P}_{\lhd p}(\phi_1 \, \mathcal{U}_J^I \phi_2) \qquad \text{iff} \quad Prob_s^{\mathcal{M}}(\phi_1 \mathcal{U}_J^I \phi_2) \lhd p \tag{1}$$

$$s \models \mathcal{E}_J(\phi) \qquad \text{iff} \quad \rho^{\mathcal{M}}(S_\phi) = \sum_{s' \in S_\phi} \mathbf{\Pi}^{\mathcal{M}}(s') \cdot \rho(s') \in J \tag{2}$$

$$s \models \mathcal{E}_J^t(\phi) \qquad \text{iff} \quad \rho_s^{\mathcal{M}}(S_\phi, t) = \sum_{s' \in S_\phi} \mathbf{\Pi}_s^{\mathcal{M}}(s', t) \cdot \rho(s') \in J \tag{3}$$

$$s \models \mathcal{C}_J^I(\phi) \qquad \text{iff} \quad \int_I \rho_s^{\mathcal{M}}(S_\phi, t)dt = \int_I \sum_{s' \in S_\phi} \mathbf{\Pi}_s^{\mathcal{M}}(s', t) \cdot \rho(s')dt \in J \tag{4}$$

where $Prob^{\mathcal{M}}(s, \phi_1 \mathcal{U}_J^I \phi_2)$ denotes the probability measure of all paths starting from s satisfying $\phi_1 \, \mathcal{U}_J^I \phi_2$.

2.3 Checking CSRL formulas

In this subsection, we briefly introduce how CSRL formulas may be checked by means of transient, steady-state and joint distributions. We refer to [6] for further information on CSRL model checking. First we explain the checking procedure for reward operators $\mathcal{E}_J(\phi)$, $\mathcal{E}_J^t(\phi)$ and $\mathcal{C}_J^I(\phi)$ by means of steady-state or transient distribution of $\mathcal{M}$ or a transformed version of it, then we consider time and reward bounded until formula $\mathcal{P}_{\lhd p}(\phi_1 \mathcal{U}_J^I \phi_2)$.

2.3.1 *Reward operators*

The verification of reward operators requires the computation of steady-state and transient distributions. Indeed, to check the steady-state operator $\mathcal{E}_J(\phi)$ (resp. the instantaneous operator $\mathcal{E}_J^t(\phi)$), we compute the steady-state distribution $\mathbf{\Pi}^{\mathcal{M}}$ (resp. transient distribution at time t, $\mathbf{\Pi}_s^{\mathcal{M}}(t)$) and then we sum over the probabilities of ϕ-states multiplied with the corresponding rewards and finally we check if the obtained reward value lies in J or not (see Eq. 2 and Eq. 3). The accumulated operator $\mathcal{C}_J^I(\phi)$ can be evaluated using a variant of uniformization technique. Indeed, the verification of the accumulated reward operator can be done simply by computing the transient distribution at each moment of time interval I (see Eq. 4). The exact verification procedure has been given in [7].

2.3.2 *Time and reward bounded until formula*

In this subsection, we consider the verification of the until formula $\mathcal{P}_{\lhd p}(\phi_1 \mathcal{U}_J^I \phi_2)$. The checking of this formula requires the computation of $Prob_s(\phi_1 \mathcal{U}_J^I \phi_2)$ (see Eq. 1). It has been shown in [6] [8] that the computation of this probability, $Prob_s(\phi_1 \mathcal{U}_J^I \phi_2)$, can be reduced to the computation of the joint distribution of state and accumulated reward $\mathbf{\Upsilon}_s^{\mathcal{M}'}(t, r)$ of a transformed MRM $\mathcal{M}'$ which is inhomogeneous with respect to time and reward derived from the original homogeneous MRM $\mathcal{M}$, where $t = sup(I)$ (resp. $r = sup(J)$) is the upper bound time (resp. reward) of the given interval I (resp. J) (see Theorem 3 of [6]). In fact, the computation is based on changing the behavior of the considered MRM $\mathcal{M}$ when both, the given lower time bound $inf(I)$ and reward bound $inf(J)$, are exceeded. Recall that the MRM is homogeneous with respect to time and reward if the transition rate matrix $\mathbf{R}$ and the reward rates assigned to states of the MRM remain unchanged with regard to the current time or the accumulated reward. The evolution of the MRM has two phases: the first phase lasts until both lower bounds $inf(I)$ and $inf(J)$ are exceeded, after then the second phase begins.

However depending on reward and time intervals (I and J), this MRM $\mathcal{M}'$ can be reduced to a homogeneous one. In these cases, only one phase (the first or the second) of the MRM $\mathcal{M}'$ is used. For instance, if $I = [t, t]$ and $J = [0, r]$, only the first phase of $\mathcal{M}'$ is used because the two bounds $inf(I)$ and $inf(J)$ are exceeded exactly at time t when we have to evaluate the probability to be in states S_{ϕ_2}. In this case there is not a second phase and hence the modified chain $\mathcal{M}'$ is homogeneous. We refer to [6] for more details, where the author has established an intuitive interpretation of the construction of the MRM $\mathcal{M}'$ for all cases of intervals I and J.

We can recapitulate that depending on reward and time intervals I and J, the considered MRM $\mathcal{M}'$ for checking $\mathcal{P}_{\lhd p}(\phi_1 \mathcal{U}_J^I \phi_2)$ can be homogeneous or inhomogeneous with respect to time and reward. In our verification approach, we consider only the interval cases where $\mathcal{M}'$ is homogeneous. Indeed, our bounding approaches consider homogeneous Markov chains.

We present here the cases which can be reduced to a homogeneous case and explain the computation of $Prob_s(\phi_1 \mathcal{U}_J^I \phi_2)$ that can be derived from the *transient accumulated reward distribution*[6] . Let us remark that in some cases when $\mathcal{M}'$ is homogeneous the verification of the until formula can be reduced to the verification of a CSL formula (for instance when $I = [t, t]$ and $J = [r, r]$).

