Perturbation of CTMC Trapping Probabilities with Application to Model Repair

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ABSTRACT

This paper studies properties of continuous-time Markov chains with one class of transient states and at least two absorbing states. We look at a perturbation of the chain that arises by uniformly decreasing all rates to absorption. For this situation, the monotonicity of the trapping probabilities is analysed, and their asymptotic limit is computed. The theoretical findings are then applied to a type of model repair problem, where a lower time-bounded and lower probability-bounded CSL until requirement needs to be satisfied. The paper presents an algorithm for this type of problem and proves its correctness.

CCS CONCEPTS

• Computing methodologies → Model verification and validation; **• Theory of computation → Verification by model checking**; **• Mathematics of computing → Markov processes**;

KEYWORDS

Markov Chain, Perturbation, Trapping Probability, Monotonicity, CSL Model Checking, Model Repair

ACM Reference format:

Alexander Gouberman, Markus Siegle, and Bharath Siva Kumar Tati. 2017. Perturbation of CTMC Trapping Probabilities with Application to Model Repair. In *Proceedings of 11th EAI International Conference on Performance Evaluation Methodologies and Tools, Venice, Italy, December 5–7, 2017 (VALUETOOLS 2017),* 8 pages.
https://doi.org/10.1145/3150928.3150947

1 INTRODUCTION

This paper studies properties of continuous-time Markov chains (CTMC) with one class of transient states and at least two absorbing states. In particular, we study a setting where all rates into the absorbing states are multiplied by a small perturbation factor. We analyze in detail the behaviour of the trapping probabilities (also known as hitting probabilities) – seen as functions of the perturbation factor – from the individual transient states to the different absorbing states. We show that, although the individual trapping probabilities are not necessarily monotonic wrt. the perturbation factor, interestingly, their enveloping functions are indeed monotonic. It is also shown that, as the perturbation factor goes to zero, in the limit the trapping probabilities from all transient states coincide, and those limits can be calculated from the unique stationary distribution of the transient class (when transitions to absorbing states are ignored) and the rates from transient to absorbing states. In order to achieve these goals, we use the concept of the Drazin inverse which allows to perform the analysis in terms of simpler algebraic manipulations.

For us, the question of monotonicity of trapping probabilites arose when we tried to solve a particular instance of model repair. In general, the model repair problem is to fix a system (or rather a model thereof), in case it does not satisfy some desirable property. Earlier work on model repair of probabilistic systems can be found, e.g. in [2, 7, 18]. We are interested in model repair problems arising in the context of CTMCs labelled with state properties, where requirements are expressed in continuous stochastic logic (CSL) [1], a temporal logic that has become very popular and can be automatically checked with tools such as PRISM [15]. We look at a typical time-bounded reach-avoid requirement that is expressed by the CSL until operator with lower time bound and lower probability bound. For the case that the requirement is violated, the paper proposes an algorithm how to repair the model, by uniformly reducing certain subsets of its transition rates. Depending on the case at hand, either one or two reduction factors are employed. It is exactly the monotonicity property derived in the earlier sections which ensures that our model repair strategy will be always successful, i.e. it is shown that the proposed algorithm will always succeed in repairing the model.

Related work: Absorbing Markov chains, also known as lossy Markov chains, have received much attention for a long time, see e.g. [9] for an early paper. Of particular interest is their quasi-stationary distribution [8], also referred to as quasi-limiting distribution, i.e. the kind of equilibrium attained after a long time, provided that absorption has not yet happened. From a different point of view, an absorbing Markov chain can also be seen as a phase-type distribution [16, 17]. This powerful class of probability distributions is frequently used for the fitting of traffic traces [13], where the matching of moments [4] and finding canonical representations [14] are prime concerns. In that context, the focus is on the distribution of the time to absorption, and there is usually only a single absorbing state. To the best of our knowledge, the question of monotonicity of trapping probabilities, in the context of perturbed absorbing

VALUETOOLS 2017, December 5–7, 2017, Venice, Italy

ACM ISBN 978-1-4503-6346-4/17/12…$15.00
https://doi.org/10.1145/3150928.3150947

Markov chains with more than one absorbing state, has not received any attention in the literature.

Finally we mention work on parameter synthesis for parametric Markov chains and Markov decision processes, which is related to model repair. In [6, 11] and the recent paper [19] strategies have been proposed to find valid parameter values in a multi-dimensional search space.

Structure of the paper: Sec. 2 introduces some terminology and notation, Sec. 3 derives the main monotonicity result for the perturbed Markov chains, and Sec. 4 is devoted to the analysis of the asymptotic limit of the trapping probabilities as the perturbation factor goes to zero. In Sec. 5, an algorithm for the model repair problem is presented and its correctness is proven with the help of the results from Sec. 3 and 4. Conclusions and future work are discussed in Sec. 6.

2 PRELIMINARIES

For a matrix $A = (A_{ij}) \in \mathbb{R}^{n\times m}$ write $A \geq 0$ if $A_{ij} \geq 0$ for all i, j and $A \gg 0$ if $A_{ij} > 0$ for all i, j. Let $\mathbf{1} \in \mathbb{R}^n$ denote the column vector with values $\mathbf{1}_i = 1$. A matrix $P \in \mathbb{R}^{n\times n}$ is *stochastic* if $P \geq 0$ and $P\mathbf{1} = \mathbf{1}$ and *substochastic* if $P \geq 0$ and $P\mathbf{1} \leq \mathbf{1}$. A matrix $Q \in \mathbb{R}^{n\times n}$ is a *generator* if $Q_{ij} \geq 0$ for all $i \neq j$ and $Q\mathbf{1} = 0$ and a *subgenerator* if $Q_{ij} \geq 0$ for all $i \neq j$ and $Q\mathbf{1} \leq 0$. There are several ways to convert a (sub-)stochastic matrix into some (sub-)generator and vice versa[1]. A matrix P is *strictly substochastic* if it is substochastic but not stochastic and similarly a matrix Q is a *strict subgenerator* if it is a subgenerator but not a generator. If Q is a generator and $D \leq 0$ is diagonal then $Q + D$ is a subgenerator. Conversely, every subgenerator S can be uniquely decomposed into $S = Q + D$ where Q is a generator and $D \leq 0$ is diagonal. In particular, S is a strict subgenerator if and only if $D \neq 0$.

A matrix $A \in \mathbb{R}^{n\times n}$ is a *nonsingular M-matrix* if $A_{ij} \leq 0$ for all $i \neq j$ and every eigenvalue of A has a strictly positive real part [3, Chapter 6]. For every nonsingular M-matrix it holds $A^{-1} \geq 0$. If A is an irreducible nonsingular M-matrix then $A^{-1} \gg 0$ [3, Chapter 6, Theorem 2.7].

For $A \in \mathbb{C}^{n\times n}$ denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the nullspace and range of A, by $\text{ind}(A) := \min\{k \in \mathbb{N} \mid \mathcal{N}(A^k) = \mathcal{N}(A^{k+1})\} < \infty$ the *index* of A and by $A^D \in \mathbb{C}^{n\times n}$ the Drazin inverse of A, i.e. the unique matrix satisfying $A^{\nu+1}A^D = A^\nu$ for $\nu = \text{ind}(A)$, $A^DAA^D = A^D$ and $AA^D = A^DA$. If A is invertible (i.e. $\text{ind}(A) = 0$) then $A^D = A^{-1}$ and if A is nilpotent then $A^D = 0$. If $\text{ind}(A) \leq 1$ then it also holds that $AA^DA = A$ (since in this case A has a group inverse which coincides with A^D). For any matrix $A \in \mathbb{C}^{n\times n}$ we can decompose $\mathbb{C}^n = \mathcal{N}(A^\nu) \oplus \mathcal{R}(A^\nu)$ where $\nu := \text{ind}(A)$ and the matrices AA^D and $I - AA^D$ are the corresponding projections to $\mathcal{R}(A^\nu)$ along $\mathcal{N}(A^\nu)$ resp. vice versa. For any projection P it holds $P^D = P$. If $A, B \in \mathbb{C}^{n\times n}$ commute then $(AB)^D = B^DA^D$.

