A Game-Theoretic Approach to Decentralized Optimal Power Allocation for Cellular Networks

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ABSTRACT

The rapidly growing demand for wireless communication makes efficient power allocation a critical factor in the network's efficient operation. Power allocation in cellular networks with interference, where users are selfish, has been recently studied by pricing methods. However, pricing methods do not result in efficient/optimal power allocations for such systems for the following reason. Because of interference, the communication between the BS and a given user is affected by that between the BS and all other users. Thus, the vector consisting of the transmission power in each BS-user link can be viewed as a public good which simultaneously affects the utilities of all the users in the network. It is well known [11, Chapter 11.C] that in public good economies, standard efficiency theorems on market equilibrium do not apply and pricing mechanisms do not result in globally optimal allocations. In this paper we study power allocation in the presence of interference for a single cell wireless CDMA network from a game theoretic perspective. We consider a network where each user knows only its own utility and the channel gain from the base station to itself. We formulate the downlink power allocation problem as a public good allocation problem. We present a game form the Nash Equilibria of which yield power allocations that are optimal solutions of the corresponding centralized downlink network.

1. INTRODUCTION

1.1 Overview and literature survey

With rapidly growing demand for wireless communication the need for efficient use of spectrum has drawn significant attention of researchers. One of the factors that governs the efficiency of spectrum usage is power and interference control. The growth in the size of wireless networks also makes it desirable to use decentralized mechanisms for power control because centrally operated mechanisms involve added infrastructure. However, the increasing intelligence of end-user/intermediate network devices which are owned by selfish users, puts decentralized mechanisms at risks of failure against strategic behavior of users. Therefore it is desirable to develop decentralized mechanisms for power allocation which are robust against the strategies of selfish users.

Decentralized mechanisms for power allocation/control in cellular networks that study game-theoretic/strategic behavior issues have received considerable attention in the literature. One of the earliest works which introduced an individual utility maximization formulation for uplink power control in a single cell Code Division Multiple Access (CDMA) data network can be found in [4]. An uplink problem similar to that of [4] in which users' utilities are taken to be functions of their respective Signal to Interference Ratio (SIR) was investigated in [8]; in this paper the existence of an equilibrium was shown and a decentralized algorithm for solving the power control problem was suggested. The problem formulated in [4] was reinvestigated in [15] using pricing; it was shown that pricing results in multiple equilibria which are Pareto superior to the equilibria obtained in [4] and [8]. Pricing-based analysis of the uplink power control problem was also done in [1]; in [1] the authors introduced user specific parametric utility functions and proposed two decentralized algorithms, the parallel update and the random update algorithms, that converge to the unique equilibrium of the problem. In [14] pricing-based ideas for uplink power control were extended to multi-cell data networks. The authors of [6] studied uplink power allocation under an Interference Temperature Constraint (ITC); the authors proposed a power auction run by a manager that achieves a power allocation arbitrarily close to the globally optimal one. The conditions under which the power auction achieves an optimal solution however requires in essence, that the manager should know the users' utility functions.

Game theoretic study of downlink CDMA data networks can be found in [9, 10, 21]. In [10] and [21], optimal power allocation strategies were determined for a single class CDMA system under the assumption that the utility functions of the users are common knowledge (see [2, 20] for the definition of common knowledge). The authors of [9] studied a downlink power allocation problem for multi-class CDMA networks; the authors proposed a decentralized mechanism based on dynamic pricing and partial cooperation between the mobiles and the base station that achieves a partial-cooperative optimal power allocation which was shown to be close to a globally-optimal power allocation. In [16] the authors presented a decentralized mechanism for power allocation that works for both uplink and downlink networks,

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and also takes into account multiple ITCs. The mechanism obtains an optimal power allocation under the assumption that the users are cooperative.

In this paper we consider a single cell wireless CDMA data network. We study power allocation for the downlink communication in a given carrier frequency where the communication between the Base Station (BS) and a user generates interference to that between the BS and other users. We consider a decentralized network where users are selfish, and where each user's utility and the channel gain between the BS and the user is that user's private information. For the downlink network, we consider a centralized [1] power allocation problem that corresponds to the decentralized problem under consideration. The objective is to develop a game form/decentralized mechanism for determining power allocations such that the allocations obtained at the Nash equilibria of the game form are optimal solutions of the corresponding centralized problem. Below we explain the motivation for considering the above problem.

