

# Game-Theoretic Deployment Design of Small-Cell OFDM Networks

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**Abstract**—We present a non-cooperative game-theoretic approach for the distributed resource allocation problem in the context of multiple transmitters communicating with multiple receivers through parallel independent fading channels, which is closely related with small-cell multi-user orthogonal frequency division multiplexing (OFDM) networks, e.g., Wi-Fi hotspots. We assume that all the transmitters are rational, selfish, and each one carries the objective of maximizing its own transmit rate, subject to its power constraint. In such a game-theoretic study, the central question is whether a Nash equilibrium (NE) exists, and if so, whether the network operates efficiently at the NE. We show, for independent fading channels, there almost surely exists a unique NE. Finally we present the behavior of average network performance at the NE through numerical results, and we compare the optimal centralized approach with our decentralized approach.

## I. INTRODUCTION

Consider a scenario that multiple transmitters simultaneously sending information to their receivers through several independent channels or resources. In this paper we will use the example of small-cell OFDM networks [1]. In such a wireless network, we assume that multiple access points (APs) serve a small area (e.g., airports, restaurants, military bases, hotels, hospitals, libraries, supermarkets, etc.), simultaneously communicating to several mobile terminals (MTs) using OFDM over a number of dedicated sub-channels. An  $N$ -carrier OFDM system [2] using a cyclic prefix or zero-padding [3], [4] to prevent inter-block interference is equivalent in the frequency domain to  $N$  flat fading parallel transmission sub-channels. In this scenario, each AP faces a problem of how to distribute the total available power among these  $N$  downlink sub-channels (subcarriers or clusters of subcarriers), i.e., should it allocate its total power to a single sub-channel, spread the power over all the sub-channels, or choose some subset of sub-channels on which to transmit? Here, we shall emphasize that a similar power allocation problem can be considered for the uplink transmission where MTs are the transmitters who decide their transmit power strategies.

When this resource allocation problem is considered centrally to maximize the total achievable rate (corresponding to Shannon capacity [5] when single user detector is applied) for the small-cell OFDM network, it is an often studied optimization problem. Given other users' strategies, the problem of how to maximize a single user's sum-rate over all the sub-channels is a convex optimization problem [6], whose solution

is "waterfilling" [7], [8], [9]. The multi-user version of this problem is a non-convex optimization problem that may have multiple local optimal points [10], [11], [12]. To solve the multi-user problem in a centralized way, it requires a scheduler to allocate the resources, which is similar to the multi-user waterfilling problem [13] in multiple access channel (MAC). This approach requires a central computing resource with perfect knowledge of the channel state information (CSIT), involving feedback and overhead communication whose load scales linearly with the number of transmitters and receivers in the network. To reduce the feedback load, selective multi-user diversity algorithms have been introduced in [14]. In contrast to centralized networks, we study in this paper distributed communication solutions that require no centralized control, reducing the overhead as well as the need for a knowledgeable powerful scheduler to allocate network resources.

Now if each AP independently allocates the total power to maximize its own total achievable rate, considering all other users' transmissions as noise, this problem can be studied as a distributed non-cooperative game [15] where the selfish players are the APs who "play" the game by choosing their transmit power levels across sub-channels. Note that a selfish player may not act as a good neighbor to each other [16], and in fact he does not care about any other player's performance neither the whole network performance at all. Nash equilibria [17], which we will formally define in Section III but which are, informally, the best mutual strategies a rational player can play assuming that other players are also rational (using their own best strategies given their information), are important to study in such a distributed non-cooperative game because they represent the natural outcomes in competitive games. It is worth to mention that two special cases of this game have been studied in [18] and [10]. In [18], the existence and uniqueness of NE were established for the two-player case. In [10], the so called *symmetric waterfilling game* was studied (they assume, for a set of subcarriers and receivers, the channel gains from all transmitters are the same). They show that there exist an infinite number (a continuum) of NE in the game. However, up to now it is still not clear how to characterize the equilibrium set in the case of general independent fading channels with multiple players. The goal of this paper is therefore to address this fundamental problem.

The paper is organized in the following form: the problem formulation is introduced in section II. In section III, we study

the existence of NE and we characterize the NE set. We shown in section IV the optimal centralized approach. Finally, some numerical results are provided in section V followed by conclusions in section VI.