A matrix $A \in \mathbb{C}^{n\times n}$ is *semistable* if $\text{ind}(A) \leq 1$ and the non-zero eigenvalues of A have strictly negative real part [5]. Equivalently, A is semistable if and only if e^{At} converges as $t \to \infty$ and in this case the limit is given by $\lim_{t\to\infty} e^{At} = I - AA^D$. For us of interest is the fact that every generator Q is semistable. The limiting matrix $\lim_{t\to\infty} e^{Qt} = I - QQ^D$ is called the *ergodic projection* of Q and its rows comprise the stationary distributions of the corresponding Markov chain.

[1] If Q is a (sub-)generator and D a non-singular matrix such that $P := DQ + I \geq 0$ then P is (sub-)stochastic. The uniformization and the embedding of a generator correspond to choosing D as a suitable diagonal matrix.

3 PERTURBED TRAPPING PROBABILITIES AND MONOTONICITY

3.1 Setting

Consider an absorbing continuous-time Markov chain (CTMC) with m absorbing states and n transient states such that all transient states communicate and all absorbing states are reachable from some transient state (and thus from all transient states). In other words, the generator $Q \in \mathbb{R}^{(m+n)\times(m+n)}$ of the Markov chain can be decomposed as $Q = Q_1 + Q_2$ with

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 0 \\ F & D \end{pmatrix} \tag{1}$$

where $E \in \mathbb{R}^{n\times n}$ is the generator of an irreducible Markov chain over n states (the transient states of Q with transitions to absorbing states omitted), $F \in \mathbb{R}^{n\times m}$ is a matrix comprising the rates for transitions to the absorbing states and $D = -\Delta(F\mathbf{1}) \in \mathbb{R}^{n\times n}$ is the diagonal matrix comprising the negative row sums of F (the operator Δ turns a column vector into a diagonal matrix). Note that D contains at least one strictly negative diagonal entry.

Let $\Pi := \lim_{t\to\infty} e^{Qt} \in \mathbb{R}^{(m+n)\times(m+n)}$ denote the ergodic projection of Q and consider the canonical decomposition $\Pi = RL$ into a matrix $R \in \mathbb{R}^{(m+n)\times m}$ which contains the trapping probabilities into the ergodic classes of Q and $L \in \mathbb{R}^{m\times(m+n)}$ which contains the stationary distributions of Q. Since we supposed that Q is absorbing and defines m absorbing states it follows that Q has m ergodic classes each consisting of a single state. Therefore L and R are of the form

$$L = \begin{pmatrix} I & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} I \\ \widetilde{R} \end{pmatrix}$$

where $I \in \mathbb{R}^{m\times m}$ is the identity matrix, $0 \in \mathbb{R}^{m\times n}$ the zero matrix and $\widetilde{R} \in \mathbb{R}^{n\times m}$ comprises the trapping probabilities from each of the n transient states into each of the m ergodic classes.

3.2 Trapping Probabilities

The following proposition provides an explicit expression for the trapping probabilities $\widetilde{R}$.

Proposition 3.1. *The matrix $-(E + D)$ is a nonsingular M-matrix and $\widetilde{R} = -(E + D)^{-1}F$.*

In order to prove the first part of this proposition we state the following well-known

Lemma 3.2. *If $P \in \mathbb{R}^{n\times n}$ is irreducible and strictly substochastic then its spectral radius $\rho(P)$ is strictly bounded by $0 < \rho(P) < 1$.*

For completeness we provide its

Proof. By the Perron-Frobenius theorem for irreducible nonnegative matrices applied to P it follows for the spectral radius $\lambda := \rho(P)$ that (i) $\lambda > 0$, (ii) λ is a simple eigenvalue of P and (iii) that there is a left eigenvector $\pi \in \mathbb{R}^n$ corresponding to λ with strictly positive components $\pi_i > 0$ (i.e. $\pi \gg 0$). We can assume π to be normalized by $\|\pi\|_1 = \sum_i \pi_i = 1$, i.e. π is stochastic. Set $v := (I - P)\mathbf{1} \in \mathbb{R}^n$ and define

$$\widehat{P} := \begin{pmatrix} P & v \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \quad \text{and} \quad \widehat{\pi} := \begin{pmatrix} \pi & 0 \end{pmatrix} \in \mathbb{R}^{1\times(n+1)}.$$

Then $\widehat{\pi}\widehat{P} = \begin{pmatrix} \pi P & \pi v \end{pmatrix}$ and the fact that P is strictly substochastic means that $P\mathbf{1} \leq \mathbf{1}$ and that there is $i \in \{1, \ldots, n\}$ such that $0 \leq (P\mathbf{1})_i < 1$ and thus $0 < v_i = 1 - (P\mathbf{1})_i \leq 1$. Since $\pi_j > 0$ for all j it follows that $\pi v = \sum_j \pi_j v_j > 0$. From $\|\widehat{\pi}\widehat{P}\|_1 = \|\pi P\|_1 + \pi v$ we deduce that

$$\lambda = \lambda\|\pi\|_1 = \|\lambda\pi\|_1 = \|\pi P\|_1 = \|\widehat{\pi}\widehat{P}\|_1 - \pi v < \|\widehat{\pi}\widehat{P}\|_1 = 1$$

where in the last equation we used that $\widehat{\pi}\widehat{P}$ is stochastic since $\widehat{\pi}$ and $\widehat{P}$ are stochastic. □

An alternative proof for Lemma 3.2 can be given by applying [3, Corollary 2.1.5].

Corollary 3.3. *If S is an irreducible strict subgenerator then $-S$ is a nonsingular M-matrix.*

Proof. Set $s := \max\{-S_{ii} \mid i = 1, \ldots, n\}$ and note that $s > 0$ since S is a strict subgenerator. Define $P := \frac{1}{s}S + I$. Then P is irreducible and strictly substochastic and it follows by Lemma 3.2 that $0 < \rho(P) < 1$. If λ is an eigenvalue of S then $\frac{1}{s}\lambda + 1$ is an eigenvalue of P and thus $\mathrm{Re}\left(\frac{1}{s}\lambda + 1\right) \leq \rho(P) < 1$ which implies that $\mathrm{Re}(\lambda) < 0$. Therefore, every eigenvalue of $-S$ has a strictly positive real part and since $(-S)_{ij} \leq 0$ for $i \neq j$ it follows that $-S$ is a nonsingular M-matrix. □

Proof of Proposition 3.1. Recall that since Q is semistable its ergodic projection Π is given by $\Pi = I - QQ^D$. In the following, we are going to compute Q^D. Instead of the decomposition $Q = Q_1 + Q_2$ as in (1) consider the decomposition $Q = B + N$ where

$$B := \begin{pmatrix} 0 & 0 \\ 0 & E + D \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}.$$

Note that $NB = 0$ and N is nilpotent of index 2 (since $N \neq 0$ and $N^2 = 0$). With this decomposition we can apply [10, Corollary 2.3] (or [12, Corollary 2.1 (iv)]) which results in

$$Q^D = (B + N)^D = B^D + (B^D)^2 N.$$

It follows that

$$\begin{aligned} \Pi &= I - QQ^D = I - (BB^D + NB^D + B(B^D)^2 N + N(B^D)^2 N) \\ &= I - BB^D(I + B^D N) \end{aligned}$$

where we have applied that $NB^D = NB^D BB^D = NB(B^D)^2 = 0$ since $NB = 0$. Since $E + D$ is an irreducible strict subgenerator, it is invertible by Corollary 3.3 and thus we have

$$B^D = \begin{pmatrix} 0 & 0 \\ 0 & (E + D)^{-1} \end{pmatrix}.$$

It follows

$$\begin{aligned} \Pi &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (E + D)^{-1}F & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} I & 0 \\ -(E + D)^{-1}F & 0 \end{pmatrix}. \end{aligned}$$