1.2 Motivation

A network resource is said to be a public good if: (i) the presence of the resource simultaneously affects the utilities of all network users without getting divided among them; and (ii) each user obtains a different individual utility from the consumption of the resource. For single cell wireless networks, power allocation problems in the presence of interference can be treated as problems of a public good allocation where the public good for the downlink network is the power vector transmitted by the BS to all the users, and that for the uplink network is the power vector received by the BS from all the users. Power allocation problems in cellular wireless networks with interference have been previously considered in the literature cited in Section 1.1. The solution approach in all the references [1,4,6,8–10,14,15,21] is based on different variations of pricing mechanisms where each user pays some money for the power allocated to it.

In general, in decentralized resource allocation problems involving a public good, pricing mechanisms that fix a common price for the public good for all the users, fail to obtain globally optimal allocations. The reason is that in a public good economy the *same* good is simultaneously consumed by users having different valuations of the good; thus, individual valuations of the public good are different from the system's valuation and this results in inefficiency. This explains why the pricing mechanisms employed in [4,6,8,15] do not achieve globally optimal allocations and why the mechanism proposed in [1] does not achieve optimal allocations unless the users vary their utilities according to their target SIRs.

The pricing mechanism proposed in [9] is different from the above references in that it obtains close to globally optimal allocations. The reason for this is the following. The authors of [9] introduce a constraint on the total power transmitted by the BS. Due to this constraint the original problem, where each user's utility depends on the entire power vector transmitted by the BS, reduces to one where each user's utility depends only on the power transmitted to it. Thus the problem changes from a public good allocation problem (when explicit interference is present) to a private good allocation problem. This is why the pricing mechanism proposed in [9] results in efficient allocations. In systems where there is no constraint on the maximum sum power, the above-stated reduction is not possible and therefore, pricing mechanisms do not yield optimal allocations. The failure of pricing mechanisms to produce globally optimal power allocations for wireless networks affected by interference, provides the key motivation for the formulation and solution methodology presented in this paper.

In [16] an optimal power allocation mechanism that is not based on pricing was proposed. However, the network studied in [16] assumes cooperative users. The results of [16] motivated us to explore optimal power allocation mechanisms for networks in a non-cooperative setup; this setup is adopted in this paper.

1.3 Contributions of the paper

The key contributions of this paper are: (i) The formulation of downlink power allocation problem with interference as a public good allocation problem; (ii) The development of a game form/decentralized mechanism (based on public good formulation) whose Nash equilibria result in the power allocation that is the optimal solution of the corresponding centralized downlink problem. Our approach to the power allocation problem is game-theoretic. As explained below, our proposed mechanism is distinctly different from the pricing mechanisms studied in the aforementioned literature. Our formulation properly captures the valuation of interference by each individual user as well as the system and hence, the proposed mechanism leads to globally optimal power allocations.

The rest of the paper is organized as follows: In Section 2 we present the downlink model and formulate an equivalent centralized power allocation problem. In Section 2.2 we model the power llcoation problem in the framework of implementation theory. In Section 3.1 we present a game form that obtains the solution of the centralized power allocation problem at its Nash equilibria. The proofs of the theorems that assert the above properties of the game form are presented in the appendix. We conclude in Section 4.

Before we present the model in Section 2, we describe here the notations that we will use throughout the paper.

<u>*Notation:*</u>

We represent vectors by bold letters and scalars by normal letters. Vector elements are represented by subscripting the vector symbol. A bold subscripted-symbol means that the vector-element is also a vector e.g. in $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N})$, each $\boldsymbol{x_i}$, $i = 1, 2, \ldots, N$, is a vector; in $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$, each x_i, $i = 1, 2, \ldots, N$, is a scalar. Unless otherwise stated, all vectors are treated as column vectors. Bold $\mathbf{0}$ is treated as a zero vector of appropriate size determined by the context. The notation $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is used to represent the following: $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is a vector of dimension same as that of $\boldsymbol{x}^*$; the ith element of $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is x_i (or $\boldsymbol{x_i}$), all other elements of it are the same as the corresponding elements of $\boldsymbol{x}^*$. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\mathrm{diag}(\boldsymbol{x})$.

2. THE MODEL (M1)

We consider a single cell CDMA wireless data network consisting of a Base Station (BS) and multiple mobile users. For simplicity of presentation we focus on the downlink trans-

9[1] A centralized problem is the one in which the BS has complete knowledge of the network.

