Whittle networks with resets*

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ABSTRACT

We present an extension of Whittle networks with multiple classes of customers and some signals as defined by Gelenbe. Customers are queued and served according to the balance rules defined for Whittle networks. Signals are not queued and interact with the customers present in the queue. We consider the reset signal previously introduced for single class networks. A reset signal entering a non empty queue deletes a customer but if the queue is empty, it fills the queue with a random number of customers. The distribution of this random variable is closely related to the steady-state distribution of the queue. We prove that these networks have a product form steady-state distribution.

CCS CONCEPTS

• **Mathematics of computing** → **Markov processes**;

KEYWORDS

Queueing Networks, Signal, Product Form Steady-State distribution, Whittle networks

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1 A BRIEF INTRODUCTION TO WHITTLE NETWORKS AND NETWORKS WITH CUSTOMERS AND SIGNALS

Whittle networks have been used in the last decade to analyze flows on Internet (see for instance [3–7, 27]). Indeed Whittle networks exhibit two very interesting properties: first, their steady-state distributions (when they exist) have a product form solution and second, they are insensitive [28, 29]. The insensitivity property implies that the probability distribution does not really depend on the service time distribution but only on the first moment of this distribution. Thus, we can replace an arbitrary distribution for the service times by an exponential distribution with the same mean which gives an easier model to analyze. Clearly, flows do not have an exponential duration but some load balancing mechanisms are very similar to the balance equation for Whittle networks and these properties and the balance among flows explains the success of Whittle networks as a modeling tool for Internet. In a Whittle network, customers of some class (say k) receive a fraction $\phi^{(k)}(\vec{x})$ of the service capacity of the server ($\vec{x}$ is the state of the queue) and these fractions obey some constraints (see definition in section 2).

In [9] we have added into Whittle networks, some signals as defined by Gelenbe for Generalized networks with customers and signals. The present paper is a sequel of this first paper and we consider a type of signal which was not studied in [9]. We prove that such a network also has a product form steady-state distribution when the flow equation has a solution.

The theory of queues with signals have received a considerable attention since the seminal paper on positive and negative customers [15] published by Gelenbe more than 20 years ago. Traditional queueing networks model systems are used to represent contention among customers for a set of resources. Customers moves form server to server, they wait for service, but they do not interact among themselves. Signals are used to change these rules. In a network of queues with signals (also denoted as a G-network of queues) customers are allowed to change to signals at the completion of their service and signals interact at their arrival into a queue with customers already backlogged in the queue. However signals are never queued. They try to interact and they may fail or succeed. In both cases they disappear immediately. Despite this deep modification of the model, G-networks still preserve the product form property for the steady-state distribution of some Markovian queueing networks.

In [15], Gelenbe introduced the first type of signal denoted as a negative customer. A negative customer deletes a positive customer at its arrival at a non empty queue. A negative customer is never queued (i.e. they are not customers, despite their name). Positive customers are ordinary customers which are queued and receive service as usual. But they can be deleted by negative customers and they can become a negative customer when they join another queue at the end of their service. Note that in this case, we may have the instantaneous departure of two customers: the positive

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customer which completes its service, becomes a negative customer which routes to another queue to delete a positive customer. Under typical assumptions for Markovian queueing networks (Poisson arrival for both types of customers, exponential service time for positive customers, Markovian routing of customers, open topology, independence of routing) and despite these new transitions in the Markov chain Gelenbe proved that such a network has a product form solution for its steady-state behavior when the chain is ergodic. Note that the open topology is mandatory because in a closed queuing network with negative customers, the positive customers disappear and all the queues are empty at steady-state with probability one under some topological constraints.

Clearly, G-networks are more complex than Jackson networks. The G-networks flow equations exhibit some uncommon properties: they are neither linear as in closed queueing networks nor contracting as in Jackson queueing networks. Therefore the existence of a solution had to be proved [22] by new techniques from the theory of fixed point equation, and a numerical algorithm was also developed to solve the flow equations [13]. G-networks have been used to model random neural networks [18, 19, 26] and complex operations involved in work deletion [1]. Many applications of G-networks have been proposed: the most notable being the CPN architecture [21] (see [12] for a bibliography).

G-networks had motivated new important results on queues, flows of customers and synchronizations. First as negative customers lead to customer deletions, the original description of quasi-reversibility does not hold anymore and new versions have been proposed. At the time being, the description proposed by Chao and his co-authors in [8] looks sufficient to study queues with customers and signals. A different approach was proposed by Harrison (see for instance [23, 24]). His theory is based on Stochastic Process Algebra. The main results (Reversed Compound Agent Theorem (i.e. RCAT) and its extensions [23–25, 2]) give some sufficient conditions for product form stationary distributions. RCAT clearly has a broader range of applications as it allows to represent component based models which are much more general than networks of queues. An interesting topic is to mix both results to obtain a quasi-reversibility characterization directly from a PEPA specification using a master-slave description of the system (like in RCAT) rather than arrivals, departure and internal transitions as proposed in [8]. Both approaches are very important to the development of new performance results.

Here we use the detailed global balance technique and the quasi-reversibility characterization by Chao [8] because they look easier in the context of an abstract queueing discipline. RCAT, quasi-reversibility and global balance equation analysis have been used to study several new types of signal which all leads to product form solution: triggers which redirect other customers among the queues, catastrophes which flush all the customers out of a queue [17, 16], resets [20], synchronized arrivals in a set of queues [11], signals which change the class of the customer in service [10]. Multiple class versions of these models have also been derived for resets [14] with a closed network topology. Note that the closed topology is possible in the reset case as the reset signal increases almost surely the number of customers in an empty queue. Therefore the empty state is not reachable (see [14] for more details).

The technical part of the paper is as follows. In section 2, we present a brief introduction to Whittle networks and we introduce the reset signal in a queue which follows the Whittle balance condition. We present how the reset signals make the population of the queue jump to any state with a distribution related to the steady-state distribution. As some proofs are based on Chao et al. version of quasi-reversibility we also give a short presentation of this new version of the theory. We first consider the model of one queue, and then we apply the approach to the network model. In section 3, we prove that the network has a product form steady-state distribution under some classical assumptions about the stochastic processes used to described the model.