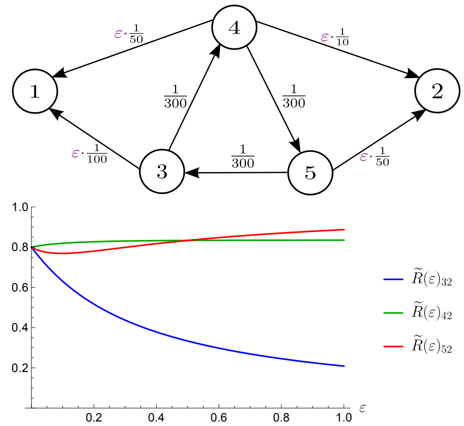

Figure 1: Top: an absorbing Markov chain with rates to absorption scaled by $\varepsilon > 0$. Bottom: the corresponding trapping probabilities $\widetilde{R}(\varepsilon)$ into state 2. The function $\widetilde{R}(\varepsilon)_{52}$ is non-monotonic, but the enveloping functions $\max_i \widetilde{R}(\varepsilon)_{i2}$ and $\min_i \widetilde{R}(\varepsilon)_{i2}$ are monotonic.

Thus when comparing with

$$\Pi = RL = \begin{pmatrix} I \\ \widetilde{R} \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \widetilde{R} & 0 \end{pmatrix}$$

we get $\widetilde{R} = -(E + D)^{-1}F$. □

Remark 3.1. For transient states i and j denote by T_{ij} the total sojourn time in state j (until absorption) when starting in state i. Then $-(E + D)^{-1} = \mathbb{E}(T_{ij})_{i,j}$ (see [9, Eq. (2.2)]) and thus $\widetilde{R}_{ik} = \sum_j \mathbb{E}(T_{ij})F_{jk}$ for an absorbing state k.

3.3 Perturbation of rates to absorbing states

Let us now scale the rates of the transitions of Q to absorption with a small factor $\varepsilon > 0$, i.e. consider the family of generators $Q^\varepsilon := Q_1 + \varepsilon Q_2$ with Q_1 and Q_2 as in Section 3.1. The generator εQ_2 can be regarded as an additive perturbation to the generator Q_1 and Q^ε as a generator of some perturbed Markov chain. Note that the number of ergodic classes of Q^ε is m for $\varepsilon > 0$ (i.e. the m absorbing states) and $m + 1$ for $\varepsilon = 0$ (the m absorbing states plus the ergodic class of Q_1 (which is transient for $\varepsilon > 0$)). When we substitute Q_2 by εQ_2 for $\varepsilon > 0$ then by Proposition 3.1 we get that the trapping probabilities $\widetilde{R}(\varepsilon)$ are also perturbed by ε and given by

$$\widetilde{R}(\varepsilon) = -(E + \varepsilon D)^{-1}\varepsilon F.$$

Note that for any $\varepsilon > 0$ the matrix $E + \varepsilon D$ is nonsingular while for $\varepsilon = 0$ it is singular. The matrix function $(E + \varepsilon D)^{-1}$ is the restriction of the *generalized resolvent* $(E + \lambda D)^{-1}$ (which is defined for all those $\lambda \in \mathbb{C}$ for which $E + \lambda D$ is nonsingular) to the positive real line $(0, \infty)$.

Figure 1 shows an example of such a perturbed CTMC and the trapping probabilities $\widetilde{R}(\varepsilon)_{i2}$ from all transient states $i = 3, 4, 5$ to the absorbing state 2. In the following, we are going to analyze the behaviour of the trapping probabilities $\widetilde{R}(\varepsilon)$. While for a fixed transient state i the trapping probability $\widetilde{R}(\varepsilon)_{ik}$ into the absorbing state k is not necessarily monotonic, we prove the monotonicity of their enveloping functions $\max_i \widetilde{R}(\varepsilon)_{ik}$ and $\min_i \widetilde{R}(\varepsilon)_{ik}$ for every absorbing state k. Following Campbell [5], it turns out to be suitable to define for $\varepsilon > 0$ the matrix functions

$$\widehat{E}_\varepsilon := (E + \varepsilon D)^{-1}E \quad \text{and} \quad \widehat{D}_\varepsilon := (E + \varepsilon D)^{-1}D.$$

Before establishing the monotonicity of the enveloping functions, we first state some helpful facts and identities involving the matrices $\widehat{E}_\varepsilon$ and $\widehat{D}_\varepsilon$.

Lemma 3.4. *(i) For any $\varepsilon > 0$ and $\delta > 0$ the matrix $\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon$ is invertible and*

$$\widehat{E}_\delta = (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)^{-1}\widehat{E}_\varepsilon \quad \text{and} \quad (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)^{-1} = \widehat{E}_\delta + \varepsilon\widehat{D}_\delta.$$

(ii) For $0 < \varepsilon \leq \delta$ it holds

$$\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon \geq I \quad \text{and} \quad \widehat{E}_\varepsilon + \varepsilon\widehat{D}_\varepsilon = I.$$

(iii) For $\varepsilon > 0$ and $\delta > 0$ the matrices $\widehat{E}_\varepsilon, \widehat{E}_\varepsilon^D, \widehat{D}_\varepsilon, \widehat{D}_\varepsilon^D, \widehat{E}_\delta, \widehat{E}_\delta^D, \widehat{D}_\delta$ and $\widehat{D}_\delta^D$ commute pairwise.

(iv) For $\varepsilon > 0$ the matrix $\varepsilon\widehat{D}_\varepsilon$ is stochastic and thus $-\widehat{E}_\varepsilon = \varepsilon\widehat{D}_\varepsilon - I$ is a generator and for both the set $\{j \mid D_{jj} < 0\}$ forms their single irreducible class.

Proof. (i) Since $E + \varepsilon D$ is invertible for any $\varepsilon > 0$ it follows that $\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon = (E + \varepsilon D)^{-1}(E + \delta D)$ is also invertible and its inverse is given by

$$(\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)^{-1} = (E + \delta D)^{-1}(E + \varepsilon D) = \widehat{E}_\delta + \varepsilon\widehat{D}_\delta.$$

The other identity follows from

$$\begin{aligned}\widehat{E}_\delta &= (E + \delta D)^{-1}E = (E + \delta D)^{-1}(E + \varepsilon D)(E + \varepsilon D)^{-1}E \\ &= ((E + \varepsilon D)^{-1}(E + \delta D))^{-1}(E + \varepsilon D)^{-1}E = (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)^{-1}\widehat{E}_\varepsilon.\end{aligned}$$