## II. SYSTEM MODEL

### A. Communication model

Consider a parallel Gaussian channels (in particular, OFDM system) with  $M$  APs simultaneously sending information to  $N$  MTs over  $N$  sub-channels. Assume that each sub-channel is pre-assigned to a different MT by a scheduler, therefore, each MT detects the signals only on the assigned sub-channel. We also assume that the channels have block fading so that the channel fading coefficients are constant during the period of each transmission block. Within a given transmission block, let  $\mathbf{G} \in \mathbb{R}_{++}^{M \times N}$  (positive real  $M \times N$  matrix) be the channel gain matrix whose  $(m, n)$  entry is  $g_{m,n}$ , the magnitude-squared of the fading channel gain on the downlink channel from AP  $m$  to MT  $n$ . We assume that  $\mathbf{G}$  is a random  $M \times N$  matrix with *i.i.d.* continuous entries (meaning that each entry  $g_{m,n}$  is independent, identically and continuously distributed) due to independent fading channels. Assuming that signals from other APs are treated as noise, the signal to interference plus noise ratio (SINR) of the signal from AP  $m$  to MT  $n$  is

$$\gamma_{m,n} = \frac{g_{m,n} p_{m,n}}{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}} \quad (1)$$

where  $p_{m,n} \geq 0$  represents the power transmitted by the  $m^{\text{th}}$  AP on the  $n^{\text{th}}$  sub-channel. For the sake of simplicity, we assume that the variance of white Gaussian noise  $\sigma^2$  is the same for each sub-channel  $n$ . The maximum achievable sum-rate for AP  $m$  is given by [7]

$$R_m = \sum_{n=1}^N \log(1 + \gamma_{m,n}), \quad \forall m \quad (2)$$

Each AP  $m$  has the power constraint

$$\sum_{n=1}^N p_{m,n} \leq \bar{P}_m, \quad \forall m \quad (3)$$

for  $\bar{P}_m > 0, \forall m$ .

### B. Game model

Here, we consider the previously mentioned model as a non-cooperative strategic game. In this game, the goal of each AP (player)  $m$  is to choose its own power vector  $\mathbf{p}_m = [p_{m,1} \dots p_{m,N}]^T$  (subject to its total power constraint (3)) to maximize the sum-rate  $R_m$ . Let the long power vector  $\mathbf{p} = [\mathbf{p}_1^T, \dots, \mathbf{p}_M^T]^T$  denote the outcome of the game in terms of transmission power levels of all the  $M$  players on  $N$  sub-channels. We can completely describe this non-cooperative OFDM game as

$$\mathcal{G} \triangleq [ \mathcal{M}, \{ \mathcal{P}_m \}_{m \in \mathcal{M}}, \{ u_m \}_{m \in \mathcal{M}} ] \quad (4)$$

where the elements of the game are

- The player set:  $\mathcal{M} = \{1, \dots, M\}$ ;

- The strategy set:  $\{ \mathcal{P}_1, \dots, \mathcal{P}_M \}$ , where the strategy of player  $m$  is

$$\mathcal{P}_m = \left\{ \mathbf{p}_m : p_{m,n} \geq 0, \forall n, \sum_{n=1}^N p_{m,n} \leq \bar{P}_m \right\}; \quad (5)$$

- The utility or payoff function set:  $\{u_1, \dots, u_M\}$ , with  $u_m(\mathbf{p}_m, \mathbf{p}_{-m}) = R_m$  expressed in (2), where  $\mathbf{p}_{-m}$  denotes the power vector of length  $(M-1)N$  consisting of elements of  $\mathbf{p}$  other than the  $m^{\text{th}}$  element, i.e.,  $\mathbf{p}_{-m} = [\mathbf{p}_1^T, \dots, \mathbf{p}_{m-1}^T, \mathbf{p}_{m+1}^T, \dots, \mathbf{p}_M^T]^T$ .

## III. NASH EQUILIBRIUM

In such a non-cooperative setting, each player  $m$  acts selfishly, aiming to maximize its own payoff, given other players' strategies and regardless of the impact of its strategy may have on other players and thus on the overall performance. The process of such selfish behaviors usually results in *Nash equilibrium*, which represents a common solution concept for non-cooperative game theoretic problems [17].

*Definition 3.1:* A power vector  $\mathbf{p}^*$  is said to be a *Nash equilibrium* if for every  $m \in \mathcal{M}$ ,

$$u_m(\mathbf{p}_m^*, \mathbf{p}_{-m}^*) \geq u_m(\mathbf{p}_m, \mathbf{p}_{-m}^*) \quad (6)$$

for all  $\mathbf{p}_m \in \mathcal{P}_m$ .

From above, it is clear that a NE simply represents a particular "steady" state for a system, in the sense that, once reached, no player has any motivation to unilaterally deviate from it. In many cases, Nash equilibria represent the result of learning and evolution of all players in a system. Therefore, it becomes fundamentally important to predict and characterize such point(s) from the system design perspective of wireless networks. In the rest context of this paper, we will focus on charactering such point(s) in the power strategy space. The following questions will be addressed one by one: 1. Does NE exist in our game? 2. Does there exist a unique or multiple NE points? 3. How does the network behave at NE?

It is known that in general NE point does not necessary exist. Therefore, our first interest turns to investigate the existence and uniqueness of NE in our game.

*Lemma 3.2:* A Nash equilibrium exists in OFDM game  $\mathcal{G}$ .