Let $\vec{x}$ be the vector of number of customers in the queues and let $\phi^{(k)}(\vec{x})$ be the service rate for class k when the network state is $\vec{x}$. Here we assume that the queues all have a constant capacity equal to 1:

$$\sum_k \phi^{(k)}(\vec{x}) = 1.$$

This implies that all the service capacity is shared between the flows.

2 RESET IN A WHITTLE MULTI-CLASS QUEUE

2.1 Whittle Queue

Whittle networks are queuing networks with exponential services and state dependent transition probabilities which obey a balance property . Again, let $\vec{x}$ be the vector of number of customers in the queues and let $\vec{x} - e^{(j)}$ be the state $\vec{x}$ with one customer less at queue j. Let $\phi^{(k)}(\vec{x})$ be the service rate for class k when the network state is $\vec{x}$.

Definition 2.1. The service rates $\phi^{(i)}(\vec{x})$, functions of the state vector $\vec{x}$, are balanced if

$$\phi^{(i)}(\vec{x})\phi^{(j)}(\vec{x} - e^{(i)}) = \phi^{(j)}(\vec{x})\phi^{(i)}(\vec{x} - e^{(j)}), \quad \forall i, j.$$

The balance property has a very simple physical interpretation. Given a state $\vec{x}$, and $< \vec{x}, \vec{x}-e^{i_1}, \vec{x}-e^{i_1}-e^{i_2}, \ldots, 0 >$ a direct path from state $\vec{x}$ to state $\vec{0}$, the balance property says that the following expression, which is the product of the service rates along the path, does not depend on the considered path:

$$\Phi(\vec{x}) = \frac{1}{\phi^{(i_1)}(\vec{x})\phi^{(i_2)}(\vec{x} - e^{(i_1)}) \ldots \phi^{(i_m)}(\vec{x} - e^{(i_1)} - \ldots - e^{(i_{m-1})})}. \quad (1)$$

In a Whittle network, the service rate functions $\phi^{(i)}(\vec{x})$ are exclusively characterized by the balance function Φ:

$$\phi^{(i)}(\vec{x}) = \frac{\Phi(\vec{x} - e^{(i)})}{\Phi(\vec{x})}, \quad i = 1, \ldots, N, x^{(i)} > 0. \quad (2)$$

Consider the example of two direct paths from X to 0 depicted in Fig. 1. Assume that the rates are $\phi^{(1)}(x1, x2)$ to

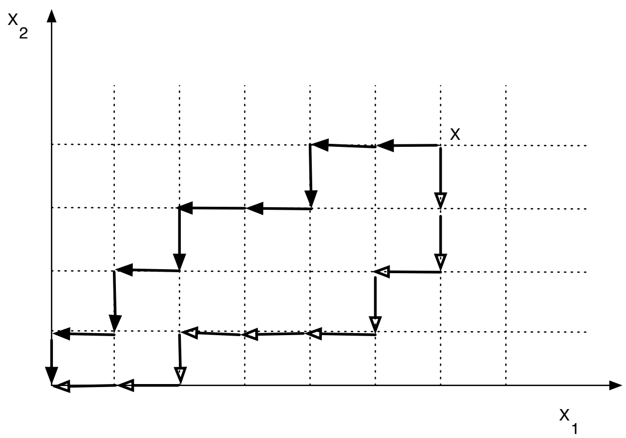

Figure 1: Two paths in the Markov chain of a Whittle network.

decrease $x1$ and $\phi^{(2)}(x1, x2)$ to decrease $x2$. For a Whittle network, the product of the rates along the path are all equal. Therefore, the product of the rates along the black arrows path is equal to the product of the rates along the white arrows path.

This balance property is very similar to the fairness property needed by flow schedulers on the Internet. A data network is considered as a set of links shared by some competing flows and fairness among the flows is mandatory. Many authors ([7], [5], [27]) have used Whittle network balance property to model balance fairness. The most famous example of Whittle networks is a network of Processor Sharing queues. In [3], many well-known models (including the Erlang model) which are all useful for studying communication networks, were shown to satisfy the Whittle network balance property, thus enhancing the potential applications of Whittle networks.

2.2 Quasi-Reversibility of Queues with signals

Let us first introduce the definition of quasi-reversibility before we can prove that the queue with resets is quasi-reversible. In [8], chap 3, Chao, Miyazawa and Pinedo gave a definition of quasi-reversibility for queue with or without trigger. Let us introduce the definition with trigger which is more general because it includes simultaneous transitions.

Consider a queue where the queue-content evolves as a continuous time Markov chain on state space $\mathcal{S}$. For a pair of states $(\vec{x}, \vec{y})$, we decompose the transition rate function $q(\vec{x}, \vec{y})$ of the queue into three types of rates: $q_u^A(\vec{x}, \vec{y}), u \in T$; $q_u^D(\vec{x}, \vec{y}), u \in T$; $q^I(\vec{x}, \vec{y})$, where T is the set of the classes of arrivals and departures, which is countable. The transition rate of the queue can be written as:

$$q(\vec{x}, \vec{y}) = \sum_{u \in T} q_u^A(\vec{x}, \vec{y}) + \sum_{u \in T} q_u^D(\vec{x}, \vec{y}) + q^I(\vec{x}, \vec{y}), \quad \vec{x}, \vec{y} \in \mathcal{S}.$$

The transition rate functions q_u^A, q_u^D and q^I generate the point processes corresponding to class u arrivals, class u departures and the internal transitions, respectively. “A”, “D” and “I” stand for “arrival”, “departure” and “internal”.