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mission from the BS to the mobiles [2] as shown in Fig. 1. We

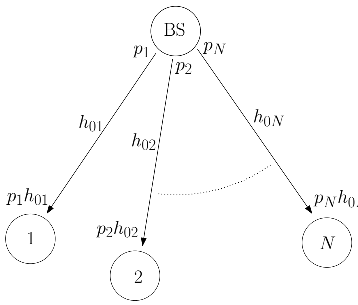

Figure 1: A downlink network with N mobile users and one base station

assume that there are N mobile users[3] in the network; we denote the set of users by $\mathcal{N} := \{1, 2, \ldots, N\}$. We consider the BS transmissions in a given carrier frequency; we assume that the signature codes used by the BS are not completely orthogonal,[4] hence each user experiences some interference due to the BS transmissions meant for other users. Due to interference the Quality of Service (QoS) received by a given user $i, i \in \mathcal{N}$, depends not only on the power p_i of the BS's transmission meant for user i but also, on power $p_j, j \neq i$, of the BS's transmissions meant for all other users $j \in \mathcal{N}\backslash\{i\}$. More specifically, due to the path loss from the BS to the mobiles, user i's QoS depends on its received powers $p_j h_{0i}, j \in \mathcal{N}$, where h_{0i} is the channel gain from the BS to user i. We assume that the BS periodically transmits pilot signals to the users so that each user $i \in \mathcal{N}$ can measure its respective channel gain h_{0i}. We also assume that the maximum power the BS can transmit to any given user is P_0^{max} and that this upper limit P_0^{max} is common knowledge in the network.

Every user pays some money to the BS which we call the *tax*, for using the wireless network. We quantify user i's, $i \in \mathcal{N}$, aggregate satisfaction from the tax t_i it pays and the QoS it obtains from the *power profile* $\boldsymbol{p} := (p_1, p_2, \ldots, p_N)$ transmitted by the base station, by a *utility function* $u_i^A : \mathbb{R}^{1+N} \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as,

$$u_i^A(t_i, \boldsymbol{p}) := -t_i + u_i(\boldsymbol{p}) - \left[\frac{1 - I_{S_i}(\boldsymbol{p})}{I_{S_i}(\boldsymbol{p})}\right] \tag{1}$$

where,

$$S_i := \{\boldsymbol{p} \mid \boldsymbol{p} \in [0, P_0^{max}]^N\} \tag{2}$$

$$I_{S_i}(\boldsymbol{p}) = \begin{cases} 1, & \text{if } \boldsymbol{p} \in S_i \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The transmission power for each user and the tax that each user must pay are determined by the BS. The tax t_i can be either positive or negative depending on whether user i pays money to the BS or receives money from the BS. Sometimes in order to compensate for the users who are in need of high QoS, the BS pays a user to accept a lower QoS and this results in a negative tax. The BS does not make any profit from the tax; it charges the tax in such a way that the users' money is just redistributed among them, i.e.,

$$\sum_{i=1}^{N} t_i = 0 \tag{4}$$

The last term in (1) signifies that an allocation $(t_i, \boldsymbol{p})$ is of no use to user i if $\boldsymbol{p} \notin S_i$, since such a power profile can never be transmitted by the BS. For $\boldsymbol{p} \in S_i$ the function $u_i : \mathbb{R}^N \rightarrow \mathbb{R}$ quantifies the influence of the BS's transmission on user i's QoS. The functional form of u_i depends on the technology user i uses to decode its received data as well as on the personal preference/perception of (human) user i for the decoded data. For $\boldsymbol{p} \notin S_i$ $u_i(\boldsymbol{p}) := 0$ since, as mentioned above, user i cannot receive any QoS from a power profile which is infeasible for the BS to transmit. It is important to note that u_i^A is known to user i but not to the other users. Specifically, only user i knows the channel gain h_{0i} that specifies how the power profile $\boldsymbol{p}$ influences its QoS and thus, specifies u_i^A. The parameters in the first and last terms of (1) are trivially known to i.

We assume that each user uses a Multi User Detector (MUD) decoder. Since in the downlink transmission the BS can synchronously transmit to all the users and it can announce the CDMA codes it uses for each user, it is feasible for the users to employ MUD [18]. In [16] it is shown that when each user uses MUD, the utilities $u_i, i \in \mathcal{N}$, can be assumed to be concave in $\boldsymbol{p}$. Hence we assume that,

Assumption 1. *for each $i \in \mathcal{N}$, $u_i(\boldsymbol{p})$ is strictly concave in $\boldsymbol{p}$ over S_i.*

We assume that the time scale in which the BS determines the transmission powers and taxes is sufficiently small and therefore, the system remains static for that period. Specifically, we assume that,

Assumption 2. *the number of users, their utilities and the channel gains from the BS to the users remain fixed throughout a power allocation period.*

We assume that before any power allocation period, the BS announces the total number of users in the network, therefore,

Assumption 3. *the number of users N is* common knowledge. [5]

Furthermore, we assume that,

Assumption 4. *the users in the network are non-cooperative and selfish.*

Assumption 4 implies that the users have an incentive to misrepresent their private information [6] so as to increase their own benefit, e.g. a user $i \in \mathcal{N}$ may not want to report to other users or to the BS its measured channel gain h_{0i} correctly or its true preference for the BS transmission, if this results in a power allocation in favor of user i. Hence we assume that,

Assumption 5. *for each $i \in \mathcal{N}$, the channel gain h_{0i} and the utility function u_i^A is user i's private information.*

9[2]In [17] we also treat the uplink transmission from the mobiles to the BS. For this problem we derive results similar to the ones for the downlink problem presented in this paper.