*Proof:* Since  $\mathcal{P}_m$  is convex, closed, and bounded for each  $m$ ;  $u_m(\mathbf{p}_m, \mathbf{p}_{-m})$  is continuous in both  $\mathbf{p}_m$  and  $\mathbf{p}_{-m}$ ; and  $u_m(\mathbf{p}_m, \mathbf{p}_{-m})$  is concave in  $\mathbf{p}_m$  for any set  $\mathbf{p}_{-m}$ , at least one Nash equilibrium point exists for  $\mathcal{G}$  [19], [20]. ■

Once existence is established, it is natural to consider the characterization of the equilibrium set. Ideally, we would prefer there to be a unique equilibrium, but unfortunately this is not true for many game problems [20]. A specific case of our game  $\mathcal{G}$ , namely *symmetric waterfilling game* is studied in [10], where the authors show infinite number of NE existing under the assumption of equal crosstalk channel coefficients. This implies that in general our game  $\mathcal{G}$  does not have unique equilibrium point. Nevertheless, we will show in the following

context that there exists a single NE almost surely for random channel gain matrix  $\mathbf{G}$  with *i.i.d.* continuous entries.

For any player  $m$ , given all other players' strategies  $\mathbf{p}_{-m}$ , the *best-response* power strategy  $\mathbf{p}_m$  can be found by solving the following optimization problem,

$$\begin{aligned} \max_{\mathbf{p}_m} \quad & u_m(\mathbf{p}_m, \mathbf{p}_{-m}) \\ \text{s.t.} \quad & \sum_n p_{m,n} \leq \bar{P}_m \\ & p_{m,n} \geq 0 \end{aligned} \quad (7)$$

which is a convex optimization, since the objective function is concave in  $\mathbf{p}_m$  and the constraint set is convex. Therefore, the Karush-Kuhn-Tucker (KKT) condition of the optimization is sufficient and necessary for the optimality [6]. To derive the KKT conditions, we form the Lagrangian for each player  $m$ ,

$$\begin{aligned} \mathcal{L}_m(\mathbf{p}, \lambda) = & \sum_{n=1}^N \log \left( 1 + \frac{g_{m,n} p_{m,n}}{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}} \right) \\ & - \lambda_m \left( \sum_{n=1}^N p_{m,n} - \bar{P}_m \right) + \sum_{n=1}^N \nu_{m,n} p_{m,n}. \end{aligned}$$

The corresponding KKT conditions are

$$\frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^M g_{j,n} p_{j,n}} - \lambda_m + \nu_{m,n} = 0, \quad \forall n \quad (8)$$

$$\lambda_m \left( \sum_{n=1}^N p_{m,n} - \bar{P}_m \right) = 0 \quad (9)$$

$$\nu_{m,n} p_{m,n} = 0, \quad \forall n, \quad (10)$$

where  $\lambda_m \geq 0$ ,  $\forall m$  and  $\nu_{m,n} \geq 0$ ,  $\forall m \forall n$  are dual variables associated with the constraints of limited power and positive power, respectively. The solution to this problem is well-known as the waterfilling algorithm [7], [8], [9]

$$p_{m,n} = \left( \frac{1}{\lambda_m} - \frac{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}}{g_{m,n}} \right)^+, \quad \forall n \quad (11)$$

where  $(x)^+ = \max\{0, x\}$  and  $\lambda_m$  satisfies

$$\sum_{n=1}^N \left( \frac{1}{\lambda_m} - \frac{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}}{g_{m,n}} \right)^+ = \bar{P}_m. \quad (12)$$

*Lemma 3.3:* The following conditions are sufficient and necessary for the Nash equilibrium in OFDM game  $\mathcal{G}$ .

$$\frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^M g_{j,n} p_{j,n}} - \lambda_m + \nu_{m,n} = 0, \quad \forall m \forall n \quad (13)$$

$$\lambda_m \left( \sum_{n=1}^N p_{m,n} - \bar{P}_m \right) = 0, \quad \forall m \quad (14)$$

$$\nu_{m,n} p_{m,n} = 0, \quad \forall m \forall n. \quad (15)$$

*Proof:* For a certain index  $m$ , the single player KKT conditions (8)-(10) are necessary and sufficient for the best response condition with index  $m$  in (6). Therefore, as a collection of single player KKT conditions from index 1 to  $M$ ,

(13)-(15) are necessary and sufficient for all the best response conditions in (6). ■

From (13), it is easy to observe  $\lambda_m > 0$ . Then from (14), we have

$$\sum_{n=1}^N p_{m,n} = \bar{P}_m, \quad \forall m \quad (16)$$

Note that (16) has an intuitive meaning: each player at NE must dedicate its total available power on all carriers, due to their "selfish instinct". However, it is still difficult to find the analytical solution for (13)-(15), since the solution form (11) and (12) construct a system of nonlinear equations. The idea to simplify this problem is therefore to consider linear equations instead of nonlinear ones, we then introduce another lemma, as follows

*Lemma 3.4:* For any realization of channel matrix  $\mathbf{G} \in \mathbb{R}_{++}^{M \times N}$ , there exist unique values of the Lagrange dual variables  $\lambda$  and  $\nu$  for any Nash equilibrium of the game  $\mathcal{G}$ . Furthermore, there is a unique vector  $\mathbf{s} = [s_1, \dots, s_N]^T$  such that any vector  $\mathbf{p}$  corresponding to a Nash equilibrium of the game satisfies

$$\sum_{m=1}^M g_{m,n} p_{m,n} \triangleq s_n, \quad \forall n. \quad (17)$$

The proof can be found in Appendix A.