Suppose that q admits a stationary distribution π. Furthermore, assume that when a class u arrives and changes the state of the queue from $\vec{x}$ to $\vec{y}$, it instantaneously triggers a class v departure with probability $f_{u,v}(\vec{x}, \vec{y})$, where:

$$\sum_v f_{u,v}(\vec{x}, \vec{y}) \leq 1, \ u \in T, \ \vec{x}, \vec{y} \in \mathcal{S}.$$

With probability $1 - \sum_v f_{u,v}(\vec{x}, \vec{y})$, the class u arrival does not trigger any departure. Function $f_{u,v}(\vec{x}, \vec{y})$ is the *triggering probability.* When $\sum_{v \in T} f_{u,v}(\vec{x}, \vec{y}) \equiv 0$ for all $u \in T, \ \vec{x}, \vec{y} \in \mathcal{S}$, no instantaneous movement between queues may occur due to signal. The quasi-reversibility of queues with instantaneous movement is defined as follows.

Definition 2.2. If there exist two sets of non-negative numbers $\{\alpha_u, u \in T\}$ and $\{\beta_u, u \in T\}$ such that: for all $\vec{x} \in \mathcal{S}$, $u \in T$,

$$\sum_{\vec{y} \in \mathcal{S}} q_u^A(\vec{x}, \vec{y}) = \alpha_u, \quad (3)$$

$$\sum_{\vec{y} \in \mathcal{S}} \pi(\vec{y}) \Big[q_u^D(\vec{y}, \vec{x}) + \sum_{v \in T} q_v^A(\vec{y}, \vec{x}) f_{v,u}(\vec{y}, \vec{x}) \Big] = \beta_u \pi(\vec{x}), \quad (4)$$

then the queue with signal is said to be *quasi-reversible with respect to* $\{q_u^A, f_{u,v}, \ u, v \in T\}$, $\{q_u^D, u \in T\}$ *and* $\{q^I\}$.

The non-negative numbers α_u and β_u are called the arrival rate and departure rate of class u customers. Chao et al. proved that this definition of queue without instantaneous movements is equivalent to the usual definition of quasi-reversibility. This implies that the arrival processes and the departure (triggered and non-triggered) of class u customers are Poisson processes.

2.3 Resets

We consider a simple Whittle multi-class queue. The arrivals of customers and signal follow independent Poisson processes and the service times distributions are exponentially distributed. The rates are $\lambda^{(k)}$ for the arrivals of class-k customers, λ^r for the arrivals of signals, and $\mu^{(k)}$ for the service of class-k customers. Upon arrival, if a queue is not empty, then the signal will delete a customer of class k with probability $\phi^{(k)}(\vec{x})$ if the state of the queue is $\vec{x}$. If the queue is empty, then the signal resets the queue. The queue jumps to an arbitrary state $\vec{x}$ with distribution $\tau(\vec{x})$ which will be precisely defined later. Note that as $\sum_k \phi^{(k)}(\vec{x}) = 1$, a signal entering the queue always have an effect.

If π is a steady state distribution, then the following conditions must be satisfied:

- When the network is empty

$$\pi(0)(\sum_k \lambda^{(k)} + \lambda^r) = \sum_k \pi(e^{(k)}) \phi^{(k)}(e^{(k)})(\lambda^r + \mu^{(k)}). \quad (5)$$

- When the network contains at least one customer

$$\pi(\vec{x})\Big\{ \sum_k \lambda^{(k)} + \lambda^r + \sum_k \mathbb{1}_{x^{(k)}\geq 1}\phi^{(k)}(\vec{x})\mu^{(k)}\Big\} = \sum_k \mathbb{1}_{x^{(k)}\geq 1}\pi(\vec{x}-e^{(k)})\lambda^{(k)} + \pi(0)\tau(\vec{x})\lambda^r + \sum_k \pi(\vec{x}+e^{(k)})\phi^{(k)}(\vec{x}+e^{(k)})(\mu^{(k)}+\lambda^r). \quad (6)$$

We would like to study if there exists a multiplicative structure form for the steady state distribution. Suppose that the stationary distribution has the form

$$\pi(\vec{x}) = \pi(0)\Phi(\vec{x})\prod_k (\rho^{(k)})^{x^{(k)}}, \quad (7)$$

where $\Phi(0) = 1$ to be consistent and $\sum_k \rho^{(k)} < 1$ for stability. Lemma 2.3 gives the relation between $\pi(0)$ and $\rho^{(k)}$ when we have constant capacity: $\sum_k \phi^{(k)}(\vec{x}) = 1, \forall \vec{x} \neq 0$.

Lemma 2.3. *Assume that the stationary distribution of a Whittle network with constant capacity 1 is given by Equation 7, then we have:*

$$\pi(0) = 1 - \sum_k \rho^{(k)}. \quad (8)$$

Proof. Denote by $\|\vec{x}\|$ the number of customer in the queue ($\|\vec{x}\| = \sum_k x^{(k)}$). We will prove by induction that, for all $n \geq 1$:

$$\sum_{\|\vec{x}\|=n} \pi(\vec{x}) = \pi(0)(\sum_k \rho^{(k)})^n. \quad (9)$$

For $n = 1$, one has that

$$\sum_{\|\vec{x}\|=1} \pi(\vec{x}) = \sum_k \pi(0)\Phi(e^{(k)})\rho^{(k)} = \sum_k \pi(0)\frac{\Phi(0)}{\phi^{(k)}(e^{(k)})}\rho^{(k)}.$$

By assumption, $\sum_l \phi^{(l)}(e^{(k)}) = 1$, and if $l \neq k$ we also have $\phi^{(l)}(e^{(k)}) = 0$. Therefore we obtain $\phi^{(k)}(e^{(k)}) = 1$. As $\Phi(0) = 1$, then we also get $\Phi(e^{(k)}) = 1$. Finally,

$$\sum_{\|\vec{x}\|=1} \pi(\vec{x}) = \sum_k \pi(0)\rho^{(k)}.$$

Hence, Equation 9 is true for $n = 1$.