9[3]In the paper we will use the terms "mobiles" and "users" interchangeably to mean mobile users.

9[4]This helps increase the capacity of the network.

9[5]See [2, 20] for the definition of common knowledge.

9[6]Private information of a user is defined as the information that is known only to that user and nobody else in the network.

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It should be noted however that none of the users $i \in \mathcal{N}$ has any incentive to measure its respective channel gain h_{0i} incorrectly because it is to each user's advantage to know the correct influence of the channel gain on its utility.

In the following section we formulate the power allocation problem for the network model (M1).

2.1 The downlink power allocation problem

For the network model (M1) we want to develop a power and tax determination mechanism that works under the constraints imposed by the model and obtains a solution to the following centralized problem corresponding to it.

Problem ($\boldsymbol{P_{CD}}$)

$$\max_{(\boldsymbol{t},\boldsymbol{p})} \sum_{i\in\mathcal{N}} u_i^A(t_i,\boldsymbol{p}) \tag{5}$$

$$\text{s.t.} \quad \sum_{i\in\mathcal{N}} t_i = 0$$

$$\equiv \quad \max_{(\boldsymbol{t},\boldsymbol{p})\,\in\, S} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{p}) \tag{6}$$

$$\text{where,} \quad S := \{(\boldsymbol{t},\boldsymbol{p}) \mid \sum_{i\in\mathcal{N}} t_i = 0, \boldsymbol{t}\in\mathbb{R}^N;\ \boldsymbol{p}\in[0,P_0^{max}]^N\} \tag{7}$$

The optimization problem (5) is equivalent to (6) because for $(\boldsymbol{t},\boldsymbol{p}) \notin S$, the objective function in (5) is negative infinity by (1). Thus S is the set of feasible solutions of Problem (P_{CD}). Because of Assumption 1, the objective function in (6) is strictly concave in $\boldsymbol{p}$. Therefore, there is a unique optimal power profile $\boldsymbol{p}^*$ of Problem (P_{CD}). Furthermore, since the objective function in (6) does not explicitly depend on $\boldsymbol{t}$, an optimal solution of Problem (P_{CD}) must be of the form $(\boldsymbol{t},\boldsymbol{p}^*)$, where $\boldsymbol{t}$ is any feasible tax profile for Problem (P_{CD}), i.e. a tax profile that satisfies (4).

As described in Section 2 it is the job of the BS to determine taxes and transmission powers for the users. Assumption 5 implies that the BS does not completely know all the parameters that describe Problem (P_{CD}). Therefore, we need to develop a mechanism which enables the BS to determine optimal solutions of Problem (P_{CD}) via some communication with the network users. Since a key assumption in Model (M1) is that the users are non-cooperative and selfish, the mechanism we develop must take into account the possible strategic behavior of the users in users to BS communication.

A systematic approach to the development of resource allocation mechanisms for informationally decentralized networks (as the one described by Model (M1)), where users behave strategically, is provided by *implementation theory* in Mathematical Economics. In the context of our problem, implementation theory deals with designing mechanisms that provide rules/guidelines for; (i) how the BS should "communicate" with the mobile users, and (ii) how the BS should "use the information communicated by the users to determine allocations" so as to induce the strategic users to send information that results in network objective maximizing allocations.

In this paper we use an implementation theory-based approach for the solution of the power allocation problem presented in this section. Therefore, in the next section we provide a brief introduction to implementation theory and set the preliminaries for our solution to the power allocation problem.

2.2 Embedding the power allocation problem for Model (M1) in the framework of implementation theory

Implementation theory is a branch of the theory of *mechanism design* developed by mathematical economists. Mechanism design provides a systematic methodology for the design of decentralized resource allocation mechanisms for informationally decentralized systems that achieve optimal allocations of the corresponding centralized systems. In the mechanism design framework, a centralized resource allocation problem is described by the triple $(\mathcal{E},\mathcal{A},\pi)$: the environment space $\mathcal{E}$, the action/allocation space $\mathcal{A}$ and the goal correspondence π.