Now, let  $\mathbf{Z}$  be the following  $(M+N)$  by  $MN$  matrix

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_M & \mathbf{I}_M & \cdots & \mathbf{I}_M \\ \mathbf{g}_1^T & \mathbf{0}_M^T & \cdots & \mathbf{0}_M^T \\ \mathbf{0}_M^T & \mathbf{g}_2^T & \cdots & \mathbf{0}_M^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_M^T & \mathbf{0}_M^T & \cdots & \mathbf{g}_N^T \end{bmatrix}_{(M+N) \times MN}$$

where  $\mathbf{g}_n$  is the  $n^{\text{th}}$  column of  $\mathbf{G}$ ,  $\mathbf{I}_M$  is the  $M$  by  $M$  identity matrix, and  $\mathbf{0}_M$  is zero vector of length  $M$ . Let  $\mathbf{c}$  be the following vector of length  $M+N$

$$\mathbf{c} = [\bar{P}_1 \ \bar{P}_2 \ \dots \ \bar{P}_M \ s_1 \ s_2 \ \dots \ s_N]^T$$

Then, (16) and (17) can be written in the form of linear matrix equation

$$\mathbf{Z}\mathbf{p} = \mathbf{c}. \quad (18)$$

Define the following sets:

$$\mathcal{X} \triangleq \{(m, n) : \nu_{m,n} = 0\},$$

$$\mathcal{N} \triangleq \{n : \exists m \text{ such that } (m, n) \in \mathcal{X}\}.$$

From (15), if an index  $(m, n) \notin \mathcal{X}$  we must have  $p_{m,n} = 0$ . Without loss of generality, we assume that  $\mathcal{N} = \{1, \dots, \tilde{N}\}$  for  $\tilde{N} \leq N$ . Let  $\tilde{\mathbf{Z}}$  be the  $M + \tilde{N}$  by  $M\tilde{N}$  matrix formed from the first  $M + \tilde{N}$  rows and first  $M\tilde{N}$  columns of  $\mathbf{Z}$ ,  $\tilde{\mathbf{p}}$  is formed from the first  $M\tilde{N}$  elements of  $\mathbf{p}$ , and  $\tilde{\mathbf{c}}$  is formed from the first  $M + \tilde{N}$  elements of  $\mathbf{c}$ , then any NE solution must satisfy

$$\tilde{\mathbf{Z}}\tilde{\mathbf{p}} = \tilde{\mathbf{c}}. \quad (19)$$

Let  $\hat{\mathbf{Z}}$  be the  $M + \tilde{N}$  by  $|\mathcal{X}|$  (the size of  $\mathcal{X}$ ) matrix formed from the columns of  $\tilde{\mathbf{Z}}$  that correspond to the elements of  $\mathcal{X}$ . Similarly, let  $\hat{\mathbf{p}}$  be the vector of length  $|\mathcal{X}|$  with the values of  $p_{m,n}$  with  $(m,n) \in \mathcal{X}$  (and in the same order as they were in  $\mathbf{p}$ ). Then any NE solution must satisfy

$$\tilde{\mathbf{Z}}\hat{\mathbf{p}} = \tilde{\mathbf{c}}. \quad (20)$$

*Lemma 3.5:* For any realization of random channel matrix  $\mathbf{G}$  with i.i.d. continuous entries, if  $M\tilde{N} > M + \tilde{N}$ ,  $|\mathcal{X}| \leq M + \tilde{N}$  with probability 1.

*Proof:* When  $\nu_{m,n} = 0$ , from (8) we have

$$\lambda_m - g_{m,n}d_n = 0, \quad \forall (m,n) \in \mathcal{X} \quad (21)$$

where  $d_n \triangleq \frac{1}{\sigma^2 + s_n}$ . From Lemma 3.4, we know that all the Nash equilibria must satisfy (21), with the same  $\lambda_m$  and  $d_n$ . In (21), the number of independent linear equations is  $|\mathcal{X}|$ , while the number of unknown parameters is  $M + \tilde{N}$  (since the rest of  $d_n$ ,  $n \notin \mathcal{X}$  is known to be  $d_n = \frac{1}{\sigma^2}$ ). It is well known that the solution to the system of linear equations is the empty set, if the number of independent equations is larger than the number of variables [21]. Since  $g_{m,n}$  is chosen randomly from a continuous distribution, it is obvious that, with probability 1, the equations in (21) are independent from each other, therefore, we must have  $|\mathcal{X}| \leq M + \tilde{N}$ . ■

*Lemma 3.6:*

- 1) If  $M\tilde{N} > M + \tilde{N}$  and  $|\mathcal{X}| \leq M + \tilde{N}$ ,  $\text{rank}(\hat{\mathbf{Z}}) = |\mathcal{X}|$  with probability 1.
- 2) If  $M\tilde{N} \leq M + \tilde{N}$ ,  $\text{rank}(\hat{\mathbf{Z}}) = M\tilde{N}$  with probability 1.