Let $\mathcal{M}[\phi]$ be the MRM defined from $\mathcal{M} = (S, \mathbf{R}, L, \rho)$, by making all ϕ-states in $\mathcal{M}$ absorbing and assigning 0 as reward, i.e.

$$\mathcal{M}' = \mathcal{M}[\phi] = (S, \mathbf{R}', L, \rho')$$

where $\mathbf{R}'(s, s') = \mathbf{R}(s, s')$, $\rho'(s) = \rho(s)$ if $s \not\models \phi$ and $\mathbf{R}'(s, s') = 0$, $\rho'(s) = 0$ otherwise.

1. Case $I = [t, t]$, $J = [0, r]$.

To determine the probability $Prob_s^{\mathcal{M}}(\phi_1 \mathcal{U}_{[0,r]}^{[t,t]} \phi_2)$, we have to compute the joint distribution $\mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1]}(t, r)$ and then we sum the probability of ϕ_2-states:

$$Prob_s^{\mathcal{M}}(\phi_1 \mathcal{U}_{[0,r]}^{[t,t]} \phi_2) = \mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1]}(S_{\phi_2}, t, r) \tag{5}$$

2. Case $I = [t, t]$, $J = (r, \infty)$.

This is almost the same as the former case, the only difference is that the accumulated reward must be larger than r instead at most r. Thus,

$$Prob_s^{\mathcal{M}}(\phi_1 \mathcal{U}_{(r,\infty)}^{[t,t]} \phi_2) = \overline{\mathbf{\Upsilon}}_s^{\mathcal{M}[\neg\phi_1]}(S_{\phi_2}, t, r)$$

3. Case $I = [t, t]$, $J = (r_1, r_2]$.

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It can be observed that:

$$Prob_s^{\mathcal{M}}(\phi_1\mathcal{U}_{(r_1,r_2]}^{[t,t]}\phi_2) = \mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1]}(S_{\phi_2},t,r_2) - \mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1]}(S_{\phi_2},t,r_1)$$

4. Case $I=[0,t], J=[0,r]$.

It has been shown in [8, Theorem 1] that s satisfies the formula $\mathcal{P}_{\lhd p}(\phi_1\mathcal{U}_{[0,r]}^{[0,t]}\phi_2)$ in $\mathcal{M}$ iff s satisfies $\mathcal{P}_{\lhd p}(true\,\mathcal{U}_{[0,r]}^{[t,t]}\phi_2)$ in $\mathcal{M}[\neg\phi_1\vee\phi_2]$, thus:

$$Prob_s^{\mathcal{M}}(\phi_1\mathcal{U}_{[0,r]}^{[0,t]}\phi_2) = \mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1\vee\phi_2]}(S_{\phi_2},t,r)$$

Let us mention that the dual formulas of the four cases presented above can be verified similarly by duality principle. Duality is based on changing reward and time constraints to facilitate the verification of $Prob_s(\phi_1\mathcal{U}_J^I\phi_2)$. It is derived from the fact that the progress of time can be regarded as the earning of reward and vice versa [2]. For instance, the dual formula of $\mathcal{P}_{\lhd p}(\phi_1\mathcal{U}_{[0,r]}^{[t,t]}\phi_2)$ is the until formula $\mathcal{P}_{\lhd p}(\phi_1\mathcal{U}_{[t,t]}^{[0,r]}\phi_2)$. So, when $I=[0,t]$ and $J=[r,r]$, the verification of $\mathcal{P}_{\lhd p}(\phi_1\mathcal{U}_{[r,r]}^{[0,t]}\phi_2)$ requires the computation of $Prob_s^{\mathcal{M}}(\phi_1\mathcal{U}_{[r,r]}^{[0,t]}\phi_2)$ that can be computed using Eq. 5 and interchanging time and reward intervals as follow [2]:

$$\begin{aligned} Prob_s^{\mathcal{M}}(\phi_1\mathcal{U}_{[r,r]}^{[0,t]}\phi_2) &= Prob_s^{\mathcal{M}^{-1}}(\phi_1\mathcal{U}_{[0,t]}^{[r,r]}\phi_2) \\ &= \mathbf{\Upsilon}_s^{\mathcal{M}^{-1}[\neg\phi_1]}(S_{\phi_2},r,t) \end{aligned}$$

where $\mathcal{M}^{-1}=(S,\mathbf{R}^{-1},L,\rho^{-1})$ is the MRM derived from $\mathcal{M}$ such that:

$$\mathbf{R}^{-1}(s,s') = \frac{\mathbf{R}(s,s')}{\rho(s)} \text{ and } \rho^{-1}(s) = \frac{1}{\rho(s)}$$

Different algorithms have been proposed to compute the joint distribution and a detailed comparison of the algorithmic intricacies can be found in [8][6]. In this paper we are interested in the analytical uniformization-based solution algorithm proposed by Sericola. In [18], Sericola derived a result for the distribution of occupation times in CTMCs for a given time t. The distribution of this occupation time can be used to compute $\mathbf{\Upsilon}_s^{\mathcal{M}}(t,r)$. It has been observed that if $\mathcal{O}(s,t)$ is the occupation time of state s prior to t then $\rho(s)\cdot\mathcal{O}(s,t)$ is the accumulated reward for this state prior to t.

Suppose that the considered MRM $\mathcal{M}$ has $m+1$ different rewards $\rho_0<\rho_1<\cdots<\rho_m$, $\rho_0=0$, and the initial distribution $\mathbf{\Pi}^{\mathcal{M}}(0)$ is defined as $\mathbf{\Pi}^{\mathcal{M}}(s,0)=1$ and $\mathbf{\Pi}^{\mathcal{M}}(s',0)=0$ for $s\neq s'$ then for $r\in[\rho_{h-1}t,\rho_h t)$ and $1\le h\le m$:

$$\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(t,r) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k}\mathbf{\Pi}^{\mathcal{M}}(0)\mathbf{C}_s^{\mathcal{M}}(h,n,k) \qquad (6)$$

where $r_h=\frac{r-\rho_{h-1}t}{\rho_h t-\rho_{h-1}t}$ and $\mathbf{C}_s^{\mathcal{M}}(h,n,k)$ is a square matrix defined recursively in terms of h, n and k[18]. It represents the complementary distribution $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(t,r)$ conditioned on n and k.