The *environment* $\boldsymbol{e}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of resources and technologies available to all the users, their utilities, and any other information available to them, taken together. These are circumstances that cannot be changed either by the users in the network or by the designer of the resource allocation mechanism. For the network described by Model (M1), the environment $\boldsymbol{e_i}$ of user i, $i \in \mathcal{N}$, consists of the channel gain h_{0i}, its utility function u_i^A, and the common knowledge about the number of users N as well as the fact that the number of users, their utilities and the channel gains remain fixed throughout a power allocation period. The environments of all the users collectively define the system environment $\boldsymbol{e} := (\boldsymbol{e_1},\boldsymbol{e_2},\dots,\boldsymbol{e_N})$. The set of all possible environments $\boldsymbol{e_i}$ of a user defines its environment space $\mathcal{E}_i$. The environment spaces of all the users collectively define the environment space $\mathcal{E} := (\mathcal{E}_1,\mathcal{E}_2,\dots,\mathcal{E}_N)$ of the system/problem.

The *action / allocation space* $\mathcal{A}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of all possible resource allocation / exchange actions that can be taken by the users. For the network described by Model (M1), $\mathcal{A}$ is the set S of all tax and power profiles $(\boldsymbol{t},\boldsymbol{p})$ that the BS can possibly allocate to the users.

The *goal correspondence* π of a centralized resource allocation problem is a map from $\mathcal{E}$ to $\mathcal{A}$ which assigns for every environment $\boldsymbol{e} \in \mathcal{E}$, the set of allocations in $\mathcal{A}$ that are solutions to the centralized resource allocation problem according to some pre-specified system goal. For the centralized power allocation problem (P_{CD}), the system goal is the maximization of the sum $\sum_{i\in\mathcal{N}} u_i^A(t_i,\boldsymbol{p})$ of users' utilities, and π is a mapping that maps every environment $\boldsymbol{e} \in \mathcal{E}$, defined in the previous paragraph, to the set of solutions of (P_{CD}). Since in a centralized scenario one of the users (or a controller such as the BS) has complete system information, i.e. it knows $\boldsymbol{e}$, it can determine optimal allocations $\pi(\boldsymbol{e})$ in $\mathcal{A}$ corresponding to any given $\boldsymbol{e}$ using centralized optimization methods (such as mathematical programming or dynamic programming).

In an informationally decentralized system as the one described by Model (M1), the controller (BS in Model (M1)) does not completely know $\boldsymbol{e}$, therefore it is not possible for the controller to determine optimal centralized allocations $\pi(\boldsymbol{e})$ by methods similar to those for the centralized problems. Therefore, for resource allocation in a decentralized system, it is desirable to devise a communication/message exchange process between the users and the controller that eventually enables the controller to determine optimal centralized allocations. However, when the users in a system are selfish, they have an incentive to misrepresent their private

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information while communicating with the controller so as to shift the allocation determined by the controller in their own favor. The users may also choose not to participate in the communication process if they know that the resulting allocation will not be in their favor (or if by not participating they are better off). This may defeat the objective of maximizing the system objective function ($\sum_{i\in\mathcal{N}} u_i^A(t_i, \boldsymbol{p})$ for the power allocation problem). Therefore, for the success of a communication process in leading to desirable outcomes it is required that the allocation rule employed by the controller induces the users to behave in a desirable manner (i.e. it ensures voluntary participation of the users in the communication process and furthermore, it induces the users to communicate information that results in system objective maximizing allocations). In the context of mechanism design, a formal treatment of the design of such communication and allocation rules is provided by implementation theory.

In implementation theory, a decentralized resource allocation mechanism is specified by a *game form*. An N-user game form is defined by the pair $(\mathcal{M}, f)$. $\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i$ is the *message space* which specifies for each $i \in \mathcal{N}$ the set of messages $\mathcal{M}_i$ that user i can communicate to other users and the controller. f is the *outcome function* which maps $\mathcal{M} \to \mathcal{A}$; it specifies for each *message profile* $\boldsymbol{m} \in \mathcal{M}$, $\big(\boldsymbol{m} := (\boldsymbol{m_1}, \boldsymbol{m_2}, \ldots, \boldsymbol{m_N}), \boldsymbol{m_i} \in \mathcal{M}_i, i \in \mathcal{N}\big)$, the resulting allocation $f(\boldsymbol{m}) \in \mathcal{A}$.

Since the participation of the users in a resource allocation mechanism requires that they be aware of its protocols, it is assumed that the game form is known to all the users in the system. In order that the decentralized mechanism specified by a game form obtains centralized solutions when the users in the system are selfish, it is required, as discussed before, that the allocation rule f induces the users to behave in a desirable manner. To specify this requirement on a game form, we must first specify the users' behavior. In the context of implementation theory, users' behavior is specified by specifying *games* and associated *equilibrium concepts* corresponding to the game forms.