(ii) The identity $\widehat{E}_\varepsilon + \varepsilon\widehat{D}_\varepsilon = I$ is clear. Since $D \leq 0$ and $(E + \varepsilon D)^{-1} \leq 0$ by Corollary 3.3 it follows that $\widehat{D}_\varepsilon \geq 0$. Thus, if $\delta \geq \varepsilon$ then $\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon \geq \widehat{E}_\varepsilon + \varepsilon\widehat{D}_\varepsilon = I$.
(iii) Since $\widehat{E}_\varepsilon + \varepsilon\widehat{D}_\varepsilon = I$ by (ii) it follows that the four ε-matrices $\widehat{E}_\varepsilon$, $\widehat{E}_\varepsilon^D$, $\widehat{D}_\varepsilon$ and $\widehat{D}_\varepsilon^D$ commute pairwise. By (i), $\widehat{E}_\delta$ is expressible in terms of $\widehat{E}_\varepsilon$ (as a power series in $\widehat{E}_\varepsilon$ for a fixed δ). Therefore, $\widehat{E}_\delta$ commutes with the ε-matrices and it then follows that $\widehat{E}_\delta^D = \widehat{E}_\varepsilon^D(\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)$, $\widehat{D}_\delta$ and $\widehat{D}_\delta^D$ also commute with the ε-matrices.
(iv) In the proof of (ii) we already mentioned that $\widehat{D}_\varepsilon \geq 0$. From $E\mathbf{1} = 0$ we get $(E+\varepsilon D)\mathbf{1} = \varepsilon D\mathbf{1}$ and thus $\varepsilon\widehat{D}_\varepsilon\mathbf{1} = (E+\varepsilon D)^{-1}\varepsilon D\mathbf{1} = \mathbf{1}$ from which we conclude that $\varepsilon\widehat{D}_\varepsilon$ is stochastic. In order to show that $J := \{j \mid D_{jj} < 0\}$ is the single irreducible class of $\varepsilon\widehat{D}_\varepsilon$ set $A := -(E + \varepsilon D)$ and note that j-th column of the matrix $\widehat{D}_\varepsilon = A^{-1}(-D)$ is of the form $(\widehat{D}_\varepsilon)_{\cdot,j} = ((A^{-1})_{ij}(-d_j))_i$ so that $(\widehat{D}_\varepsilon)_{\cdot,j} = 0$ if $d_j = 0$ and $(\widehat{D}_\varepsilon)_{\cdot,j} \gg 0$ if $d_j \neq 0$ since A is an irreducible nonsingular M-matrix and thus $A^{-1} \gg 0$. Therefore, the stochastic matrix $\varepsilon\widehat{D}_\varepsilon$ has exactly $|J|$ strictly positive columns and $n - |J|$ zero columns from which we deduce that J is the single irreducible class for $\varepsilon\widehat{D}_\varepsilon$. Since $-\widehat{E}_\varepsilon = \varepsilon\widehat{D}_\varepsilon - I$ it follows that $-\widehat{E}_\varepsilon$ is an irreducible generator with the same single irreducible class J. □

We are now ready to establish the monotonicity of the enveloping functions. This will allow us to compare the trapping probabilities $\widetilde{R}(\varepsilon)$ for different perturbation values ε.

Theorem 3.5. *Fix an absorbing state $1 \leq k \leq m$ and consider the perturbed trapping probabilities $\widetilde{R}(\varepsilon)_{ik}$ from all transient states $1 \leq i \leq n$ to k for $\varepsilon > 0$. Define the functions*

$$M_k(\varepsilon) := \max_{i=1,\dots,n} \widetilde{R}(\varepsilon)_{ik} \quad \text{and} \quad m_k(\varepsilon) := \min_{i=1,\dots,n} \widetilde{R}(\varepsilon)_{ik} \quad .$$

Then $M_k(\varepsilon)$ is monotonically increasing and $m_k(\varepsilon)$ is monotonically decreasing.

Proof. Consider the k-th column of $\widetilde{R}$. We show that for all transient states $i \in \{1, \dots, n\}$, for all $\delta > 0$ and for all $0 < \varepsilon \leq \delta$ the i-th component $\widetilde{R}(\varepsilon)_{ik}$ is a convex combination of all the $\widetilde{R}(\delta)_{jk}$, $j = 1, \dots, n$. It then follows that

$$m_k(\delta) = \min_j \widetilde{R}(\delta)_{jk} \leq \widetilde{R}(\varepsilon)_{ik} \leq \max_j \widetilde{R}(\delta)_{jk} = M_k(\delta).$$

Since these inequalities hold for an arbitrary transient state i it follows that

$$m_k(\delta) \leq m_k(\varepsilon) = \min_i \widetilde{R}_{ik}(\varepsilon) \leq \max_i \widetilde{R}_{ik}(\varepsilon) = M_k(\varepsilon) \leq M_k(\delta)$$

and the conclusion follows. So let $\delta > 0$ and $0 < \varepsilon < \delta$. Then

$$\begin{aligned}\widetilde{R}(\varepsilon) &= -(E + \varepsilon D)^{-1}\varepsilon F = (E + \varepsilon D)^{-1}(E + \delta D)(E + \delta D)^{-1}(-\varepsilon F) \\ &= (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)\frac{\varepsilon}{\delta}\widetilde{R}(\delta).\end{aligned}$$

Now note that $P(\varepsilon, \delta) := (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)\frac{\varepsilon}{\delta}$ is stochastic. Indeed, since $0 < \varepsilon \leq \delta$ we get by Lemma 3.4(ii) that $\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon \geq I \geq 0$ and thus $P(\varepsilon, \delta) \geq 0$ and moreover $P(\varepsilon, \delta)\mathbf{1} = \frac{\varepsilon}{\delta}\widehat{E}_\varepsilon\mathbf{1} + \varepsilon\widehat{D}_\varepsilon\mathbf{1} = \mathbf{1}$ where we applied that $\varepsilon\widehat{D}_\varepsilon$ is stochastic by Lemma 3.4(iv) and $\widehat{E}_\varepsilon\mathbf{1} = (E + \varepsilon D)^{-1}E\mathbf{1} = 0$ since $E\mathbf{1} = 0$. Therefore $\widetilde{R}(\varepsilon)_{ik} = \sum_j P(\varepsilon, \delta)_{ij}\widetilde{R}(\delta)_{jk}$ is a convex combination of all the $\widetilde{R}(\delta)_{jk}$ for any i. □

Remark 3.2. (1) We recall that in contrast to the enveloping functions $m_k(\varepsilon)$ and $M_k(\varepsilon)$, a fixed component function $\widetilde{R}_{ik}(\varepsilon)$ of $\widetilde{R}(\varepsilon)$ need not be increasing or decreasing (Fig. 1).
(2) The lower bound $\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon \geq I$ from Lemma 3.4(ii) provides additional information on the behaviour of the trapping probabilities $\widetilde{R}(\varepsilon)$: from $\widetilde{R}(\varepsilon) = (\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon)\frac{\varepsilon}{\delta}\widetilde{R}(\delta)$ and $\widetilde{R}(\delta) \geq 0$ it follows that $\widetilde{R}(\varepsilon) \geq \frac{\varepsilon}{\delta}\widetilde{R}(\delta)$ for $0 < \varepsilon \leq \delta$. In other words, any component of $\frac{1}{\varepsilon}\widetilde{R}(\varepsilon)$ is decreasing in ε and by differentiating it we deduce that $\frac{d}{d\varepsilon}\widetilde{R}(\varepsilon) \leq \frac{1}{\varepsilon}\widetilde{R}(\varepsilon)$ for each $\varepsilon > 0$. The inequality $\widetilde{R}(\varepsilon) \geq \frac{\varepsilon}{\delta}\widetilde{R}(\delta)$ for $0 < \varepsilon \leq \delta$ also implies that for any $\delta > 0$ the graph of $\widetilde{R}(\varepsilon)_{ik}$ in the interval $(0, \delta]$ is always above the line connecting the origin with the point $(\delta, \widetilde{R}(\delta)_{ik})$.

4 ASYMPTOTIC LIMIT OF TRAPPING PROBABILITIES

In this section we analyze the limiting behaviour of the perturbed trapping probabilities $\widetilde{R}(\varepsilon)$, i.e. we establish the limit of $\widetilde{R}(\varepsilon)$ as $\varepsilon \to 0$. We begin with the following

Lemma 4.1. *Fix $\varepsilon > 0$. Then*
(i) $\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$ and $\widehat{D}_\varepsilon\widehat{E}_\varepsilon^D$ do not depend on ε.
(ii) $\operatorname{ind}(\widehat{E}_\varepsilon) = 1$ and thus $\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D\widehat{E}_\varepsilon = \widehat{E}_\varepsilon$.
(iii) $\varepsilon\widehat{D}_\varepsilon(I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D) = I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$
(iv) $(I + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon)^{-1}\widehat{E}_\varepsilon^D = \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$