The main result that we use in this paper to propose our bounding approach to check $\mathcal{P}_p(\phi_1\mathcal{U}_J^I\phi_2)$ is given in [18, corollary 5.8] where author proves that if $\mathbf{P}_\lambda^{\mathcal{M}}$ is the uniformized matrix of $\mathcal{M}$ then $\mathbf{C}_s^{\mathcal{M}}(h,n,k)$ is positif and smaller than the power of n of $\mathbf{P}_\lambda^{\mathcal{M}}$ i.e.

$$\mathbf{C}_s^{\mathcal{M}}(h,n,k) \le (\mathbf{P}_\lambda^{\mathcal{M}})^n$$

It can be deduced from the above inequality and Eq. 6 that if $\mathbf{\Pi}_s(n)$ is the transient distribution at time n of $\mathbf{P}_\lambda^{\mathcal{M}}$, then:

$$\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}}(t,r) \le \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k}\mathbf{\Pi}_s(n) \qquad (7)$$

$$\mathbf{\Upsilon}_s^{\mathcal{M}}(t,r) \ge \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-\sum_{k=0}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k})\mathbf{\Pi}_s(n) \qquad (8)$$

3. STOCHASTIC COMPARISON BY CLASS $\mathcal{C}$ MATRICES

In this section we first give a brief overview on stochastic comparison [14] and then we introduce the class $\mathcal{C}$ matrices and their main properties [11].

3.1 Stochastic comparison

Let denote by $\mathcal{F}_{st}$ the class of all increasing real functions on a totally ordered state space E and by $\le_{st}$ the strong stochastic order relation.

Definition 1. *Let X and Y be two random variables taking values on a totally ordered space E,*

$$X\le_{st}Y \iff Ef(X)\le Ef(Y), \quad \forall f\in\mathcal{F}_{st}$$

whenever the expectations exist.

In the case of finite state space $\{1,2,\ldots,N\}$, the comparison of random variables are defined through following probability inequalities.

Proposition 1. *Let X and Y be two random variables taking values on $\{1,2,\ldots,N\}$, and $p=[p_1\ldots p_i\ldots p_N]$, $q=[q_1\ldots q_i\ldots q_N]$ be probability vectors which are respectively denoting distributions of X and Y.*

$$X\le_{st}Y \Leftrightarrow \sum_{k=i}^{N}p_k \le \sum_{k=i}^{N}q_k \qquad 1\le i\le N \qquad (9)$$

We apply the following definition to compare Markov chains:

Definition 2. *Let $\{X(n),n\ge 0\}$ (resp. $\{Y(n),n\ge 0\}$) be a DTMC. We say $\{X(n)\}\le_{st}\{Y(n)\}$, if :*

$$X(n)\le_{st}Y(n), \quad \forall n.$$

Let $\mathbf{\Pi_X^n}$ (resp. $\mathbf{\Pi_Y^n}$) be transient distribution at time n, and $\mathbf{\Pi_X}$ (resp. $\mathbf{\Pi_Y}$) its steady-state distribution (if it exists).

If $\{X(t)\}\le_{st}\{Y(t)\}$ then $\mathbf{\Pi_X^n}\le_{st}\mathbf{\Pi_Y^n}$, $\forall n$ and $\mathbf{\Pi_X}\le_{st}\mathbf{\Pi_Y}$.

The comparison of CTMCs can be established through the embedded DTMCs associated to them using uniformization technique.

Theorem 1. *Let $\{X(t),t\ge 0\}$ and $\{Y(t),t\ge 0\}$ be two uniformizable CTMCs and $\{X_\lambda(n),n\ge 0\}$ and $\{Y_\lambda(n),n\ge 0\}$ be the uniformized DTMCs associated to them. We have:*

If $\{X_\lambda(n)\}\le_{st}\{Y_\lambda(n)\}$ then $\{X(t)\}\le_{st}\{Y(t)\}$.

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In this work, the stochastic comparison is applied to construct a bounding chain within a particular class called class $\mathcal{C}$ chains by means of the stochastic monotonicity. We introduce this class in the following subsection and present its main properties. We first give the monotonicity and the comparability of transition matrices yield sufficient conditions to compare stochastically the underlying chains [14, p.186].

THEOREM 2. *Let* $\mathbf{P}$ *(resp.* $\mathbf{Q}$*) be the probability transition matrix of the time-homogeneous Markov chain* $\{X(n), n \geq 0\}$ *(resp.* $\{Y(n), n \geq 0\}$*). The comparison of Markov chains is established (*$\{X(n)\} \leq_{st} \{Y(n)\}$*), if the following conditions are satisfied :*

- $X(0) \leq_{st} Y(0)$,
- *at least one of the probability transition matrices is monotone, that is, either* $\mathbf{P}$ *or* $\mathbf{Q}$ *(say* $\mathbf{P}$*) is* $\leq_{st}$ *monotone, if for all probability vectors* p *and* q,

$$p \leq_{st} q \implies p\mathbf{P} \leq_{st} q\mathbf{P}$$

- *the transition matrices are comparable in the sense of the* $\leq_{st}$ *order :*

$$\mathbf{P} \leq_{st} \mathbf{Q} \Longleftrightarrow \mathbf{P}[i,*] \leq_{st} \mathbf{Q}[i,*], \quad \forall i \in E$$

where $\mathbf{P}[i,*]$ *denotes the ith row of matrix* $\mathbf{P}$.

In this paper, we apply the stochastic comparison to construct a bounding chain within a particular class called class $\mathcal{C}$ chains. In the following subsection we present this class and we give its main properties.

3.2 Class $\mathcal{C}$ matrices and closed-form distributions

We first introduce class $\mathcal{C}$ stochastic matrices and then give the closed-form solution for transient and steady-state distributions of time-homogeneous discrete (resp. continuous) time Markov chains for which probability transition (resp. the uniformized) matrices belong to this class.