A game is specified by a game form $(\mathcal{M}, f)$ together with the utilities $u_i^A, i \in \mathcal{N}$, specified by an environment $\boldsymbol{e} \in \mathcal{E}$. Such a game is represented by $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$. In this game the players are the users in $\mathcal{N}$, the set of strategies of a user is its respective message space $\mathcal{M}_i, i \in \mathcal{N}$, and the payoff of a user corresponding to a given strategy/message profile $\boldsymbol{m}$ is the utility $u_i^A(f(\boldsymbol{m})), i \in \mathcal{N}$, it obtains from the resulting allocation $f(\boldsymbol{m})$. Given a game, the users can be assumed to behave according to different "behavioral concepts" which lead to different types of "equilibrium concepts". One such equilibrium concept is *Nash Equilibrium* (NE). A Nash Equilibrium of a game is defined as a message profile $\boldsymbol{m}^*$ such that none of the users finds it profitable to unilaterally deviate to any other message. Mathematically, $\boldsymbol{m}^*$ is a NE of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ if, [7]

$$u_i^A(f(\boldsymbol{m}^*)) \geq u_i^A(f((\boldsymbol{m_i}, \boldsymbol{m}^*/i))) \quad \forall\, \boldsymbol{m_i} \in \mathcal{M}_i, \ \forall\, i \in \mathcal{N} \tag{8}$$

Let $NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ represent the set of all Nash equilibria of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$, and let,

$$\mathcal{A}_{NE}(\mathcal{M}, f, \{u_i^A\}_{i=1}^N) := \Big\{\boldsymbol{a} \in \mathcal{A} \mid \boldsymbol{a} = f(\boldsymbol{m}) \text{ for some } \boldsymbol{m} \in NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)\Big\}, \tag{9}$$

that is, $\mathcal{A}_{NE}$ is the set of allocations corresponding to all Nash equilibria of the game.

Now consider a decentralized resource allocation problem. Let $\mathcal{E} = \prod_{i=0}^{N} \mathcal{E}_i$ be the environment space and $\mathcal{A}$ the allocation space associated with the problem, let $\pi : \mathcal{E} \to \mathcal{A}$ be a goal correspondence, and let $u_1^A, u_2^A, \ldots, u_N^A$, be the users' utilities corresponding to a given environment $\boldsymbol{e} \in \mathcal{E}$. Then, we have the following:

DEFINITION 1.
Implementation in Nash equilibria: *π is said to be "fully implemented in Nash equilibria" by the game form $(\mathcal{M}, f)$ if $\mathcal{A}_{NE}(\mathcal{M}, f, \{u_i^A\}_{i=1}^N) = \pi(\boldsymbol{e})$ for all $\boldsymbol{e} \in \mathcal{E}$, i.e. for any given environment, the set of allocations resulting (through the outcome function f) from the Nash equilibria of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ is exactly the same as the set of allocations $\pi(\boldsymbol{e})$ that are optimal solutions of the corresponding centralized problem $(\boldsymbol{e}, \mathcal{A}, \pi)$.*

Definition 1 implies that a game form that implements in NE a goal correspondence, takes into account users' strategic behavior and obtains centralized solutions, given that the users participate in the message exchange process specified by the game form. However, in order that the users voluntarily participate in a mechanism specified by a game form, the game form must satisfy an additional property defined as follows. Let the *initial endowment* of a user be defined as the amount of resources the user has before participating in a game form; e.g. for the network model (M1), the initial endowment $\boldsymbol{f_i^0}$ of user $i, i \in \mathcal{N}$, is the tax and transmission power profile before the power allocation mechanism is run, i.e. $\boldsymbol{f_i^0} = (t_i^0, \boldsymbol{p}^0) = (0, \boldsymbol{0}), \ \forall\, \mathbf{i} \in \mathcal{N}$. We then have the following,

DEFINITION 2.
Individual rationality: *A game form $(\mathcal{M}, f)$ is said to be individually rational if $\forall\, i \in \mathcal{N}$, $u_i^A(f(\boldsymbol{m})) \geq u_i^A(\boldsymbol{f_i^0})$ for all $\boldsymbol{m} \in NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$, i.e. at any NE allocation the utility of each user is at least as much as its utility before participating in the game form.*

Definitions 1 and 2 imply that a game form that is individually rational and implements in NE a goal correspondence, obtains optimal allocations of a centralized system (corresponding to a decentralized one) by having the users voluntarily participate in the allocation process. These are exactly the properties that we want in a tax and power allocation mechanism for the network model (M1). The theory of implementation introduced above thus provides us with a framework to develop the desired decentralized power allocation mechanism for the network model (M1).

In light of the discussion provided in this section, we now state our objective for the power allocation problem presented in Section 2.1.