Proof. (i) can be found in [5, Theorem 3.1.2, p. 36].
(ii) Since E is irreducible and $E + \varepsilon D$ is invertible we get $\mathrm{ind}(\widehat{E}_\varepsilon) = \mathrm{ind}(E) = 1$. Therefore $\widehat{E}_\varepsilon^D$ is the group inverse of $\widehat{E}_\varepsilon$ and thus $\widehat{E}_\varepsilon \widehat{E}_\varepsilon^D \widehat{E}_\varepsilon = \widehat{E}_\varepsilon$.
(iii) By applying Lemma 3.4(ii, iii) and this lemma (ii) we compute

$$\varepsilon\widehat{D}_\varepsilon(I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D) = (I - \widehat{E}_\varepsilon)(I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D) = I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D - \widehat{E}_\varepsilon + \widehat{E}_\varepsilon^2\widehat{E}_\varepsilon^D = I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D.$$

(iv) From Lemma 3.4(ii, iii) and this lemma (ii) it follows that

$$\widehat{E}_\varepsilon(I + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon) = \widehat{E}_\varepsilon(I + \widehat{E}_\varepsilon^D(I - \widehat{E}_\varepsilon)) = \widehat{E}_\varepsilon + \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D - \widehat{E}_\varepsilon^2\widehat{E}_\varepsilon^D = \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D.$$

Since $\widehat{E}_\varepsilon$ and $\widehat{D}_\varepsilon$ commute it follows that

$$(I + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon)^{-1}\widehat{E}_\varepsilon^D = (\widehat{E}_\varepsilon(I + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon))^D = (\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D)^D = \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$$

where in the last step we used that $\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$ is a projection. □

In order to establish the limit of $\widetilde{R}(\varepsilon)$ as $\varepsilon \to 0$ we show that $\widetilde{R}(\varepsilon)$ can be extended to an analytic function on $\mathbb{R}$ and establish its power series expansion at 0. For this purpose, we first state the following

Proposition 4.2. *The generalized resolvent $(E + \varepsilon D)^{-1}$ satisfies*

$$(E + \varepsilon D)^{-1} = \left((I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} + \widehat{\Pi}\frac{\delta}{\varepsilon}\right)(E + \delta D)^{-1} \tag{2}$$

where $\delta > 0$ is arbitrary and $\widehat{\Pi} := I - \widehat{E}_\delta\widehat{E}_\delta^D = I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$.

Proof. First write

$$\begin{aligned}(E + \varepsilon D)^{-1} &= (E + \varepsilon D)^{-1}(E + \delta D)(E + \delta D)^{-1}\\ &= ((E + \delta D)^{-1}(E + \varepsilon D))^{-1}(E + \delta D)^{-1}\\ &= (\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1}(E + \delta D)^{-1}.\end{aligned}$$

Decompose $(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1}$ with respect to the projection $\widehat{\Pi} = I - \widehat{E}_\delta\widehat{E}_\delta^D$:

$$(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} = (I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} + \widehat{\Pi}(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1}.$$

Simplify the right hand side as required by applying Lemma 3.4(i) which gives

$$\widehat{\Pi}(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} = \widehat{\Pi}(\widehat{E}_\varepsilon + \delta\widehat{D}_\varepsilon) = \delta\widehat{D}_\varepsilon\widehat{\Pi} = \frac{\delta}{\varepsilon}\widehat{\Pi}$$

where in the last two steps we also applied Lemma 3.4(iii) and Lemma 4.1(ii, iii). □

Remark 4.1. In [5, Proof of Theorem 4.2.1, p. 80] one can also find the Laurent series expansion at 0 of the generalized resolvent $(E + \varepsilon D)^{-1}$ which takes the form

$$(E + \varepsilon D)^{-1} = \left(\widehat{E}_\delta^D\sum_{k=0}^{\infty}(-\widehat{E}_\delta^D\widehat{D}_\delta)^k\varepsilon^k + \widehat{D}_\delta^D\widehat{\Pi}\frac{1}{\varepsilon}\right)(E + \delta D)^{-1}.$$

This expansion can be also deduced from (2) by using the equalities $\widehat{D}_\delta^D\widehat{\Pi} = \delta\widehat{\Pi}$ (which follows from Lemma 4.1(iii)) together with $(I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} = \widehat{E}_\delta^D(I + \varepsilon\widehat{E}_\delta^D\widehat{D}_\delta)^{-1}$ (see proof of Theorem 4.3) and its Neumann series expansion.

In the following, we are going to apply the preceding proposition in order to establish the power series expansion at 0 of the trapping probabilities $\widetilde{R}(\varepsilon)$ which then can be used to compute their asymptotic behaviour as $\varepsilon \to 0$.

Theorem 4.3. *$\widetilde{R}(\varepsilon)$ can be extended to an analytic function on $\mathbb{R}$ and its power series expansion at 0 can be written as*

$$\widetilde{R}(\varepsilon) = \left(\widehat{E}_\delta^D(I + \varepsilon\widehat{M})^{-1}\varepsilon + \widehat{\Pi}\delta\right)\cdot\left(\frac{1}{\delta}\widetilde{R}(\delta)\right) \tag{3}$$

where $\delta > 0$ is arbitrary and $\widehat{M} := \widehat{E}_\delta^D\widehat{D}_\delta$ (independent of δ).

Proof. From Proposition 4.2 we have

$$\begin{aligned}\widetilde{R}(\varepsilon) &= -(E + \varepsilon D)^{-1}\varepsilon F\\ &= -\left((I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} + \widehat{\Pi}\frac{\delta}{\varepsilon}\right)(E + \delta D)^{-1}\varepsilon F\\ &= \left((I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1}\varepsilon + \widehat{\Pi}\delta\right)\frac{1}{\delta}\widetilde{R}(\delta).\end{aligned}$$

We show that

$$(I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1} = \widehat{E}_\delta^D(I + \varepsilon\widehat{E}_\delta^D\widehat{D}_\delta)^{-1}.$$

For this purpose, note that

$$\begin{aligned}\widehat{E}_\delta^D(I + \varepsilon\widehat{E}_\delta^D\widehat{D}_\delta)^{-1} &= \widehat{E}_\delta^D\widehat{E}_\delta\widehat{E}_\delta^D(I + \varepsilon\widehat{E}_\delta^D\widehat{D}_\delta)^{-1}\\ &= \widehat{E}_\delta^D\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D(I + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon)^{-1}\\ &= \widehat{E}_\delta^D\widehat{E}_\varepsilon\widehat{E}_\varepsilon\widehat{E}_\varepsilon^D = \widehat{E}_\delta^D\widehat{E}_\varepsilon\end{aligned}$$

where we used Lemma 4.1(i, iv, ii). In order to show $\widehat{E}_\delta^D\widehat{E}_\varepsilon = (I - \widehat{\Pi})(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)^{-1}$ we show that $I - \widehat{\Pi} = \widehat{E}_\delta^D\widehat{E}_\varepsilon(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta)$:

$$\begin{aligned}\widehat{E}_\delta^D\widehat{E}_\varepsilon(\widehat{E}_\delta + \varepsilon\widehat{D}_\delta) &= \widehat{E}_\varepsilon(\widehat{E}_\delta^D\widehat{E}_\delta + \varepsilon\widehat{E}_\delta^D\widehat{D}_\delta) = \widehat{E}_\varepsilon(\widehat{E}_\varepsilon^D\widehat{E}_\varepsilon + \varepsilon\widehat{E}_\varepsilon^D\widehat{D}_\varepsilon)\\ &= \widehat{E}_\varepsilon^D\widehat{E}_\varepsilon(\widehat{E}_\varepsilon + \varepsilon\widehat{D}_\varepsilon) = \widehat{E}_\varepsilon\widehat{E}_\varepsilon = I - \widehat{\Pi}\end{aligned}$$

where we applied Lemma 3.4(ii, iii) and Lemma 4.1(i). Finally, the desired identity in (3) follows. □

Corollary 4.4. *The componentwise limit of $\widetilde{R}(\varepsilon)$ as $\varepsilon \to 0$ exists and we denote it by $\widetilde{R}(0) := \lim_{\varepsilon\to 0}\widetilde{R}(\varepsilon)$. If π is the unique stationary distribution of the irreducible generator E then the ergodic projection $\Pi := I - EE^D$ of E has equal rows π (i.e. $\Pi = \mathbf{1}\pi$) and*

$$\widetilde{R}(0) = \frac{1}{\|\pi D\|_1}\Pi F = \frac{1}{\|\pi D\|_1}\mathbf{1}\pi F.$$

In particular, $\widetilde{R}(0)$ is a stochastic matrix with equal rows $\widetilde{R}(0)_{i\,\cdot} = \frac{1}{\|\pi D\|_1}\pi F$ (for each i) and thus constant columns $\widetilde{R}(0)_{\cdot\,k} = \mathbf{1}\frac{(\pi F)_k}{\|\pi D\|_1}$.