Suppose that Equation 9 holds for n. We will prove that it is also true for $n+1$. As $\sum_c \phi^{(c)}(\vec{x}) = 1$, we have:

$$\pi(\vec{x}) = \pi(0)\Phi(\vec{x})\sum_c \phi^{(c)}(\vec{x})\prod_k (\rho^{(k)})^{x^{(k)}}.$$

But the balance function is such that $\Phi(\vec{x})\phi^{(c)}(\vec{x}) = \Phi(\vec{x} - e^{(c)})\mathbb{1}_{x^{(c)}\geq 1}$. Thus,

$$\begin{aligned}\sum_{\|\vec{x}\|=n+1} \pi(\vec{x}) &= \sum_{\|\vec{x}\|=n+1} \pi(0)\sum_c \mathbb{1}_{x^{(c)}\geq 1}\Phi(\vec{x}-e^{(c)})\rho^{(c)} \prod_{k\neq c}(\rho^{(k)})^{x^{(k)}}(\rho_c)^{x^{(c)}-1} \\ &= \sum_c \rho^{(c)}\sum_{\|\vec{x}\|=n+1}\pi(0)\mathbb{1}_{x^{(c)}\geq 1}\Phi(\vec{x}-e^{(c)}) \prod_{k\neq c}(\rho^{(k)})^{x^{(k)}}(\rho_c)^{x^{(c)}-1} \\ &= \sum_c \rho^{(c)}\sum_{\|\vec{y}\|=n}\pi(y).\end{aligned} \quad (10)$$

By induction Equation 9 is true for all n. It yields that Equation 8 holds as $\sum_k \rho^{(k)} < 1$. This completes the proof of the Lemma.

□

Let us now find the form for probability distribution τ. Substitute π in Equation 5, one has:

$$\sum_k \lambda^{(k)} + \lambda^r = \sum_k \rho^{(k)}(\lambda^r + \mu^{(k)}). \quad (11)$$

Using the form of π, the relation $\Phi(\vec{x}) = \Phi(\vec{x}+e^{(k)})\phi^{(k)}(\vec{x}+e^{(k)})$ and Equation 11 one has:

$$\pi(\vec{x})(\sum_k \lambda^{(k)} + \lambda^r) = \sum_k \pi(\vec{x}+e^{(k)})\phi^{(k)}(\vec{x}+e^{(k)})(\mu^{(k)}+\lambda^r) .$$

Then Equation 6 can be simplified as:

$$\begin{aligned}\tau(\vec{x})\lambda^r &= \sum_k \mathbb{1}_{x^{(k)}\geq 1}\tfrac{\pi(\vec{x})}{\pi(0)}\phi^{(k)}(\vec{k})\mu^{(k)} \\ &- \sum_k \mathbb{1}_{x^{(k)}\geq 1}\lambda^{(k)}\tfrac{\pi(\vec{x}-e^{(k)})}{\pi(0)}.\end{aligned} \quad (12)$$

Hence, taking into account the multiplicative solution of π, the form of τ is given by

$$\tau(\vec{x}) = \sum_k \mathbb{1}_{x^{(k)}\geq 1}\Phi(\vec{x}-e^{(k)})\prod_c (\rho^{(c)})^{(x^{(c)})}\frac{\mu^{(k)}\rho^{(k)} - \lambda^{(k)}}{\lambda^r\rho^{(k)}}.$$

And it suggests a relation between $\tau(\vec{x})$ and $\pi(\vec{x} - e^{(k)})$. We would like to find ρ in the form:

$$\rho^{(k)} = \frac{\lambda^{(k)} + \gamma^{(k)}\lambda^r}{\mu^{(k)} + \lambda^r}, \quad (13)$$

with $\sum_k \gamma^{(k)} = 1$. Note that if the values of $\rho^{(k)}$ satisfy Equation 13 with $\sum_k \gamma^{(k)} = 1$, Equation 11 also holds. Moreover, one has

$$\frac{\mu^{(k)}\rho^{(k)} - \lambda^{(k)}}{\lambda^r} = \gamma^{(k)} - \rho^k.$$

Assume that $\gamma^{(k)} = \frac{\rho^{(k)}}{\sum_c \rho^{(c)}}$. Clearly, $\sum_k \gamma^{(k)} = 1$, and

$$\begin{aligned}\tfrac{\mu^{(k)}\rho^{(k)}-\lambda^{(k)}}{\lambda^r}\tfrac{1}{\pi(0)} &= (\tfrac{\rho^{(k)}}{\sum_c \rho^{(c)}} - \rho^{(k)})\tfrac{1}{1-\sum_c \rho^{(c)}} \\ &= \tfrac{\rho^{(k)}}{\sum_c \rho^{(c)}} \\ &= \gamma^{(k)}.\end{aligned}$$

Finally, after substitution, τ is given by:

$$\tau(\vec{x}) = \sum_k \mathbb{1}_{x^{(k)}\geq 1}\pi(\vec{x}-e^{(k)})\gamma^{(k)}. \quad (14)$$

We must now check that τ is a distribution of probability. We have that

$$\sum_{\vec{x}\neq 0}\tau(\vec{x}) = \sum_{\vec{x}\neq 0}\sum_{k}\mathbb{1}_{x^{(k)}\geq 1}\pi(\vec{x}-e^{(k)})\beta^{(k)} = \sum_{k,\vec{y}}\gamma^{(k)}\pi(\vec{y}) = 1.$$

Hence, if $\rho^{(k)}$ is given by Equation 13, where $\gamma^{(k)} = (\rho^{(k)})/(\sum_c \rho^{(c)})$ and τ is given by Equation 14, then the balance equations 5 and 6 are satisfied by a stationary distribution π given in Equation 7.

LEMMA 2.4. *The system of equations*

$$\rho^{(k)} = \frac{\lambda^{(k)} + (\rho^{(k)}/\sum_c \rho^{(c)})\lambda^r}{\mu^{(k)} + \lambda^r},$$

has a positive solution.