9[7]Nash equilibria describe strategic behavior in games of complete information. Since the users in Model (M1) do not know each other's utilities, the NE (8) in the context of Model (M1) should be interpreted as follows: The NE can be viewed as a "stationary point of an iterative adjustment process" in which the users continually revise (the revision of messages can be done via *best response* or any other strategy) their messages until a point is reached where unilateral deviation no longer pays [12, 13].

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The objective:

Let $\mathcal{E}$ and $\mathcal{A}$ be respectively the environment space and the allocation space corresponding to the downlink network model (M1) as defined in Section 2.2. Let $\pi : \mathcal{E} \to \mathcal{A}$ be the goal correspondence corresponding to Problem (P_{CD}) as defined in Section 2.2. Our objective is to design an individually rational game form $(\mathcal{M}, f)$ that implements in NE the goal correspondence π.

In the next section, we present a game form that achieves the above objective.

3. SOLUTION OF THE DOWNLINK POWER ALLOCATION PROBLEM

In this section we present a game form that provides a decentralized mechanism for solving the downlink power allocation problem presented in Section 2.1. We first present the structure of the message space $\mathcal{M}$ and the outcome function f that constitute the game form. We then present theorems that assert that the proposed game form is individually rational and that it fully implements in NE the goal correspondence π corresponding to Problem (P_{CD}). At the end we present a discussion on the intuition behind the structure of the proposed game form.

3.1 The Game form

To obtain an appropriate game form for the power allocation problem it is useful to observe that in the downlink network, the power profile $\boldsymbol{p} = (p_1, p_2, \ldots, p_N)$ transmitted by the BS can be treated as a *public good* [11]. This is because, analogous to a public good in an economy, the same vector $\boldsymbol{p}$ affects the utility of all the users in the network. Furthermore, like a public good, the exact amount of utility a user obtains from $\boldsymbol{p}$ differs from user to user and depends on its individual utility function $u_i, i \in \mathcal{N}$. Game forms that implement in NE efficient allocation of public goods have been proposed by Groves and Ledyard [5], Hurwicz [7] and Walker [19]. In this section we present a game form for the downlink power allocation problem that is derived from Hurwicz' mechanism [7]. Below we specify each of the elements of the proposed game form, the message space and the outcome function.

The message space:

Since we are interested in determining transmission powers and taxes for the network model (M1), the communication between the users and the BS should contain information that is helpful in determining the optimal amounts of each of these. We let each user $i \in \mathcal{N}$ send to the BS a message $\boldsymbol{m_i} \in \mathcal{M}_i := \mathbb{R}_+^N \times \mathbb{R}^N$ that has the following form:

$$\boldsymbol{m_i} := (\boldsymbol{\pi_i}, \boldsymbol{p_i}); \qquad \boldsymbol{\pi_i} \in \mathbb{R}_+^N, \ \boldsymbol{p_i} \in \mathbb{R}^N \tag{10}$$

The message $\boldsymbol{m_i}$ consists of two elements: $\boldsymbol{p_i} = (p_{i1}, p_{i2}, \ldots, p_{iN})$ which can be interpreted as the power profile that user $i \, (i \in \mathcal{N})$ suggests to be allocated to all the users $j \in \mathcal{N}$; and $\boldsymbol{\pi_i} = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{iN})$ which can be interpreted as the price that user $i \, (i \in \mathcal{N})$ suggests to be charged to the users $j \in \mathcal{N}$ for using the network.

Outcome function:

Based on the message profile $\boldsymbol{m} = (\boldsymbol{m_1}, \boldsymbol{m_2}, \ldots, \boldsymbol{m_N})$, the BS sets the taxes $\hat{t}_i(\boldsymbol{m}), i \in \mathcal{N}$, and determines transmission powers $\hat{\boldsymbol{p}}(\boldsymbol{m}) = (\hat{p}_1(\boldsymbol{m}), \hat{p}_2(\boldsymbol{m}), \ldots, \hat{p}_N(\boldsymbol{m}))$ for the users as follows:

$$\hat{\boldsymbol{p}}(\boldsymbol{m}) = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{p_i}, \tag{11}$$

$$\begin{aligned} \hat{t}_i(\boldsymbol{m}) = \boldsymbol{l_i}^T(\boldsymbol{m}) \hat{\boldsymbol{p}}(\boldsymbol{m}) + (\boldsymbol{p_i} - \boldsymbol{p_{i+1}})^T \operatorname{diag}(\boldsymbol{\pi_i}) \\ (\boldsymbol{p_i} - \boldsymbol{p_{i+1}}) - (\boldsymbol{p_{i+1}} - \boldsymbol{p_{i+2}})^T \operatorname{diag}(\boldsymbol{\pi_{i+1}}) \\ (\boldsymbol{p_{i+1}} - \boldsymbol{p_{i+2}}), \qquad i \in \mathcal{N}, \end{aligned} \tag{12}$$

$$\text{where} \quad \boldsymbol{l_i}(\boldsymbol{m}) = \boldsymbol{\pi_{i+1}} - \boldsymbol{\pi_{i+2}} \tag{13}$$

In (12) and (13), $i+2 \equiv 1$ for $i = N-1$, and for $i = N$, $i+1 \equiv 1$ and $i+2 \equiv 2$.