*Proof:* We only give the proof for case 1)  $M\tilde{N} > M + \tilde{N}$ , case 2)  $M\tilde{N} \leq M + \tilde{N}$  can be proved in a similar way. Matrix  $\hat{\mathbf{Z}}$  can be transformed into a  $2 \times 2$  block matrices, by applying some elementary column and row operations, as follows,

$$\hat{\mathbf{Z}} \xrightarrow{\text{column}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{A}_{\tau \times \xi_2} \\ \mathbf{B}_{\xi_1 \times \tau} & \mathbf{C}_{\xi_1 \times \xi_2} \end{bmatrix} \xrightarrow{\text{column}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{0}_{\tau \times \xi_2} \\ \mathbf{B}_{\xi_1 \times \tau} & \hat{\mathbf{C}}_{\xi_1 \times \xi_2} \end{bmatrix} \\ \xrightarrow{\text{row}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{0}_{\tau \times \xi_2} \\ \mathbf{0}_{\xi_1 \times \tau} & \hat{\mathbf{C}}_{\xi_1 \times \xi_2} \end{bmatrix}$$

where  $\tau = \min\{M, \tilde{N}\}$ ,  $\xi_1 = M + \tilde{N} - \tau \geq \tau$ ,  $\xi_2 = |\mathcal{X}| - \tau$ .  $\hat{\mathbf{C}}$  is a  $\xi_1 \times \xi_2$  matrix, where each column contains one or two random variables, and each row contains at least one random variable. Again we can transform  $\hat{\mathbf{C}}$  in row echelon form, denoted as  $\hat{\mathbf{C}}_r$ . Note that the rank of  $\hat{\mathbf{C}}_r$  is  $\xi_2$  with probability 1, since each leading coefficient of a row is a random variable or the linear combination of two continuously distributed random variables. So, with probability 0, any leading coefficient takes the value of 0. Therefore, we have  $\text{rank}(\hat{\mathbf{Z}}) = \tau + \xi_2 = |\mathcal{X}|$  with probability 1. ■

*Theorem 3.7:* For any realization of random channel matrix  $\mathbf{G}$  with i.i.d. continuous entries, the OFDM game  $\mathcal{G}$  has exactly one Nash equilibrium.

*Proof:* If  $M\tilde{N} > M + \tilde{N}$ , we have from Lemma 3.6 that, with probability 1,  $\text{rank}(\hat{\mathbf{Z}}) = |\mathcal{X}|$ . Any NE must satisfy (20); assume that two different power strategies  $\hat{\mathbf{p}}$  and  $\hat{\mathbf{p}}'$  are both

solutions to (20). Then  $\hat{\mathbf{Z}}(\hat{\mathbf{p}} - \hat{\mathbf{p}}') = \mathbf{0}$ . By the rank-nullity theorem [21], since the rank of  $\hat{\mathbf{Z}}$  is equal to the number of its columns, this implies  $\hat{\mathbf{p}} - \hat{\mathbf{p}}' = \mathbf{0}$ , which means there must be exactly one NE.

If  $M\tilde{N} \leq M + \tilde{N}$ , we have from Lemma 3.6 that, with probability 1, there is at most one solution to (19). Since any NE must satisfy (19) and we know that there is at least one NE solution, so the NE must be unique. ■

#### IV. PARETO OPTIMALITY

To measure the inefficiency of Nash equilibrium, we consider in this section the Pareto boundary for the total network rate. The total network rate maximization problem can be formulated as

$$\begin{aligned} \max_{\mathbf{p}_m} \quad & \sum_{m=1}^M u_m(\mathbf{p}) \\ \text{s.t.} \quad & \sum_m p_{m,n} \leq \bar{P}_m, \quad m = 1, \dots, M \\ & p_{m,n} \geq 0 \end{aligned} \quad (22)$$

which unfortunately is a difficult problem, since the objective function is non-convex in  $\mathbf{p}$ . However, a relaxation of this optimization problem (see in [12]) can be considered as a geometric programming problem [22], therefore, can be transformed into a convex optimization problem. A low complexity algorithm was proposed in [12] to solve the dual problem by updating dual variables through a gradient descent. Note that this approach enable us to find a tight *lower-bound* for the Pareto boundary that is sufficient for the performance measurement presented in the next section.