PROOF. Consider the compact and convex set $\mathcal{B}$ defined by:

$$\mathcal{B} = \{\vec{\rho} \in R^{\mathcal{C}} \mid \rho^{(k)} \geq 0, \sum_k \rho^{(k)}(\mu^k + \lambda^r) = \sum_k \lambda^{(k)} + \lambda^r\}.$$

Consider the function $\Psi : \mathcal{B} \to R^{\mathcal{C}}$, where:

$$\Psi(\rho)^{(k)} = \frac{\lambda^{(k)} + (\rho^{(k)}/\sum_c \rho^{(c)})\lambda^r}{\mu^{(k)} + \lambda^r}.$$

One has that:

$$\Psi(\rho)^{(k)}(\mu^k + \lambda^r) = \sum_k \lambda^{(k)} + \lambda^r,$$

then $\Psi(\mathcal{B}) \subset \mathcal{B}$. Moreover, Ψ is a continuous function.

Applying the Brouwer's Fixed point theorem, one has that there exists a solution in $\mathcal{B}$ to the function $\rho^{(k)} = \Psi(\rho)^{(k)}$. This completes the proof.

□

LEMMA 2.5. *The stationary queue is quasi-reversible in the sense defined by Chao et al in [8] as for all $\vec{x}$, one has*

$$\frac{\sum_{\vec{y}}\pi(\vec{y})Q^{D,(k)}(\vec{y},\vec{x})}{\pi(\vec{x})} = \frac{\Phi(\vec{x}+e^{(k)})\rho^{(k)}\mu^{(k)}\phi^{(k)}(\vec{x})}{\Phi(\vec{x})} = \mu^{(k)}\rho^{(k)},$$

where $Q^{D,(k)}(\vec{y},\vec{x})$ is the transition rate caused by a departure of class k from state $\vec{y}$ to state $\vec{x}$.

We now have all the ingredients to state the theorem on the stationary distribution of the queue. For the sake of readability, we give again all the assumptions.

THEOREM 2.6. *Consider a simple Whittle queue with multiple classes of customers and with constant capacity 1 and the balance function $\Phi(\vec{x})$ ($\vec{x} = (x^{(k)})$). The arrival rate and service rate for class-k customers are respectively $\lambda^{(k)}$ and $\mu^{(k)}$. The reset signals arrive to the queue according to a Poisson process of r ate λ^r. If the state of the queue is $\vec{x}$, then the signal will chose a customer of a class k as a target to delete with probability $\phi^{(k)}(\vec{x})$.*

Assume that the system of equations

$$\rho^{(k)} = \frac{\lambda^{(k)} + (\rho^{(k)}/\sum_c \rho^{(c)})\lambda^r}{\mu^{(k)} + \lambda^r}$$

has a solution $(\rho^{(k)})$ such that $\sum_k \rho^{(k)} < 1$. Let

$$\gamma^{(k)} = \frac{\rho^{(k)}}{\sum_c \rho^{(c)}} \ .$$

Consider the measure π defined by

$$\pi(\vec{x}) = \pi(0)\Phi(\vec{x})\prod_k (\rho^{(k)})^{x^{(k)}},$$

where $\pi(0) = 1 - \sum_k \rho^{(k)}$.

When a reset signal arrives to an empty queue, we assume that the queue jumps to state $\vec{x}$ with distribution of probability:

$$\tau(\vec{x}) = \sum_k \mathbb{1}_{x^{(k)}\geq 1}\pi(\vec{x}-e^{(k)})\gamma^{(k)}.$$

Then the stationary distribution of the network is $\pi(\vec{x})$ and the queue is quasi-reversible with the departure rate for class-k customers equal to $\mu^{(k)}\rho^{(k)}$.

PROPOSITION 2.7. *Note that in the case where the capacity is a constant C (not equal to 1), one has that*

$$\sum_{\|\vec{x}\|=n}\pi(\vec{x}) = \pi(0)\Big(\frac{\sum_k \rho^{(k)}}{C}\Big)^n,$$

and

$$\pi(0) = 1 - \frac{\sum_k \rho^{(k)}}{C}.$$

In this case, we modify the system as follows: when a signal arrives to a nonempty queue, it will chose a customer of class k to cancel with rate:

$$\frac{\lambda^r}{C}\phi^{(k)}(x),$$

and all γ, τ are defined as before, ρ is given by:

$$\rho^{(k)} = \frac{\lambda^{(k)} + \gamma^{(k)\lambda^r}}{\mu^{(k)} + \lambda^r/C},$$

and we still have the same result.

3 RESET IN A NETWORK OF WHITTLE MULTI-CLASS QUEUES WITH CONSTANT CAPACITY

We consider a network of N quasi-reversible queues and we see how we can combine these queues into a network with a product form solution. Again, this part is a short presentation extracted from [8] for the sake of completeness. We consider the same set of arrival and departure classes, T. Let $\vec{x}_i$ be the state of queue i. Let $\mathcal{S}_i$ be the state space. The Poisson source has index 0 and we assume that the source has only one state which is denoted as 0. For each queue, we need to specify the arrival effects, the departure rate, the internal transition rate and the triggering probability. For queue i, we introduce functions p^A_{iu}, q^D_{iu}, q^I_i and $f_{iu,v}$ on the state space $\mathcal{S}_i$:

- $p^A_{iu}(\vec{x}_i,\vec{y}_i)$ is the probability that a class u arrival at queue i changes the state from $\vec{x}_i$ to $\vec{y}_i$, where it is assumed that $\sum_{\vec{y}\in\mathcal{S}_i} p^A_{iu}(\vec{x}_i,\vec{y}_i) = 1$, $\vec{x}_i \in \mathcal{S}_i$; We simply have:

$$p_u^A(\vec{x},\vec{y}) = \frac{q_u^A(\vec{x},\vec{y})}{\sum_z q_u^A(\vec{x},\vec{z})} \ ,$$

- $q_{iu}^D(\vec{x}_i,\vec{y}_i)$ is the rate at which class u departures change the state of queue i from $\vec{x}_i$ to $\vec{y}_i$;
- $q_i^I(\vec{x}_i,\vec{y}_i)$ is the rate at which internal transitions change the state of queue i from $\vec{x}_i$ to $\vec{y}_i$;
- $f_{iu,v}(\vec{x}_i,\vec{y}_i)$ is the triggering probability that when a class u arrival occurs at queue i and the state changes from $\vec{x}_i$ to $\vec{y}_i$, it simultaneously induces a class v departure, where $\sum_{v\in T} f_{iu,v}(\vec{x}_i,\vec{y}_i) \leq 1,\ i \leq N,\ u \in T,\ \vec{x}_i,\vec{y}_i \in \mathcal{S}_i$.
- q_u^D and q^I are the departure and internal transition functions already defined in the previous section.