DEFINITION 3 (CLASS $\mathcal{C}$ MATRIX). *A stochastic matrix* $\mathbf{P}$ *belongs to class* $\mathcal{C}$ *, if for each column* j *there exists a real constant* c_j *such that :*

$$\mathbf{P}(i,j) = \mathbf{P}(1,j) + (i-1)\ c_j, \qquad 1 \leq i,j \leq N. \qquad (10)$$

In fact, stochastic matrices of class $\mathcal{C}$ are defined by their first row and a set of real constants c_j, $1 \leq j \leq N$. This regular form yields interesting properties as the closed-form solution to compute transient distributions [12] and the steady-state distribution [11]. A stochastic matrix $\mathbf{P}$ in class $\mathcal{C}$ can be also represented by means of vectors:

$$\mathbf{P} = \mathbf{e}\ \mathbf{p} + \mathbf{d}\ \mathbf{c}$$

where $\mathbf{p}$ is the row vector representing the first row of $\mathbf{P}$; $\mathbf{c}$ is the row vector for constants c_j. The column vectors $\mathbf{e}$ and $\mathbf{d}$ are defined as follows : $e_i = 1, \quad d_i = (i-1), \qquad 1 \leq i \leq N$.

Since $\mathbf{P}$ is a stochastic matrix, $\mathbf{c}\ \mathbf{e} = 0$. The following theorems give the closed-form computation for transient and steady-state distributions in case time-homogeneous discrete (and continuous) time Markov chain of this class. The proof of these theorems can be found in [13][11].

THEOREM 3 (TRANSIENT DISTRIBUTION FOR DTMC). *Let* a*,* b *and* g *be the constants defined as follows :*

$$a = \mathbf{c}\ \mathbf{d} = \sum_{k=1}^{N} (k-1)\ c_k, \quad b = \mathbf{p}\ \mathbf{d} = \sum_{k=1}^{N} (k-1)\ p_{1,k}$$

$$g = \mathbf{\Pi^0}\ \mathbf{d} = \sum_{k=1}^{N} (k-1)\ \pi_k^0$$

Let $\{X(n),\ n \geq 0\}$ *be a time-homogeneous discrete time Markov chain with probability transition matrix* $\mathbf{P}$*. Let us note by* $\mathbf{\Pi^n}$ *the transient distribution of* $\{X(n), n \geq 0\}$ *at time* n*. If* $\mathbf{P}$ *belongs to class* $\mathcal{C}$ *, then for all* $n \geq 0$,

$$\mathbf{\Pi^n} = \mathbf{p} + \alpha_n\ \mathbf{c} \qquad (11)$$

where α_n *is the constant defined as*

$$\alpha_n = b \sum_{k=0}^{n-2} a^k + g\ a^{n-1} = \begin{cases} b\ \frac{(1-a^{n-1})}{1-a} + g\ a^{n-1}, & a \neq 1 \\ b\ (n-1) + g, & a = 1. \end{cases}$$

THEOREM 4 (TRANSIENT DISTRIBUTION FOR CTMC). *Let* a*,* b *and* g *be the constants defined in the previous theorem and* $\{X(t), t \geq 0\}$ *be a time-homogeneous continuous time Markov chain with infinitesimal generator* $\mathbf{Q}$*. Let us note by* $\mathbf{P}_\lambda$ *it uniformized matrix and by* $\mathbf{\Pi(t)}$ *the transient distribution at time* t *. If* $\mathbf{P}_\lambda$ *belongs to class* $\mathcal{C}$ *, with row vectors* $\mathbf{p}$ *representing the first row of* $\mathbf{P}_\lambda$ *and* $\mathbf{c}$ *representing column constants* c_j*, then for all* $t \geq 0$,

$$\mathbf{\Pi(t)} = e^{-\lambda t}\mathbf{\Pi(0)} + (1 - e^{-\lambda t})\mathbf{p} + \alpha(t)\ \mathbf{c} \qquad (12)$$

where $\alpha(t)$ *is defined as*

$$\begin{aligned} \alpha(t) &= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \alpha_n \\ &= \begin{cases} b\frac{1-e^{-\lambda t}}{1-a} + \\ (\frac{g}{a} - \frac{b}{a(1-a)})e^{-\lambda t}(e^{\lambda t a} - 1), & \text{if } a \neq \{1, 0\} \\ e^{-\lambda t}(\lambda t g - \lambda t b - b) + b, & \text{if } a = 0 \\ b\lambda t + (g-b)(1 - e^{-\lambda t}), & \text{if } a = 1 \end{cases} \end{aligned}$$

THEOREM 5 (STEADY-STATE DISTRIBUTION). *Let* $\mathbf{\Pi}$ *the stationary distribution of* $\{X(n), n \geq 0\}$*. If* $\mathbf{\Pi}$ *exists and transition probability matrix* $\mathbf{P}$ *belongs to class* $\mathcal{C}$ *then*

$$\mathbf{\Pi} = \mathbf{p} + \frac{b}{1-a}\ \mathbf{c} \qquad (13)$$

The proof of this theorem can be found in [11] where it has been demonstrated that a can never be equal to 1 if $\mathbf{P}$ is irreducible. Moreover let us remark here that in the case of CTMCs, the closed-form solution for steady-state distribution can be applied if the uniformized matrix $\mathbf{P}_\lambda$ belongs to class $\mathcal{C}$.

Obviously, in general the underlying model does not belong to class $\mathcal{C}$. We propose to construct class $\mathcal{C}$ bounding chain for the underlying Markov chain. Construction algorithms can be found in [11] for the $\leq_{st}$ order case.

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4. PROPOSED CHECKING METHOD

We have proposed in [13] to check CSL formulas with bounding distributions. We extend this approach to CSRL formulas. In this section we propose our methodology based on the stochastic comparison method to check reward operators $\mathcal{E}_J(\phi)$, $\mathcal{E}^t_J(\phi)$, and $\mathcal{C}^I_J(\phi)$ and time and reward bounded until operator $\mathcal{P}_{\triangleright p}(\phi_1 \mathcal{U}^I_J \phi_2)$. It can be seen from subsection 2.3 that model checking of these operators requires the computation of steady-state, transient or joint distribution in the considered Markov chain.