#### V. NUMERICAL EVALUATION

In this part, numerical results are provided to demonstrate the network performance at the unique Nash equilibrium (outcome of decentralized networks). To be precise, we are interested in comparing the “average total network rate” (average over the distribution of fading channel gains, and we will use the short term “total network rate” to represent it) instead of the instantaneous one. We denote by  $\bar{u}(M, N)$  this total network rate,

$$\bar{u}(M, N) = \mathbb{E}_{\mathbf{G}} \left[ \sum_{m=1}^M \sum_{n=1}^N \log \left( 1 + \frac{p_{m,n} g_{m,n}}{\sigma^2 + \sum_{j \neq m} p_{j,n} g_{j,n}} \right) \right]$$

As a basis for comparison, the Pareto-maximum total network rate will also be provided and considered as an upper bound for the decentralized settings. Parameters are selected as: the number of transmitters is  $M \in [1, 25]$ ; the number of receivers  $N$  takes several representative values as 5, 10 and 15; the power constraint of each AP is  $\bar{P}_m = 1$ ,  $\forall m$ ; the variance of additive white Gaussian noise is set to  $\sigma^2 = 0.1$  and 1, respectively.

Fig. 1 and Fig. 2 show the total network rate for  $\sigma^2 = 0.1$  and 1, respectively. As expected, the curves of centralized

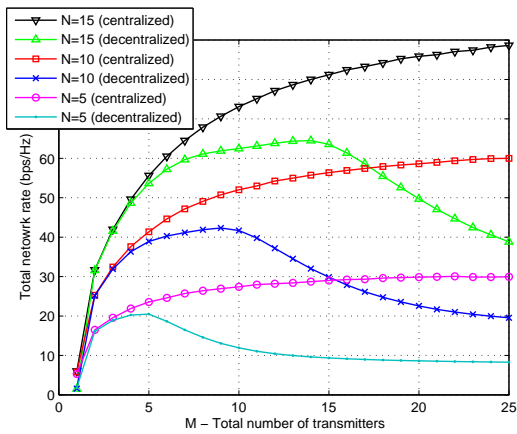


Fig. 1. Total network rate, centralized vs. decentralized,  $\sigma^2 = 0.1$

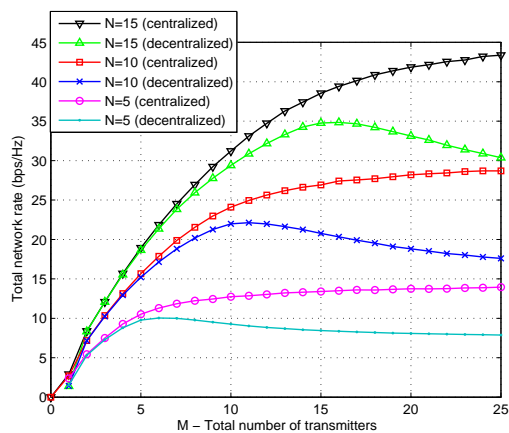


Fig. 2. Total network rate, centralized vs. decentralized,  $\sigma^2 = 1$

networks always outperform the decentralized ones. More precisely, for a fixed number of receivers  $N$ , as the number of transmitters  $M$  increases, the performance loss of decentralized networks (compared to centralized networks) becomes more and more apparent. This phenomenon can be easily understood as: *when there are a great number of selfish players, the hostile competition turns the multi-user communication system into an interference-limited environment, where interference begins to dominate the performance efficiency.*

In Fig. 1 and Fig. 2, we also find that the total network rate of centralized network is an increasing function of  $M$  (for a fixed value of  $N$ ), and the total network rates of decentralized networks corresponding to Nash equilibrium show an increasing slope before diminishing and reaching convergence. For some typical values of  $N$ , i.e.,  $N = 5, 10$  and  $15$ , in Fig. 1, when  $\sigma^2 = 0.1$  the total network rate of decentralized networks are maximized approximately at  $M = 4, 9, 14$ , respectively; in Fig. 2, when  $\sigma^2 = 1$  the total network rate of decentralized networks are maximized

approximately at  $M = 6, 11, 16$ , respectively. It simply shows that different noise variance (in general channel condition) has a different impact on the decentralized network performance. This observation is fundamentally important for improving the energy efficiency in such a multi-user decentralized network: *for a given area (given the number of receivers  $N$  and the current channel condition), there exists an optimal choice for the number of transmitters (denoted as  $M^*$ ) to be put in the network.* Roughly saying: when  $M > M^*$ , the network is *overloaded* due to the increase of competition over limited spectrum resources; when  $M < M^*$ , the network is operated at an *unsaturated* state, since the spectrum resources are not fully exploited.

## VI. CONCLUSIONS AND FUTURE WORKS

In this paper we described the scenario of multiple transmitters communicating with multiple receivers through independent parallel sub-channels as a strategic non-cooperative game. Each transmitter is modeled as a player in the game who decides, in a distributed way, the strategy of how to allocate its total power through these sub-channels. We studied the existence and uniqueness of Nash equilibrium which represents a natural outcome of the game. For any realization of a random channel matrix with *i.i.d.* continuous entries, we proved that there exists almost surely a unique Nash equilibrium. This result is a fundamental step to understand the resource allocation conflicts in a decentralized small-cell multi-user OFDM network, and moreover, it offers the possibility to predict the network performance outcomes. Finally, in our simulations results, we show how the average performance of the decentralized network behaves, and we compared it with the centralized network. Future works shall focus on the scaling analysis of the decentralized network performance.