Proof. Letting $\varepsilon \to 0$ in (3) we note that since $(I + \varepsilon\widehat{M})^{-1}$ is bounded in a neighborhood of 0 the componentwise limit $\widetilde{R}(0)$ of $\widetilde{R}(\varepsilon)$ as $\varepsilon \to 0$ exists and is given by $\widetilde{R}(0) = \widehat{\Pi}\widetilde{R}(\delta)$ for any $\delta > 0$. By Lemma 3.4(iv), $-\widehat{E}_\varepsilon$ is a generator with a single irreducible class for any $\varepsilon > 0$. Recall that in contrast to $-\widehat{E}_\varepsilon$, its ergodic projection $\widehat{\Pi} = I - \widehat{E}_\varepsilon\widehat{E}_\varepsilon^D$ does not depend on ε. Since $-\widehat{E}_\varepsilon$ is irreducible it has a unique stationary distribution $\widehat{\pi}$ and thus $\widehat{\Pi} = \mathbf{1}\widehat{\pi}$. Since E is irreducible it holds for its unique stationary distribution π that $\pi E = 0$ and $\pi \gg 0$. Since D is diagonal and $D \neq 0$ it follows that $\pi D \neq 0$. Now note that $\widehat{\pi} = -\frac{1}{\|\pi D\|_1}\pi D$ since (by setting $\varepsilon := 1$)

$$\pi D\widehat{E}_1 = \pi D(E + D)^{-1}E = \pi(E + D)(E + D)^{-1}E = \pi E = 0$$

where we have applied that $\pi D = \pi(E + D)$. Finally, from $\widehat{\pi} = -\frac{1}{\|\pi D\|_1}\pi(E+D)$ it follows that $-\widehat{\pi}(E+D)^{-1} = \frac{1}{\|\pi D\|_1}\pi$ and since $\widehat{\Pi} = \mathbf{1}\widehat{\pi}$ we deduce that

$$\widetilde{R}(0) = \widehat{\Pi}\widetilde{R}(1) = \mathbf{1}\widehat{\pi}(E+D)^{-1}(-F) = \frac{1}{\|\pi D\|_1}\mathbf{1}\pi F = \frac{1}{\|\pi D\|_1}\Pi F.$$

□

Returning to the example from Fig. 1, where obviously $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we can calculate the limit as

$$\widetilde{R}(0)_{i2} = \lim_{\varepsilon \to 0} \widetilde{R}(\varepsilon)_{i2} = \frac{\frac{1}{3} \cdot \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{50}}{\frac{1}{3} \cdot \frac{1}{100} + \frac{1}{3}(\frac{1}{50} + \frac{1}{10}) + \frac{1}{3} \cdot \frac{1}{50}} = 0.8$$

for $i \in \{3, 4, 5\}$, which is the value that can also be read from the figure.

5 APPLICATION TO MODEL REPAIR

In this section, the theoretical results derived in the previous sections are employed to solve a model repair problem.

5.1 Setting and Approach

Consider a CTMC with generator Q, with one irreducible class comprising n transient states and $m = 2$ absorbing states $\{failed, done\}$ which are both reachable from the transient states. The CTMC represents a system that performs useful work for some time (all transient states carry the label *work*) and – if all goes well – eventually finishes by moving to the inactive state *done*. However, it is possible that an error occurs during the working phase, which will lead the system to the undesirable *failed* state. A typical requirement for such a system would be that it works for a sufficiently long period of time and then finishes without an error. This can be expressed formally with the help of the following CSL [1] time-bounded Until formula: $\Phi = P_{\geq b}(work\ \mathcal{U}^{>t}\ done)$, with lower probability bound $0 < b < 1$ and lower time bound $t > 0$. It is required that each transient state should satisfy Φ.

If some of the transient states violate requirement Φ, the system should be "repaired", i.e. modified according to some strategy. Among the many possible approaches to model repair, such as adding / removing states or transitions, we advocate a scheme where the structure of the CTMC remains untouched, but transition rates may be reduced. The rationale behind rate reduction is that in most real systems, slowing down a process (a processor, a machine, etc.) is possible, while acceleration may not be feasible. However, rate reduction still leaves many degrees of freedom. For example, each transition could be reduced by its individual reduction factor, which could lead to good solutions but would open a possibly huge multidimensional search space. Therefore we restrict ourselves further by only allowing for common reduction factors applied to sets of transitions.

Basically, for a transient state s, there can be two reasons (or a combination of the two) for violating requirement Φ:

(1) The trapping probability from s to state *done* is too low (in other words, the trapping probability to state *failed* is too high).
(2) The trapping probability to state *done* is high enough, but the time to absorption (starting from s) is too short.

We propose a general solution which takes into account (1) and (2) and is guaranteed to lead to a solution for all transient states.

(I) We first try to deal with both (1) and (2) at the same time by applying the common reduction factor $0 < \eta \leq 1$ to all transitions from the transient class to state *failed*. As η is reduced, the probability of getting absorbed in state *done* can be made arbitrarily close to 1, and at the same time the system will become "slower", since the exit rates of the transient states are reduced. Depending on the case at hand, it may be possible to find some $0 < \eta \leq 1$ such that Φ will be satisfied for all transient states, in which case we are done. But it is also possible that no such η exists (since the system goes to absorption too early, even though some rates were reduced), in which case we need to proceed.

(IIa) If step (I) was not successful, we first concentrate on the time-unbounded problem, i.e. we deal exclusively with issue (1). The weakened requirement for this step is $\Phi' = P_{>b}(work\ \mathcal{U}\ done)$, where the time bound has been removed and the probability bound has been changed from $\geq b$ to $> b$. As shown in [20], one can always find a common reduction factor η_{ut} (applied simultaneously to all transitions from transient states to state *failed*) such that all transient states satisfy the time-unbounded requirement Φ'. After step (IIa) we always move to (IIb).

(IIb) In this final step, we deal with issue (2). We keep factor η_{ut} fixed and return to the original time-bounded requirement Φ. We now introduce a second common reduction factor $0 < \varepsilon \leq 1$ to all transitions from transient states to all absorbing states (*failed* and *done*). The purpose is to slow down the system, such that absorption before t becomes less likely. It is essential that this perturbation by factor ε does not destroy the trapping probabilities which were already fixed in step (IIa). This is where we need Theorem 3.5 from Sec. 3.3, which guarantees that during this slow-down the trapping probabilities are preserved in the admissible range.