For source 0, we set $p_{0u}^A(0,0)=1$, $p_{0u}^A(0,0)=\beta_{0u}$, $q_0^I(0,0)=0$ and $f_{0u,v}\equiv 0$. Here, β_{0u}^A is the arrival rate to the network from the outside (the source).

The dynamics of the network are described as follows. Customers of class u arrive to the network from the outside according to a Poisson process with rate β_{0u}, and are routed to queue i as a class v arrival with probability $r_{0u,iv}$. A class u departure from queue i, either trigger or non-trigger, enters queues j as a class v arrival with probability $r_{iu,jv}$. Furthermore, whenever there is a class u arrival at queue i, either from the outside or from other queues, it makes the state of the queue change from $\vec{x}_i$ to $\vec{y}_i$ with probability $p_{iu}^A(\vec{x}_i,\vec{y}_i)$, it also triggers a class u departure with probability $f_{iu,v}(\vec{x}_i,\vec{y}_i)$, and it triggers no departure from queue i with probability $1-\sum_{v\in T} f_{iu,v}(\vec{x}_i,\vec{y}_i)$, $i=0,1,\ldots,N$.

The transition rate function of the network is denoted by $q(\vec{x},\vec{y})$, $\vec{x},\vec{y}\in\mathcal{S}=\mathcal{S}_1\times\cdots\times\mathcal{S}_N$ (note that we accept the case where $q(\vec{x},\vec{x})\neq 0$).

Consider for each queue i the following auxiliary process:

$$q_i^{(\vec{\alpha}_i)}(\vec{x}_i,\vec{y}_i) = \sum_{u\in T}\Big(\alpha_{iu}p_{iu}^A(\vec{x}_i,\vec{y}_i)+q_{iu}^D(\vec{x}_i,\vec{y}_i)\Big)+q_i^D(\vec{x}_i,\vec{y}_i),$$

where $(\vec{\alpha}_i)=(\alpha_{iu},u\in T)$ are considered as dummy parameters. Suppose that $q_i^{(\vec{\alpha}_i)}$ has a stationary distribution $\pi_i^{(\vec{\alpha}_i)}$. We always have:

$$\sum_{\vec{y}_i\in\mathcal{S}_i}\alpha_{iu}p_{iu}^A(\vec{x}_i,\vec{y}_i)=\alpha_{iu}, \qquad i=1,\ldots N,\ u\in T.$$

Hence, the quasi-reversibility of $q_i^{(\vec{\alpha}_i)}$ for $i=1,\ldots,N$ is equivalent to the existence of a set of non-negative numbers $\beta_{iu}, u\in T$ such that:

$$\sum_{\vec{y}_i}\pi_i^{(\vec{\alpha}_i)}(\vec{y}_i)\Big[q_{iu}^D(\vec{y}_i,\vec{x}_i)+\sum_{v\in T}\alpha_{iv}p_{iv}^A(\vec{y}_i,\vec{x}_i)f_{iv,u}(\vec{y}_i,\vec{x}_i)\Big] = \beta_{iu}\pi_i^{(\vec{\alpha}_i)}(\vec{x}_i), \quad (15)$$

for all $\vec{x}_i\in\mathcal{S}_i$, $i=1,\ldots,N$ and $u\in T$.

Queue i in isolation is said to be quasi-reversible with $\vec{\alpha}_i$ if 15 is satisfied. Since α_{iu} and β_{iu} are the arrival and the departure rates of class u customers at queue i, we have the following traffic equations:

$$\alpha_{iu}=\sum_{j=0}^{N}\sum_{v}\beta_{jv}r_{jv,iu}\ , \qquad i=0,1,\ldots,N. \quad (16)$$

The stationary distribution of the network process has product form (see [8] for a proof).

THEOREM 3.1. *Under the assumptions on the routing, if each queue with signals is quasi-reversible (with $\vec{\alpha}_i$ as the solution to the traffic equations 16), then the queueing network with signal has the product form stationary distribution*

$$\pi(\vec{x})=\prod_{i=1}^{N}\pi_i^{(\vec{\alpha}_i)}(\vec{x}_i),$$

where $\pi_i^{(\vec{\alpha}_i)}$ is the stationary distribution of $q_i^{(\vec{\alpha}_i)}$, $i=1,\ldots,N$.

Let us turn back now to the network of queues with resets. We have proved that the queues are quasi-reversible in the previous section. One can use Theorem 3.1 to conclude. However, we think that the proof of the product using the Global Balance Equation technique is more informative in that case. Thus, we establish the main theorem of this paper using the Chapman Kolmogorov equation. Let us first describe how the queues are combined in the network.

We still assume that the capacity of each queue is 1 and that the balance function only depends on the state of the queue.

$$\phi_i^{(k)}(\vec{x}_i)=\frac{\Phi_i(\vec{x}_i-e_i^{(k)})}{\Phi_i(\vec{x}_i)}. \quad (17)$$

As before, the reset signals arrive to queue i according to a Poisson process of rate λ_i^r. At its arrival, a signal chooses a class k-customer of queue i as a target to delete with probability $\phi_i^{(k)}(\vec{x}_i)$. If a reset signal arrives to an empty queue i, the queue jumps to the $\vec{x}_i$ with probability:

$$\tau_i(\vec{x}_i)=\sum_k \mathbb{1}_{x_i^{(k)}\geq 1}\pi_i(\vec{x}_i-e_i^{(k)})\beta_i^{(k)},$$

which will be determined later.