In our approach we will avoid to compute the required exact distributions, but using stochastic comparison technique (see section 3), we determine bounding distributions by means of closed-form solution of class C Markov chains. The proposed checking procedure exploits the quickness of computing the steady-state, the transient and therefore the joint bounding distributions through the closed-form solutions of class $\mathcal{C}$ matrices ($\theta(N)$ for computation and memory complexities). To do so, we compute class $\mathcal{C}$ bounding distributions rather than the exact distributions to check the underlying operators. The overall complexity to check the considered CSRL formula is determined by the complexity to construct bounding matrices which is $\theta(N^2)$ in the worst case [11]. Thus the computation complexity is largely diminished: $\theta(N^2)$ instead of $\theta(N^3)$ for steady-state reward operator $\mathcal{E}_J(\phi)$, and $\theta(N^2)$ instead of $\theta(\lambda t N^3)$ for transient reward operator $\mathcal{E}^t_J(\phi)$, accumulated reward operator $\mathcal{C}^I_J(\phi)$ and until operator $\mathcal{P}_{\triangleleft p}(\phi_1\ \mathcal{U}^I \phi_2)$.

Let us emphasize here that the proposed method constitutes a first step rapid model checking. Since we check a CSRL formula through a bounding distribution, it is not always possible to conclude if the underlying formula is checked or not. In the case if we can not conclude, the model checking must be performed by the usual methods. However to include the proposed method as a first step checking would not increase significantly the complexity but may let to decrease largely the overall complexity for some cases. Let us give the proposed model checking approach for reward operators $\mathcal{E}_J(\phi)$, $\mathcal{E}^t_J(\phi)$ and $\mathcal{C}^I_J(\phi)$ and then we give for until operator $\mathcal{P}_{\triangleright p}(\phi_1 \mathcal{U}^I_J \phi_2)$

4.1 Model checking of reward operators

It can be seen from Eq. 2, Eq. 3 and Eq. 4 that to check $\mathcal{E}_J(\phi)$, $\mathcal{E}^t_J(\phi)$ and $\mathcal{C}^I_J(\phi)$ we should compute respectively the steady-state distribution $\mathbf{\Pi}^{\mathcal{M}}$, the transient distribution at time t, $\mathbf{\Pi}^{\mathcal{M}}_s(t)$ and the accumulated transient distributions during time interval I,$\int_I \mathbf{\Pi}^{\mathcal{M}}_s(t)dt$ of the underlying Markov chain $\mathcal{M}$. Then we sum under S_ϕ states, the steady-state, transient or accumulated transient probabilities multiplied with their corresponding rewards to check if these bounding reward values meets the bound of J or not. We denote by $R(\phi)$ this reward value. Recall that in our approach we will compute bounding distributions rather than the exact distributions. Let us denote by $\mathcal{M}_{sup}$ the upper bounding chain of $\mathcal{M}$ in the sense of $\leq_{st}$ order and by $\mathbf{\Pi}^{\mathcal{M}_{sup}}_s(t)$ (resp. $\mathbf{\Pi}^{\mathcal{M}_{sup}}$) its transient (resp. steady-state) distribution.

Proposition 2. *Assume that the chains $\mathcal{M}$ and $\mathcal{M}_{sup}$ are comparable in the sens $\leq_{st}$ (see definition 2). Moreover assume that the state space is reordered to put S_ϕ in the last and states belonging to S_ϕ are reordered according to their increasing rewards. We have:*

- *Bound on steady-state reward rate:*

$$\rho^{\mathcal{M}}_s(S_\phi) = \sum_{s' \in S_\phi} \mathbf{\Pi}^{\mathcal{M}}(s') \cdot \rho(s') \leq \sum_{s' \in S_\phi} \mathbf{\Pi}^{\mathcal{M}_{sup}}(s') \cdot \rho(s')$$

- *Bound on instantaneous reward rate at time t:*

$$\rho^{\mathcal{M}}_s(S_\phi, t) = \sum_{s' \in S_\phi} \mathbf{\Pi}^{\mathcal{M}}_s(s', t) \cdot \rho(s') \leq \sum_{s' \in S_\phi} \mathbf{\Pi}^{\mathcal{M}_{sup}}_s(s', t) \cdot \rho(s')$$

- *Bound on accumulated reward during the interval I:*

$$\int_I \rho^{\mathcal{M}}_s(S_\phi, t) = \sum_{s' \in S_\phi} \rho(s') \cdot \int_I \mathbf{\Pi}^{\mathcal{M}}_s(s', t)dt \leq \sum_{s' \in S_\phi} \rho(s') \cdot \int_I \mathbf{\Pi}^{\mathcal{M}_{sup}}_s(s', t)dt$$

Proof. By construction the Markov chain $\mathcal{M}_{sup}$ is an upper bound to $\mathcal{M}$ in the sens of $\leq_{st}$:

$$\mathcal{M} \leq_{st} \mathcal{M}_{sup}$$

We can deduce so from definition 2 that the transient distributions of $\mathcal{M}$ and $\mathcal{M}_{sup}$ are $\leq_{st}$ comparable:

$$\mathbf{\Pi}^{\mathcal{M}}_s(t) \leq_{st} \mathbf{\Pi}^{\mathcal{M}_{sup}}_s(t)$$

and the steady-state distributions of $\mathcal{M}$ and $\mathcal{M}_{sup}$ are $\leq_{st}$ comparable:

$$\mathbf{\Pi}^{\mathcal{M}} \leq_{st} \mathbf{\Pi}^{\mathcal{M}_{sup}}$$

Therefore we have the inequalities between the increasing functionals of these distributions (see definition 1) for states S_ϕ which are put at the end of the state space and ordered according to their increasing rewards.