## APPENDIX

### A. Proof of Lemma 3.4

*Proof:* Consider a Nash equilibrium  $\mathbf{p} \in \mathbb{R}_+^{MN \times 1}$  (non-negative real vector of length  $MN$ ), from Lemma 3.3, we have the following equation

$$\phi(\mathbf{p}) + \nu - \lambda = \mathbf{0}$$

where

$$\phi(\mathbf{p}) = \begin{bmatrix} \frac{g_{1,1}}{\sigma^2 + \sum_j p_{j,1} g_{j,1}} \\ \frac{g_{1,2}}{\sigma^2 + \sum_j p_{j,1} g_{j,1}} \\ \vdots \\ \frac{g_{M,N}}{\sigma^2 + \sum_j p_{j,N} g_{j,N}} \end{bmatrix} \quad \nu = \begin{bmatrix} \nu_{1,1} \\ \nu_{1,2} \\ \vdots \\ \nu_{M,N} \end{bmatrix} \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_M \end{bmatrix}$$

Now, assume that there exist two different Nash equilibria, e.g.,  $\mathbf{p}^0, \mathbf{p}^1$  ( $\mathbf{p}^0 \neq \mathbf{p}^1$ ), the following equation must also hold

$$\underbrace{[(\mathbf{p}^1 - \mathbf{p}^0)^T (\mathbf{p}^0 - \mathbf{p}^1)^T]}_{\alpha^T} \left( \underbrace{\begin{bmatrix} \phi(\mathbf{p}^0) \\ \phi(\mathbf{p}^1) \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} \nu^0 - \lambda^0 \\ \nu^1 - \lambda^1 \end{bmatrix}}_{\gamma} \right) = 0 \quad (23)$$

from where we have

$$\begin{aligned}
\alpha^T \beta &= (\mathbf{p}^1 - \mathbf{p}^0)^T \phi(\mathbf{p}^0) + (\mathbf{p}^0 - \mathbf{p}^1)^T \phi(\mathbf{p}^1) \\
&= \sum_{n=1}^N \sum_{m=1}^M \left[ (p_{m,n}^1 - p_{m,n}^0) \frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^M p_{j,n}^0 g_{j,n}} \right] \\
&\quad + \sum_{n=1}^N \sum_{m=1}^M \left[ (p_{m,n}^0 - p_{m,n}^1) \frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^M p_{j,n}^1 g_{j,n}} \right] \\
&= \sum_{n=1}^N \sum_{m=1}^M \frac{g_{m,n} (p_{m,n}^0 - p_{m,n}^1) \left[ \sum_{j=1}^M g_{j,n} (p_{j,n}^0 - p_{j,n}^1) \right]}{\left( \sigma^2 + \sum_{j=1}^M p_{j,n}^0 g_{j,n} \right) \left( \sigma^2 + \sum_{j=1}^M p_{j,n}^1 g_{j,n} \right)} \\
&= \sum_{n=1}^N \frac{\left[ \sum_{j=1}^M g_{j,n} (p_{j,n}^0 - p_{j,n}^1) \right]^2}{\left( \sigma^2 + \sum_{j=1}^M p_{j,n}^0 g_{j,n} \right) \left( \sigma^2 + \sum_{j=1}^M p_{j,n}^1 g_{j,n} \right)} \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
\alpha^T \gamma &= (\mathbf{p}^1 - \mathbf{p}^0)^T (\nu^0 - \lambda^0) + (\mathbf{p}^0 - \mathbf{p}^1)^T (\nu^1 - \lambda^1) \\
&= \sum_{n=1}^N \sum_{m=1}^M [(p_{m,n}^1 - p_{m,n}^0)(\nu_{m,n}^0 - \lambda_m^0)] \\
&\quad + \sum_{n=1}^N \sum_{m=1}^M [(p_{m,n}^0 - p_{m,n}^1)(\nu_{m,n}^1 - \lambda_m^1)] \\
&= \sum_{m=1}^M \left[ \underbrace{\left( \sum_{n=1}^N p_{m,n}^1 - \sum_{n=1}^N p_{m,n}^0 \right)}_{\bar{P}_m - \bar{P}_m = 0} (\lambda_m^1 - \lambda_m^0) \right] \\
&\quad + \sum_{n=1}^N \sum_{m=1}^M (p_{m,n}^0 \nu_{m,n}^1 + p_{m,n}^1 \nu_{m,n}^0) \\
&= \sum_{n=1}^N \sum_{m=1}^M (p_{m,n}^0 \nu_{m,n}^1 + p_{m,n}^1 \nu_{m,n}^0) \\
&\geq 0
\end{aligned}$$

From above, we find that (23) holds if and only if the two equalities are satisfied:  $\alpha^T \beta = 0$  and  $\alpha^T \gamma = 0$ , which are equivalent to the following two equations, respectively:

$$\sum_{m=1}^M g_{m,n} p_{m,n}^0 - \sum_{m=1}^M g_{m,n} p_{m,n}^1 = 0, \quad \forall n \quad (24)$$

$$p_{m,n}^0 \nu_{m,n}^1 = p_{m,n}^1 \nu_{m,n}^0 = 0, \quad \forall m \quad \forall n \quad (25)$$