The routing probabilities were defined in the previous sections: $d_i^{(k)}$ and matrices $P_{i,j}^{(k,c)}$. We now also consider the probability that a customer of class k in queue i at its service completion jump to queue j as a reset signal denoted as the entries of matrix $P_{i,j}^{(k),r}$. Of course, one still has that: for all i and k:

$$d_i^{(k)}+\sum_j\sum_c P_{i,j}^{(k,c)}+\sum_j P_{i,j}^{(k),r}=1.$$

Consider the Traffic Equations

$$\Lambda_i^{(k)} = \lambda_i^{(k)}+\sum_{j,c\in\mathcal{C}}\mu_j^{(c)}\rho_j^{(c)}P_{j,i}^{(c,k)}, \quad (18)$$

$$\Lambda_i^{r} = \lambda_i^{r}+\sum_{j,c\in\mathcal{C}}\mu_j^{(c)}\rho_j^{(c)}P_{j,i}^{(c),r}, \quad (19)$$

where $\vec{\rho}_i$ is the solution to the system

$$\rho_i^{(k)}=\frac{\Lambda_i^{(k)}+\beta_i^{(k)}\Lambda_i^r}{\mu^{(k)}+\Lambda_i^r}\ ,$$

and

$$\beta_i^{(k)} = \frac{\rho_i^{(k)}}{\sum_c \rho_i^{(c)}}.$$

Using the result in Theorem 2.6, and by applying the Theorem of network with quasi-reversible queues in [8] to the network model, one has the following theorem.

THEOREM 3.2. *Consider a network with reset signals where each queue is a Whittle-queue. If the Traffic Equations has a solution such that* $\sum_k \rho_i^{(k)} < 1$ *for all* i*, then the steady-state distribution of the network is given by*

$$\pi(\vec{x}) = \prod_{i \le N} \pi_i(\vec{x}_i),$$

where

$$\pi_i(\vec{x}_i) = (1 - \sum_k \rho_i^{(k)}) \Phi_i(\vec{x}_i) \prod_k (\rho_i^{(k)})^{\vec{x}_i^{(k)}}.$$

Let us give the proof by writing now the Chapman Kolmogorov equation:

$$\pi(\vec{x}) \textstyle\sum_{i;k \in \mathcal{C}} \left\{ \lambda_i^{(k)} + \mathbb{1}_{x_i^{(k)} \ge 1} \phi_i^{(k)}(\vec{x}) \big(\mu_i^{(k)} + \lambda_i^r \big) + \mathbb{1}_{\vec{x}_i = \vec{0}} \lambda_i^r \right\} =$$

$$\textstyle\sum_{i;k \in \mathcal{C}} \mathbb{1}_{x_i^{(k)} \ge 1} \pi(\vec{x} - e_i^{(k)}) \lambda_i^{(k)} \quad [1]$$

$$+ \textstyle\sum_{i;k \in \mathcal{C}} \pi(\vec{x} + e_i^{(k)}) \phi_i^{(k)}(\vec{x} + e_i^{(k)}) \lambda_i^r \quad [2]$$

$$+ \textstyle\sum_{i;k \in \mathcal{C}} \mathbb{1}_{\vec{x}_i \ne \vec{0}} \pi(\vec{x} \oplus (i, \vec{0})) \tau_i(\vec{x}_i) \lambda_i^r \quad [3]$$

$$+ \textstyle\sum_{i;k \in \mathcal{C}} \pi(\vec{x} + e_i^{(k)}) \phi_i^{(k)}(\vec{x} + e_i^{(k)}) \mu_i^{(k)} d_i^{(k)} \quad [4]$$

$$+ \textstyle\sum_{i,j;k,c \in \mathcal{C}} \mathbb{1}_{x_j^{(c)} \ge 1} \pi(\vec{x} + e_i^{(k)} - e_j^{(c)}) \phi_i^{(k)}(\vec{x} + e_i^{(k)} - e_j^{(c)}) \mu_i^{(k)} P_{i,j}^{(k,c)} \quad [5]$$

$$+ \textstyle\sum_{i;k \in \mathcal{C}} \pi(\vec{x} + e_i^{(k)} + e_i^{(c)}) \phi_i^{(k)}(\vec{x} + e_i^{(k)} + e_i^{(c)}) \mu_i^{(k)} P_{i,j}^{(k),r} \phi_j^{(c)}(\vec{x} + e_i^{(c)}) \quad [6]$$

$$+ \textstyle\sum_{i;k \in \mathcal{C}} \mathbb{1}_{\vec{x}_j \ne \vec{0}} \pi\big((\vec{x} + e_i^{(k)}) \oplus (j, \vec{0})\big) \phi_i^{(k)}\big((\vec{x} + e_i^{(k)}) \oplus (j, \vec{0})\big) \mu_i^{(k)} P_{i,j}^{(k),r} \tau_j(\vec{x}_j), \quad [7]$$

where $\vec{x} \oplus (i, \vec{y}_i)$ denote for the vector which is equal to $\vec{x}$ except component $\vec{x}_i$ which is equal to vector $\vec{y}_i$.

Let us give some explanations about the right-hand side of this equation. The first term corresponds to external arrivals of customers of class k in queue i. The next two terms correspond to an arrival of a signal at queue i (a cancellation if queue i is not empty (i.e. Term 2), or a reset if queue i is empty (Term 3)). The fourth term corresponds to an end of service and a departure to the outside. Term 5 represents the migration of a customer of class k leaving queue i for queue j as a customer of class c. In the last two terms, we describe a customer of class k leaving queue i for queue j as a signal which deletes one customer if queue j is not empty (Term 6) or resetting empty queue j (Term 7).