□

Let us recall here that in [11], authors have proposed algorithms to construct upper and lower bounding chains in the sens of $\leq_{st}$ belonging to class $\mathcal{C}$. Therefore, lower bounding chain $\mathcal{M}_{inf}$, that is $\leq_{st}$ monotone and belongs to class $\mathcal{C}$ can be provided. Hence, we can deduce lower bounds to the steady-state reward rate, $\rho^{\mathcal{M}}_s(S_\phi)$, to the instantaneous reward rate at time t, $\rho^{\mathcal{M}}_s(S_\phi, t)$, and to the accumulated reward during the interval I, $\int_I \rho^{\mathcal{M}}_s(S_\phi, t)$. We give only the upper bounding case in the verification of reward operators, as the lower bounding case is similar.

Suppose now that we want to check a reward formula Fr. First we start by reordering the state space by putting S_ϕ states at the end and states belonging to S_ϕ are reordered according to increasing rewards since $\leq_{st}$ stochastic ordering is associated to increasing reward functions. Then we construct the uniformized matrix for the obtained MRM that is denoted by $\mathbf{P}^{\mathcal{M}}_\lambda$. Once the state space is reordered and the uniformized matrix is computed, we construct for the uniformized matrix $\mathbf{P}_\lambda$, a monotone, bounding matrix in the sense of $\leq_{st}$ order which belongs to class $\mathcal{C}$. The construction algorithms are not given here because of the lack of space but they can be found in [11]. We denote by

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$\mathbf{P}_\lambda^{\mathcal{M}_{sup}}$ the upper bounding matrix and discuss only upper bounding case, since lower bounding case is similar. Since bounding matrix $\mathbf{P}_\lambda^{\mathcal{M}_{sup}}$ belongs to class $\mathcal{C}$, we compute the closed-form bounding transient distribution $\mathbf{\Pi}_s^{\mathcal{M}_{sup}}(t)$ and steady-state distribution $\mathbf{\Pi}^{\mathcal{M}_{sup}}$ by means of Eqs. 12 and 13.

In the case of checking the accumulated reward operator $\mathcal{C}_J^I(\phi)$, $\int_I \mathbf{\Pi}_s^{\mathcal{M}_{sup}}(t)dt$, it can be deduced easily by applying $\int_I$ to the closed-form transient distribution. Once bounding distributions are computed, we sum under S_ϕ bounding probabilities multiplied with the corresponding rewards to obtain bounding reward values $R_{inf}(\phi)$ and $R_{sup}(\phi)$ for $R(\phi)$. The following proposition gives how we can check the underlying reward formula Fr to check if $R(\phi)$ meets the bound of reward interval $J \in [r_{min}, r_{max}]$ or not.

PROPOSITION 3. *1. if $R_{inf}(\phi) \geq r_{min}$ and $R_{sup}(\phi) \leq r_{max}$ then we can conclude that Fr is true*

2. if $R_{inf}(\phi) > r_{max}$ or $R_{sup}(\phi) < r_{min}$ then we can conclude that Fr is false

3. otherwise, we cannot conclude if Fr is true or not, through these bounding distributions.

PROOF. It follows obviously from the fact that by construction (see proposition 2), we have:

$$R_{inf}(\phi) \leq R(\phi) \leq R_{sup}(\phi)$$

Thus, case 1 allows us to conclude that the considered formula Fr is satisfied:

$$r_{min} \leq R_{inf}(\phi) \leq R(\phi) \leq R_{sup}(\phi) \leq r_{max}$$

Similarly, case 2 lets us to conclude that Fr is not satisfied. Otherwise the reward computed on closed-form bounding distributions do not let us to conclude if the formula Fr is satisfied or not.

□

4.2 Model checking of until operator

In Subsection 2.3, we have summarized how model checking of $\mathcal{P}_{\triangleright p}(\phi_1 \mathcal{U}_J^I \phi_2)$ over a MRM $\mathcal{M}$ can be reduced to the computation of joint distribution of another MRM $\mathcal{M}'$ obtained by transforming $\mathcal{M}$. For instance, if the time interval $I = [0, t]$ and the reward interval $J = [0, r]$ then to check $\mathcal{P}_{\triangleright p}(\phi_1 \mathcal{U}_{[0,r]}^{[0,t]} \phi_2)$, we must compute the joint distribution at time t and for reward r, $\mathbf{\Upsilon}_s^{\mathcal{M}[\neg\phi_1 \vee \phi_2]}$(t,r) of the Markov chain $\mathcal{M}[\neg\phi_1 \vee \phi_2]$ and then we sum probabilities of states belonging to S_{ϕ_2}. Moreover, it can be seen from Eq. 8 and Eq. 7 that there is a lower bound to the joint distribution $\mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r)$ and an upper bound to its complementary $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$ that we will denote by respectively $\mathbf{B}$ and $\overline{\mathbf{B}}$. $\mathbf{B}$ and $\overline{\mathbf{B}}$ are equal respectively to:

$$\mathbf{B} = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1 - \sum_{k=0}^{n}\binom{n}{k} r_h^k (1-r_h)^{n-k})\mathbf{\Pi}_s(n)$$

$$\overline{\mathbf{B}} = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} r_h^k (1-r_h)^{n-k}\mathbf{\Pi}_s(n)$$

Let us remark here that these bounds are not necessarily probability vectors but positive vectors. Moreover we extend the definition of the $\leq_{st}$ order to positive vectors as the satisfaction of inequalities given in Proposition 1. These bounds require the computation of transient distributions $\mathbf{\Pi}_s^{\mathcal{M}'}(n)$ of the uniformized matrix $\mathbf{P}_\lambda^{\mathcal{M}'}$ of $\mathcal{M}'$. Likewise, using stochastic comparison technique we will not compute the exact value of $\mathbf{\Pi}_s^{\mathcal{M}'}(n)$ and so for $\mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r)$ and $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$, but we will compute bounding distributions to $\mathbf{\Pi}_s^{\mathcal{M}'}(n)$ that we denote by $\mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$ (resp. $\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n)$) its upper (resp. lower) bounding distribution. Since by construction bounding chains belong to class $\mathcal{C}$, bounding distributions have closed-form solution and thus we can deduce closed-form bounds to $\mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r)$ and $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$ as we give in the following theorem.