Proposition 5.1. *The procedure described in steps (I), (IIa) and (IIb) solves the model repair problem for the given requirement* $\Phi = P_{\geq b}(work\ \mathcal{U}^{>t}\ done)$*, for all transient states.*

Proof. The goal is that all transient states s should satisfy $P_{\geq b}(work\ \mathcal{U}^{>t}\ done)$, where b and t are fixed. From [20] we know that one can find a solution for the corresponding time-unbounded problem, i.e. one can find a reduction factor η_{ut} s.t. $s \models P_{>b}(work\ \mathcal{U}\ done)$ for all $0 < \eta \leq \eta_{ut}$, or in other words, that $b < Pr^{\eta_{ut}}(s, work\ \mathcal{U}\ done)$ (the superscript indicates the probability measure for the Markov chain modified according to (IIa)). Keeping η_{ut} fixed, we know by Theorem 3.5, shown in Sec. 3.3, that for all transient states s of the CTMC in which all rates to absorption are further reduced by the common reduction factor $0 < \varepsilon \leq 1$ according to (IIb) the following inequality also holds:

$$b < Pr^{\eta_{ut},\varepsilon}(s, work\ \mathcal{U}\ done).$$

The right hand side converges to some value $p \geq b$ as $\varepsilon \to 0$ and this limit is the same for all transient states s (Corollary 4.4). Since $\min_s Pr^{\eta_{ut},\varepsilon}(s, work\ \mathcal{U}\ done)$ (taken over all transient states s) is decreasing in ε it follows that $p > b$. Now, for any Markov chain it

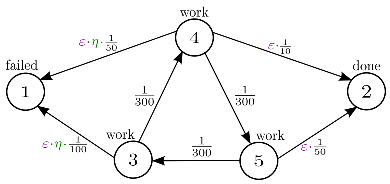

Figure 2: Example Markov chain, also showing the application of the reduction factors η and ε.

trivially holds that

$$Pr(s, work\ \mathcal{U}\ done) = Pr(s, work\ \mathcal{U}^{\leq t}\ done) + Pr(s, work\ \mathcal{U}^{>t}\ done).$$

We apply this to the Markov chain modified by both reduction factors η_{ut} and ε and combine it with the previous inequality:

$$\begin{aligned} b &< Pr^{\eta_{ut},\varepsilon}(s, work\ \mathcal{U}\ done) \\ &= Pr^{\eta_{ut},\varepsilon}(s, work\ \mathcal{U}^{\leq t}\ done) + Pr^{\eta_{ut},\varepsilon}(s, work\ \mathcal{U}^{>t}\ done). \end{aligned}$$

Since as $\varepsilon \longrightarrow 0$ the first term of the sum vanishes, the second term of the sum converges to p. From $p > b$ it follows that there is $\varepsilon > 0$ for which the second term is $\geq b$ for all transient states s. □

5.2 Illustration by Example

We now study a concrete example, in order to demonstrate the different situations that may occur during the algorithm proposed in the previous section. Consider an example with three transient states (labelled by *work*) plus the two absorbing states labelled by *failed* and *done*, as shown in Fig. 2. (The example is the same as the one from Fig. 1, but now the figure shows the application of both reduction factors η and ε.) Furthermore, consider three instances of a time-bounded requirement $\Phi_i = P_{\geq b_i}(work\ \mathcal{U}^{>5}\ done)$, where the probability bounds are chosen as follows: $b_1 = 0.2$, $b_2 = 0.5$ and $b_3 = 0.7$.

The probabilities for states 3, 4 and 5 to satisfy the path formula $\varphi = (work\ \mathcal{U}^{>5}\ done)$ are $Pr(3, \varphi) = 0.2053$, $Pr(4, \varphi) = 0.4609$ and $Pr(5, \varphi) = 0.7925$. Therefore, when model checking Φ_1, all three values are greater than $b_1 = 0.2$ and thus all three transient states satisfy Φ_1. There is no need for model repair in this case.

Turning to requirement Φ_2, which contains the same path formula $\varphi = (work\ \mathcal{U}^{>5}\ done)$, model checking yields the same probabilities as for formula Φ_1. But since the probability bound $b_2 = 0.5$ is higher, states 3 and 4 now violate that probability bound, such that model repair is needed. Figure 3 shows that the model can be repaired according to step (I) of the general solution by using the common reduction factor η. The figure shows how probabilities increase beyond $b_2 = 0.5$ for all three transient states while reducing η, such that Φ_2 is satisfied in the range $0 < \eta \leq 0.29$. At $\eta = 0.29$, the probabilities are $Pr^{\eta}(3, \varphi) = 0.5017$, $Pr^{\eta}(4, \varphi) = 0.5586$ and $Pr^{\eta}(5, \varphi) = 0.8349$. We refer to [21] for more details on finding η and how to deal with intersecting curves as in Fig. 3. (Note, however, that in [21] only the upper time-bounded case was considered since we did not know yet how to solve the lower time-bounded case.)

Finally, model checking Φ_3, again containing the same path formula φ, gives the original probabilities as in the case Φ_1. But now the probability bound $b_3 = 0.7$ is higher again, which means that

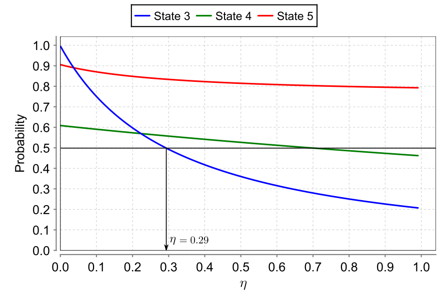

Figure 3: Step (I): Probability curves for $\varphi = work\ \mathcal{U}^{>5}\ done$ (time-bounded requirement), depending on η. If $b = 0.5$ then model repair is successful with $\eta = 0.29$; if $b = 0.7$ then one needs to move on to step (IIa).

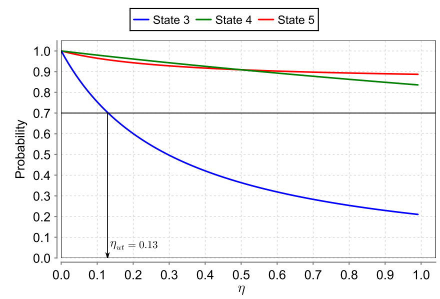

Figure 4: Step (IIa): Probability curves for $\varphi' = work\ \mathcal{U}\ done$ (time-unbounded requirement), depending on η. Determining $\eta_{ut} = 0.13$ for $b = 0.7$.

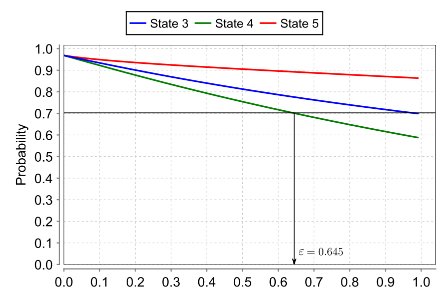

Figure 5: Step (IIb): Probability curves for $\varphi = work\ \mathcal{U}^{>5}\ done$ (time-bounded requirement) at $\eta_{ut} = 0.13$, depending on ε. Model repair is successful with $\varepsilon = 0.645$ (and $\eta = \eta_{ut} = 0.13$).

reducing η alone as for Φ_2 will not be sufficient. This is evident from Fig. 3, as the probability of state 4 can reach a maximum of 0.61 as $\eta \longrightarrow 0$. Therefore, we follow step (IIa) of the general procedure and momentarily focus on the time-unbounded requirement $\Phi_3' = P_{\geq 0.7}(work\ \mathcal{U}\ done)$. We reduce factor η as shown in Fig. 4, thereby finding $\eta_{ut} = 0.13$. Afterwards, while keeping η_{ut} fixed, we return to the original time-bounded requirement Φ_3 and introduce reduction factor ε. The behaviour of the satisfaction probabilities while reducing ε is shown in Fig. 5. Hence, in the range $0 < \varepsilon \leq 0.645$, all three transient states have satisfaction probability higher than 0.7 and therefore satisfy Φ_3. The probabilities at $\eta_{ut} = 0.13$ and $\varepsilon = 0.645$ are $Pr^{\eta_{ut},\varepsilon}(3,\varphi) = 0.7764$, $Pr^{\eta_{ut},\varepsilon}(4,\varphi) = 0.7002$ and $Pr^{\eta_{ut},\varepsilon}(5,\varphi) = 0.8925$. Thus, with a combination of reduction factors η_{ut} and ε, the model repair problem has been solved also for Φ_3.