Divide both side of the equation by $\pi(\vec{x})$, using the balance equation (i.e. Eq. 17), the routing condition, the form of π, then the left-hand-side is given by:

$$\begin{aligned} L_1 &= \textstyle\sum_{i;k \in \mathcal{C}} \left\{ \lambda_i^{(k)} + \mathbb{1}_{x_i^{(k)} \ge 1} \phi_i^{(k)}(\vec{x}) \big(\mu_i^{(k)} + \lambda_i^r \big) + \mathbb{1}_{\vec{x}_i = \vec{0}} \lambda_i^r \right\} \\ &= \textstyle\sum_{i,k} \lambda_i^{(k)} + \sum_i \lambda_i^r + \sum_{i,k} \mathbb{1}_{x_i^{(k)} \ge 1} \phi_i^{(k)}(\vec{x}) \mu_i^{(k)}. \end{aligned}$$

And the right-hand-side is given by:

$$\begin{aligned} R_1 &= \textstyle\sum_{i;k \in \mathcal{C}} \mathbb{1}_{x_i^{(k)} \ge 1} \frac{1}{\rho_i^{(k)}} \phi_i^{(k)}(\vec{x}_i) \lambda_i^{(k)} + \sum_{i,k} \rho_i^{(k)} \lambda_i^r \\ &+ \textstyle\sum_i \mathbb{1}_{\vec{x}_i \ne \vec{0}} \frac{\pi_i(0)}{\pi_i(\vec{x}_i)} \tau_i(\vec{x}_i) \lambda_i^r + \sum_{i,k} \rho_i^{(k)} \mu_i^{(k)} d_i^{(k)} \\ &+ \textstyle\sum_{i,j;k,c \in \mathcal{C}} \mathbb{1}_{x_j^{(c)} \ge 1} \frac{\rho_i^{(k)}}{\rho_j^{(c)}} \phi_j^{(c)}(\vec{x}_j) \mu_i^{(k)} P_{i,j}^{(k,c)} \\ &+ \textstyle\sum_{i,j;k,c \in \mathcal{C}} \rho_i^{(k)} \rho_j^{(c)} \mu_i^{(k)} P_{i,j}^{(k),r} \\ &+ \textstyle\sum_{i,k} \sum_j \mathbb{1}_{\vec{x}_j \ne \vec{0}} \rho_i^{(k)} \mu_i^{(k)} P_{i,j}^{(k),r} \frac{\pi_j(0)}{\pi_j(\vec{x}_i)} \tau_j(\vec{x}_j). \end{aligned}$$

Using the form of τ and $\pi_i(0)$, one has that

$$\begin{aligned} \tfrac{\pi_i(0)}{\pi_i(\vec{x}_i)} \tau_i(\vec{x}_i) &= \textstyle\sum_k \mathbb{1}_{x_i^k \ge 1} \phi_i^{(k)}(\vec{x}_i) \frac{1}{\rho_i^{(k)}} \beta_i^{(k)} (1 - \sum_k \rho_i^{(k)}) \\ &= \textstyle\sum_k \mathbb{1}_{x_i^k \ge 1} \frac{\phi_i^{(k)}(\vec{x}_i)}{\rho_i^{(k)}} (\beta_i^{(k)} - \rho_i^{(k)}). \end{aligned}$$

The form of the Traffic Equations implies that:

$$\begin{aligned} R_1 &= \textstyle\sum_{i,k} \rho_i^{(k)} \Lambda_i^r + \sum_{i,k} \rho_i^{(k)} \mu_i^{(k)} d_i^{(k)} \\ &\quad + \textstyle\sum_{i,k} \mathbb{1}_{x_i^k \ge 1} \frac{\phi_i^{(k)}(\vec{x}_i)}{\rho_i^{(k)}} \Lambda_i^{(k)} \\ &\quad + \textstyle\sum_k \mathbb{1}_{x_i^k \ge 1} \frac{\phi_i^{(k)}(\vec{x}_i)}{\rho_i^{(k)}} (\beta_i^{(k)} - \rho_i^{(k)}) \Lambda_i^r \\ &= \textstyle\sum_{i,k} \Lambda_i^{(k)} + \sum_{i,k} \beta_i^{(k)} \Lambda_i^r - \sum_{i,k} \rho_i^{(k)} \mu_i^{(k)} (1 - d_i^{(k)}) \\ &\quad + \textstyle\sum_{i,k} \mathbb{1}_{x_i^k \ge 1} \phi_i^{(k)}(\vec{x}_i) (\mu_i^{(k)} \\ &\quad + \Lambda_i^r) - \textstyle\sum_{i,k} \mathbb{1}_{x_i^k \ge 1} \phi_i^{(k)}(\vec{x}_i) \Lambda_i^r. \end{aligned}$$

The Traffic Equations also implies that:

$$\begin{aligned} \textstyle\sum_{i,k} \Lambda_i^{(k)} + \sum_i \Lambda_i^r &= \textstyle\sum_{i,k} \lambda_i^{(k)} + \sum_i \lambda_i^r \\ &\quad + \textstyle\sum_{i,j,k} \rho_j^{(c)} \mu_j^{(c)} P_{j,i}^{(c,k)} \\ &\quad + \textstyle\sum_{i,j,k} \rho_j^{(c)} \mu_j^{(c)} P_{j,i}^{(c),r} \\ &= \textstyle\sum_{i,k} \lambda_i^{(k)} + \sum_i \lambda_i^r \\ &\quad + \textstyle\sum_{i,k} \rho_i^{(k)} \mu_i^{(k)} (1 - d_i^{(k)}). \end{aligned}$$

It yields that

$$R_1 = \sum_{i,k} \lambda_i^{(k)} + \sum_i \lambda_i^r + \sum_{i,k} \mathbb{1}_{x_i^k \geq 1} \phi_i^{(k)}(\vec{x}_i)\mu_i^{(k)},$$

the balance equation $L_1 = R_1$ is proved and the theorem is established.

4 CONCLUSION

Note that in [9], we have considered a slightly more general balance condition: system state $\vec{x}$ takes into account not only the customer number in each queue, but the customer number $x_i^{(k)}$ of each class k in every queue i. The service rates are then functions of these variables and the balance property is defined with respect to classes. The global balance approach in section 3 can be generalized to deal with this more general balance property.

As previously mentioned both G-networks and Whittle networks have many applications. Whittle networks have been used to model flows over the Internet while G-networks received considerable attention and allowed many applications, most of them to model random neural networks and to build cognitive packet networks.

This paper is mainly theoretical, extending the first results on the introduction of signals to Whittle networks presented in [9]. We hope this new extension will allow new theoretical developments (for instance on insensitivity as ordinary Whittle networks are insensitive) and new applications.

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