THEOREM 6. *Let $\mathcal{M}'_{inf}$ (resp. $\mathcal{M}'_{sup}$) the lower (resp. upper) bounding chain to $\mathcal{M}'$ that belongs to class $\mathcal{C}$, we have:*

- *Stochastic upper bound to $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$ defined as:*

$$\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r) \leq_{st} \overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r) \tag{14}$$

$$\begin{aligned}\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r) \;&=\; e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(0) + (1 - e^{-\lambda r_h t})\mathbf{p} \\ &\quad +(\alpha(t) - \psi(t))\ \mathbf{c}\end{aligned}$$

- *Stochastic lower bound to $\mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r)$ defined as:*

$$\mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r) \geq_{st} \mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r) \tag{15}$$

$$\begin{aligned}\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r) \;&=\; e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(0) + (e^{-\lambda r_h t} - e^{-\lambda t})\mathbf{p} \\ &\quad +\psi(t)\ \mathbf{c}\end{aligned}$$

where $\mathbf{p}$, $\mathbf{c}$ and $\alpha(t)$ were already defined in subsection 3.2, $\psi(t)$ is defined as:

$$\begin{aligned}\psi(t) \;&=\; e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda(1-r_h)t)^n}{n!}\alpha_n \\ &=\; \begin{cases} \frac{b}{1-a}(e^{-\lambda r_h t} - e^{-\lambda t}) \\ +(\frac{g}{a} - \frac{b}{a(1-a)})e^{-\lambda t}(e^{\lambda(1-r_h)ta} - 1), & if a \neq \{1,0\} \\ e^{-\lambda t}(\lambda(1-r_h)tg \\ +be^{\lambda(1-r_h)t} - (1-r_h)\lambda t b - b), & if\ a = 0 \\ b\lambda(1-r_h)te^{-\lambda r_h t} \\ +(g-b)(e^{-\lambda r_h t} - e^{-\lambda t}), & if\ a = 1 \end{cases}\end{aligned}$$

PROOF. By construction $\mathcal{M}'_{inf}$ and $\mathcal{M}'_{sup}$ are monotone bounding Markov chain to $\mathcal{M}'$ in the sense of $\leq_{st}$ and they belong to class $\mathcal{C}$. This involves that their corresponding uniformized transient distributions at time n are comparable (see Definition 2):

$$\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n) \leq_{st} \mathbf{\Pi}_s^{\mathcal{M}'}(n) \leq_{st} \mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$$

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and $\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n)$, $\mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$ have closed-form (see Eq. 11) .

Let us denote respectively by $\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r)$ and $\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r)$ the following vectors:

$$\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-\sum_{k=0}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k})\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n)$$

and

$$\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k}\mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$$

From the fact that terms that multiply $\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n)$ and the distribution $\mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$ are positif in each time n we can conclude that:

$$\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r) \leq_{st} \mathbf{B} \ \text{ and } \ \overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r) \geq_{st} \overline{\mathbf{B}}$$

Recall that:

$$\mathbf{B} \leq \mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r) \ \text{ and } \overline{\mathbf{B}} \geq \overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$$

We can deduce so that:

$$\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r) \leq_{st} \mathbf{\Upsilon}_s^{\mathcal{M}'}(t,r) \ \text{ and } \overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r) \geq_{st} \overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(t,r)$$

Using the closed-form that have bounding distributions and taking under account that:

$$\sum_{k=1}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k} = 1-(1-r_h)^n$$

we obtain easily a following closed-form to $\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r)$ and $\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r)$:

$$\begin{aligned}\mathbf{R}^{\mathcal{M}'_{inf}}(t,r) &= e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(0) + \\ &\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-\sum_{k=1}^{n}\binom{n}{k} r_h^k(1-r_h)^{n-k})\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n) \\ &= e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(0) + \\ &\quad \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-r_h)^n\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n) \\ &= e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(0) + \\ &\quad \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-r_h)^n(\mathbf{p}+\alpha_n\ \mathbf{c}) \\ &= e^{-\lambda t}\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(0) + (e^{-\lambda r_h t}-e^{-\lambda t})\mathbf{p} + \psi(t)\ \mathbf{c}\end{aligned}$$

Closed-form solution for $\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r)$ is similarly derived. □

Moreover to check the until operator, we have to sum the probability of states belonging to S_{ϕ_2} and we check if this probability meets the threshold p. Indeed,

$$\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(n) \leq_{st} \mathbf{\Pi}_s^{\mathcal{M}'}(n) \leq_{st} \mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(n)$$

we can deduce that:

$$\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(S_{\phi_2},n) \leq \mathbf{\Pi}_s^{\mathcal{M}'}(S_{\phi_2},n) \leq \mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(S_{\phi_2},n)$$

if we reorder the state space and we put S_{ϕ_2} states at the end (see Eq. 9).

If we denote by $\mathbf{R}_s^{\mathcal{M}'_{inf}}(S_{\phi_2},t,r)$ (resp. $\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(S_{\phi_2},t,r)$) the probability sum in vector $\mathbf{R}_s^{\mathcal{M}'_{inf}}(t,r)$ (resp. $\overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(t,r)$) to be in states belonging to S_{ϕ_2}, we can deduce from the following proposition an upper bound to $\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(S_{\phi_2},t,r)$ and a lower bound to $\mathbf{\Upsilon}_s^{\mathcal{M}'}(S_{\phi_2},t,r)$:

Proposition 4. *By reordering the state space and putting S_{ϕ_2} states at the end, we have:*

$$\overline{\mathbf{\Upsilon}}_s^{\mathcal{M}'}(S_{\phi_2},t,r) \leq \overline{\mathbf{R}}_s^{\mathcal{M}'_{sup}}(S_{\phi_2},t,r) \tag{16}$$

$$\mathbf{\Upsilon}_s^{\mathcal{M}'}(S_{\phi_2},t,r) \geq \mathbf{R}_s^{\mathcal{M}'_{inf}}(S_{\phi_2},t,r) \tag{17}$$