After having solved the model repair problem for all three problem instances, we shall once more return to the trapping probabilities and their limit (purely for illustration purposes). As we had seen in Fig. 1, when applying reduction factor ε to the original CTMC, we could observe both non-monotonicity and intersection of the trapping probability curves $\widetilde{R}(\varepsilon)_{i2}$. As shown in Fig. 6, for the modified system, where already $\eta_{ut} = 0.13$ was applied before reducing ε, we still get very slight non-monotonicity (for $\widetilde{R}(\varepsilon)_{52}$) but no intersection for $0 < \varepsilon \leq 1$. Of course, the enveloping functions in this case are also monotonic, as we know from Theorem 3.5. Figure 6 also illustrates Corollary 4.4, which states that the perturbed trapping probabilities $\widetilde{R}(\varepsilon)_{i2}$ all converge to the same value as $\varepsilon \to 0$. For this system where already $\eta_{ut} = 0.13$ was applied, we can calculate the limiting trapping probability by Corollary 4.4 which gives

$$\widetilde{R}(0)_{i2} = \frac{\frac{1}{3}\cdot\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{50}}{\frac{1}{3}\cdot\frac{1}{100}\cdot\frac{13}{100} + \frac{1}{3}\left(\frac{1}{50}\cdot\frac{13}{100} + \frac{1}{10}\right) + \frac{1}{3}\cdot\frac{1}{50}} \approx 0.9685 \,.$$

In fact, this limit is identical to the limit observed in Fig. 5, since for very small values of ε the probability of absorption before $t = 5$ goes to zero, see proof of Proposition 5.1.

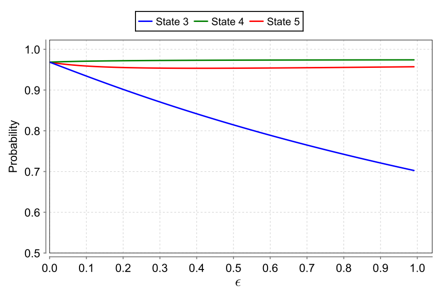

Figure 6: Probability curves for $\varphi' = work\ \mathcal{U}\ done$ (time-unbounded requirement) at $\eta_{ut} = 0.13$, depending on ε. These are the trapping probabilities $\widetilde{R}(\varepsilon)_{i2}$ for the Markov chain modified according to (IIa) with $\eta_{ut} = 0.13$. Here, $\widetilde{R}(\varepsilon)_{52}$ is nonmonotonic, but the curves do not intersect (as opposed to the situation in Fig. 1).

6 CONCLUSION AND FUTURE WORK

This paper considered absorbing CTMCs with multiple sink states for which the transient class would form an irreducible Markov chain if the transitions to absorbing states were ignored. For such Markov chains, we studied the behaviour of the trapping probabilities as the rates to absorption are scaled by a paramter $\varepsilon > 0$, for which setting we were able to prove monotonicity and limiting results. The paper also presented an application of these theoretical findings to a certain type of model repair problem. As future work, we plan to generalize the established results without imposing any such reducibility restriction on the structure of the transient class. Moreover, we intend to check whether our results also generalize to certain infinite-state absorbing Markov chains. Regarding the application to model repair which we presented, we are currently in the process of improving the efficiency of our prototype implementation, such that the algorithm will run faster on models with a large number of states.

REFERENCES

[1] C. Baier, B. Haverkort, H. Hermanns, and J.-P. Katoen. 2003. Model-Checking Algorithms for Continuous-Time Markov Chains. *IEEE Transactions on Software Engineering* 29, 6 (2003), 524–541.
[2] E. Bartocci, R. Grosu, P. Katsaros, C.R. Ramakrishnan, and S.A. Smolka. 2011. Model Repair for Probabilistic Systems. In *TACAS'11*. Springer LNCS 6605, 326–340.
[3] A. Berman and R.J. Plemmons. 1979. *Nonnegative matrices in the mathematical sciences*. Academic Press.
[4] A. et al. Bobbio. 2005. Matching Three Moments with Minimal Acyclic Phase Type Distributions. *Stochastic Models* 2-3, 21 (2005), 303–326.
[5] S. L. Campbell. 1980. *Singular systems of differential equations*. Pitman, London.
[6] M. Ceska, F. Dannenberg, M. Z. Kwiatkowska, and N. Paoletti. 2014. Precise Parameter Synthesis for Stochastic Biochemical Systems. In *Computational Methods in Systems Biology - 12th International Conference, CMSB*. 86–98.
[7] T. Chen, E. M. Hahn, T. Han, M. Kwiatkowska, H. Qu, and L. Zhang. 2013. Model Repair for Markov Decision Processes. In *Proc. 7th Int. Symp. Theor. Aspects of Software Engineering (TASE)*. IEEE CS Press, 85–92.
[8] P. Collet, S. Martinez, and J. San Martin. 2013. *Quasi-Stationary Distributions – Markov Chains, Diffusion and Dynamical Systems*. Springer.
[9] J. N. Darroch and E. Seneta. 1967. On Quasi-Stationary Distributions in Absorbing Continuous-Time Finite Markov Chains. *J. Appl. Probab.* 4, 1 (1967), 192–196.
[10] N. Castro González. 2005. Additive perturbation results for the Drazin inverse. *Linear Algebra Appl.* 397, 0 (2005), 279 – 297.
[11] T. Han, J.-P. Katoen, and A. Mereacre. 2008. Approximate Parameter Synthesis for Probabilistic Time-Bounded Reachability. In *Proceedings of the 29th IEEE Real-Time Systems Symposium, RTSS*. 173–182.
[12] Robert E. Hartwig, Guorong Wang, and Yimin Wei. 2001. Some additive results on Drazin inverse. *Linear Algebra Appl.* 322, 1 - 3 (2001), 207 – 217.
[13] A. Horvath and M. Telek. 2002. PhFit: A General Phase-Type Fitting Tool. In *TOOLS 2002*, T. Field et al. (Ed.). Springer LNCS 2324, 82–91.
[14] G. Horvath and M. Telek. 2009. On the canonical representation of phase type distributions. *Performance Evaluation* 66, 8 (2009), 396 – 409. Selected papers of the Fourth European Performance Engineering Workshop (EPEW) 2007.
[15] M. Kwiatkowska, G. Norman, and D. Parker. 2011. PRISM 4.0: Verification of Probabilistic Real-time Systems. In *Proc. 23rd International Conference on Computer Aided Verification (CAV'11)*. Springer LNCS 6806, 585–591.
[16] G. Latouche and V. Ramaswami. 1999. *Introduction to Matrix Analytic Methods in Stochastic Modeling*. Society for Industrial and Applied Mathematics.
[17] C. A. et al. O'Cinneide. 1967. Characterization of Phase-Type Distributions. *Communications in Statistics: Stochastic Models* 1, 6 (1967), 1–57.
[18] S. Pathak, E. Ábrahám, N. Jansen, A. Tacchella, and J.-P. Katoen. 2015. A Greedy Approach for the Efficient Repair of Stochastic Models. In *NASA Formal Methods - 7th Int. Symposium*. 295–309.
[19] T. Quatmann, C. Dehnert, N. Jansen, S. Junges, and J.-P. Katoen. 2016. Parameter Synthesis for Markov Models: Faster Than Ever. In *ATVA 2016*. Springer LNCS 9938, 50–67.
[20] B. Tati and M. Siegle. 2015. Parameter and Controller Synthesis for Markov Chains with Actions and State Labels. In *2nd International Workshop on Synthesis of Complex Parameters (SynCoP'15) (OpenAccess Series in Informatics (OASIcs))*, É. André and G. Frehse (Eds.), Vol. 44. Dagstuhl, Germany, 63–76.
[21] B. Tati and M. Siegle. 2016. Rate Reduction for State-labelled Markov Chains with Upper Time-bounded CSL Requirements. In Proc. Cassting Workshop on *Games for the Synthesis of Complex Systems* and 3rd Int. Workshop on *Synthesis of Complex Parameters,* Eindhoven, NL, April 2-3, 2016, T. Brihaye, B. Delahaye, L. Jezequel, N. Markey, and J. Srba (Eds.). Open Publishing Assoc., El. Proc. in Theoret. Computer Science Vol. 220, 77–89.