First, from equation (24) one can easily find that the value of  $s_n (= \sum_m g_{m,n} p_{m,n})$  is unique for any Nash equilibrium point. Second, from equation (16) we know that for any player  $m$  there must be a (positive) power allocated on a certain sub-channel  $n'$ , e.g.,  $p_{m,n'}^0 > 0$ , and we have two important observations:

- 1) Using (15), we have  $p_{m,n'}^0 \nu_{m,n'}^0 = 0 \Rightarrow \nu_{m,n'}^0 = 0$ .
- 2) Using (25), we have  $p_{m,n'}^0 \nu_{m,n'}^1 = 0 \Rightarrow \nu_{m,n'}^1 = 0$ .

Then (13) yields

$$\lambda_m^0 = \lambda_m^1 = -\frac{g_{m,n'}}{\sigma^2 + s_{n'}}, \quad \forall m \quad (26)$$

which shows that the dual variable  $\lambda$  is unique for any Nash equilibrium. Furthermore, by using (26) into (13), we derive

$$\nu_{m,n}^0 = \lambda_m^0 - \frac{g_{m,n}}{\sigma^2 + s_n} = \lambda_m^1 - \frac{g_{m,n}}{\sigma^2 + s_n} = \nu_{m,n}^1, \quad \forall m \quad \forall n$$

which confirms that the dual variable  $\nu$  is also unique for any Nash equilibrium.

Finally, we conclude that both Lagrange dual variable  $\lambda$  and  $\nu$  are unique for any Nash equilibrium in our game  $\mathcal{G}$ . This completes the proof. ■

## REFERENCES

- [1] M. Debbah, "Mobile Flexible Networks: the route towards autonomic communications," submitted to *IEEE Trans. on Vehicular Technology*, 2009.
- [2] J. Bingham, "Multicarrier Modulation for Data Transmission: An Idea Whose Time Has Come," *IEEE Communications Magazine*, vol. 28, no. 5, pp. 514, May 1990.
- [3] B. Muquet, "Novel Receiver and Decoding Schemes for Wireless OFDM Systems with Cyclic Prefix or Zero-padding," PhD dissertation, École Nationale Supérieure des Télécommunications, June 2001.
- [4] B. Muquet, Z. Wang, G. B. Giannakis, M. de Courville, P. Duhamel, "Cyclic-Prefixing or Zero-Padding for Wireless Multicarrier Transmissions?," *IEEE Trans. on Communications*, vol. 50, pp. 2136-2148, Dec. 2002.
- [5] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379-423 and 623-656, July and October, 1948.
- [6] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
- [7] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, 1991.
- [8] A.J. Goldsmith and P.P. Varaiya, "Capacity of Fading Channels with Channel Side Information," *IEEE Trans. on Information Theory*, vol. 43, pp. 1986-1992, 1997.
- [9] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.
- [10] E. Altman, K. Avrachenkov and A. Garnaev, "Closed Form Solutions for Symmetric Water Filling Games," *IEEE Conference on Computer Communications (INFOCOM)*, pp. 673-681, Apr. 2008.
- [11] K.B. Song, S.T. Chung, G. Ginis and J.M. Cioffi, "Dynamic Spectrum Management for Next-generation DSL Systems," *IEEE Communications Magazine*, vol. 40, pp. 101-109, 2002.
- [12] J. Papandriopoulos and J. S. Evans, "Low-Complexity Distributed Algorithms for Spectrum Balancing in Multi-User DSL Networks," *IEEE International Conference on Communications (ICC)*, vol. 7, pp. 3270-3275, June 2006.
- [13] W. Yu, *Competition and cooperation in multi-user communication environments*, PhD Thesis, Stanford University, June 2002.
- [14] D. Gesbert and M. S. Alouini, "How Much Feedback is Multi-User Diversity Really Worth?" *IEEE International Conference on Communications (ICC)*, Paris, France, vol. 1, pp. 234-238, June 2004.
- [15] N. John, "Non-cooperative Games", *The Annals of Mathematics*, vol. 54, no. 2, pp. 286-295, 1951.
- [16] O. Popescu and C. Rose, "Water Filling May Not Good Neighbors Make," *IEEE Global Telecommunications Conference (GLOBECOM)*, vol. 3, pp. 1766-1770, 2003.
- [17] N. John, "Equilibrium Points in N-person Games" *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48-49, 1950.
- [18] W. Yu, G. Ginis and J.M. Cioffi, "Distributed Multiuser Power Control for Digital Subscriber Lines," *IEEE Journal on Selected Areas in Communications*, vol. 20, pp.1105-1115, 2002.
- [19] J. B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," *Econometrica*, vol. 33, pp. 520-534, 1965.
- [20] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1991.
- [21] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Philadelphia, Society for Industrial and Applied Mathematics (SIAM), 2000.
- [22] M. Chiang, "Geometric Programming for Communication Systems," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 1/2, pp. 1-153, 2005.