Proof. It follows obviously from the fact that:

$$\mathbf{\Pi}_s^{\mathcal{M}'_{inf}}(S_{\phi_2},n) \leq \mathbf{\Pi}_s^{\mathcal{M}'}(S_{\phi_2},n) \leq \mathbf{\Pi}_s^{\mathcal{M}'_{sup}}(S_{\phi_2},n)$$

□

5. EXAMPLE

We illustrate our proposed approach under a cellular mobile system where the effect of handoff arrivals and the use of guard channels are included [16]. The phenomenon of handoff occurs when a mobile station moves across a cell boundary. The channel in the earlier cell is released and an idle channel is required. If there is no idle channel offered by the new base station, the handoff call is *dropped*. There is a fixed number of channels called guard channels which are reserved to handoff calls. We consider in this example that the mobile station has to treat two types of calls: best effort calls (or new calls) that have low priority and handoff calls that have the high priority. A best effort call is said to be *blocked*, if there are not enough channels available for them. Moreover, due to the quality of service (QoS) requirements of advanced applications, some calls claim more bandwidth (expressed by channels in this example) than others to be established. So, in the considered model, we distinguish two types of best efforts calls and two types of handoff calls.

In the sequel, we design by b_1 (resp. b_2) best effort calls that use one (resp. two) channels and by h_1 (resp. h_2) handoff calls that need one (resp. two) channels. Let N be the total number of idle channels of the base station and g the number of guard channels reserved to handoff calls (h_1 and h_2). We consider Poisson arrivals and exponential service times for all types of calls.

Under these Markovian arrival hypothesis, the considered system can be modelled as a CTMC with state space $S = \{(n_{b_1}, n_{b_2}, n_{h_1}, n_{h_2}) \mid C = n_{b_1} + 2n_{b_2} + n_{h_1} + 2n_{h_2} \leq N\}$ where C represents the total number of channel occupied, n_{b_1} (resp. n_{b_2}, n_{h_1} and n_{h_2}) represents the number of calls b_1 (resp. b_2, h_1 and h_2) currently in the cell.

We note that the size of the underlying Markov chain increases if more service class calls are considered. We associate to each state s atomic propositions that characterize the state s. In this example, we assign the following atomic propositions

- we assign *guardused* to states in which number of channels used (i.e. $C > N - g$).
- we assign *canalbusy* for states that all channels are busy (i.e. $C = N$).

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Reward values assigned to states of the considered system depends on the reward measure that we want to evaluate (see first column of table 1).

Based on these atomic propositions and the considered reward function, the considered chain is a labelled MRM characterized by the state space S, the rate matrix, the set of atomic propositions and the considered reward structure.

Different performance measures can be checked through checking CSRL formulas for the considered system. For instance, we check $\mathcal{E}^{50}_{[0,0.6]}(true)$ to evaluate the occupation channel rate at time 50 and the occupation rate of guard channel at time 60 by checking $\mathcal{E}^{60}_{[0,0.05]}(guardused)$. The formula $\mathcal{C}^{[0,50]}_{[0,8]}(canalbusy)$ is checked to estimate the accumulated time during $[0, 50]$ that all channels are busy. By checking the until formula $\mathcal{P}_{\leq 0.01}(true\mathcal{U}^{[15,15]}_{[30,30]}canalbusy)$, we evaluate the probability that at time 15 all channels are busy and in that time the number of channels occupied by handoff calls is equal to 30. Moreover the until formula $\mathcal{P}_{\leq 0.1}(true\mathcal{U}^{[20,20]}_{[30,30]}canalbusy)$ is checked to evaluate the probability at time 20. We give in table 1 some numerical results obtained when we check some CSRL formulas using our proposed approach. In the last column of the table, the symbol ? indicates that we cannot conclude whether the formula is satisfied or not through these bounding distributions.

We can observe that if the bound computed is not sufficently accurate, we cannot conclude if the considered formula is verified or not. Contrary to the other bounding approaches that we have employed in the verification of model checking formulas [15], it is not possible to refine the class $\mathcal{C}$ bounding models considered in this paper. So if we cannot conclude if the considered formula is verified or not we have to use classical model checking algorithms or use another bounding approach.

We note that numerical results have been obtained without the use of a particular model checker but it based on the use of Markov chain resolution tool. We have considered that $N = 30$, $g = 5$, the arrival rate $\lambda_{b_1} = \lambda_{h_1} = 0.001$, $\lambda_{b_2} = \lambda_{h_2} = 0.0005$, and the service rate of b_1, b_2, h_1 and h_2 calls equal to 0.0001.

6. CONCLUSIONS

In this paper, we presented a bounding approach based on stochastic comparison to check CSRL operators. By constructing bounding class $\mathcal{C}$ Markov chains, we can compute transient and steady-state distributions through closed-form solutions which reduces significantly memory and computation complexities for checking CSRL operators. We can not always conclude if the studied property is validated or violated from bounding distributions. However this approach provides a first step model checking algorithm, if we can not concluded, then we must apply classical model checking algorithms.

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Table 1: Checking CSRL formulas

Reward	Formulas	Exact	Bound	Valid?
Occupation channel rate	$\mathcal{E}^{50}_{[0,0.6]}(true)$	$4.5e^{-1}$	$5.4e^{-1}$	yes
Occupation guard channel rate	$\mathcal{E}^{60}_{[0,0.05]}(guardused)$	$3.1e^{-2}$	$4.6e^{-2}$	yes
Occupation interval time	$\mathcal{C}^{[0,50]}_{[0,8]}(canalbusy)$	0.44	3.83	yes
Number of channels used by handoff calls	$\mathcal{P}_{\leq 0.01}(true\mathcal{U}^{[15,15]}_{[30,30]}canalbusy)$	$7.7e^{-5}$	$2.3e^{-2}$	?
Number of channels used by handoff calls	$\mathcal{P}_{\leq 0.1}(true\mathcal{U}^{[20,20]}_{[30,30]}canalbusy)$	$3.2e^{-4}$	$3.1e^{-2}$	yes

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