The game form defined by (10)–(13) together with the users' utility functions in (1) specifies a game. The strategy of user $i, i \in \mathcal{N}$, in this game is its message $\boldsymbol{m_i}$. It should be noted that the message $\boldsymbol{m_i}$ of user $i, i \in \mathcal{N}$, is allowed to take any value (which can be unboundedly large) in the space $\mathbb{R}_+^N \times \mathbb{R}^N$; in particular $\boldsymbol{p_i}$ is not restricted to lie in S_i. Thus, a Nash equilibrium [8] of the above game is a message profile $\boldsymbol{m}^*$ from which no user wants to unilaterally deviate (see (8)) even when arbitrary deviations are possible by unbounded magnitude of messages.

As discussed in Section 2.2, our objective is to develop a game form for which the set of tax and power allocations obtained at all its NE is the same as the set of optimal tax and power allocations for the centralized problem (P_{CD}). Below we present theorems that assert that the proposed game form achieves this goal.

3.2 Optimality of the game form

The main results of this paper are summarized by Theorems 1 and 2 below.

THEOREM 1. *Let $\boldsymbol{m}^*$ be a NE of the game specified by the game form presented in Section 3.1 and the users' utility functions (1). Let $(\hat{\boldsymbol{t}}(\boldsymbol{m}^*), \hat{\boldsymbol{p}}(\boldsymbol{m}^*)) =: (\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ be the tax and power allocation at $\boldsymbol{m}^*$ determined by the game form. Then,*

(a) $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is individually rational, i.e. all users weakly prefer $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ to the initial allocation $(\mathbf{0}, \mathbf{0})$, and

(b) $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is an optimal solution of the centralized problem (P_{CD}).

Furthermore, all NE of the game result in the same optimal power allocation, i.e. if $\overline{\boldsymbol{m}}$ is any other NE, then, $\hat{\boldsymbol{p}}(\overline{\boldsymbol{m}}) = \hat{\boldsymbol{p}}(\boldsymbol{m}^*)$.

THEOREM 2. *Given the optimum power profile $\hat{\boldsymbol{p}}^*$ of Problem (P_{CD}), there exists at least one NE $\boldsymbol{m}^*$ of the game corresponding to the game form presented in Section 3.1 such that, $\hat{\boldsymbol{p}}(\boldsymbol{m}^*) = \hat{\boldsymbol{p}}^*$. Furthermore, given $\hat{\boldsymbol{p}}^*$, the set of all NE that result in $\hat{\boldsymbol{p}}^*$ can be characterized.*

The proofs of Theorem 1 and Theorem 2 are given in the Appendix. In the next section we provide a brief discussion on the intuition behind the structure of the proposed game form.

9[8]See footnote 8 for a discussion on the interpretation of Nash equilibria.

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3.3 Intuition behind the game form

To understand how the proposed structure of the game form achieves the desired goal, let us look at the properties the game form induces in its NE. A NE of the game corresponding to the proposed game form can be interpreted as follows: Since the allocated power profile, given the users' messages $\boldsymbol{m_j}, j \in \mathcal{N}$, is $1/N \sum_{i=1}^{N} \boldsymbol{p_i}$, user i's proposal $\boldsymbol{p_i}$ can be interpreted as the increment user i desires in everybody's power over the sum of other users' proposals so as to bring the allocated power profile $\hat{\boldsymbol{p}}(\boldsymbol{m})$ to i's desired value. Thus, if the average of the power profiles proposed by users other than user i does not lie in S_i, user i can propose an appropriate power profile and bring the allocated profile within S_i. It should be noted that the flexibility of proposing any power profile in $\mathbb{R}^N$ gives each user $i \in \mathcal{N}$ the capability to make the constraint $\boldsymbol{p} \in S_i$ satisfy by unilateral deviation. It follows that any NE power profile must lie in $\cap_{i\in\mathcal{N}} S_i$. Furthermore, it can be seen from (12) that the game form formulation enures that the allocated tax profile satisfies (4) (even at off-NE messages). The above two features imply that all NE allocations $(\boldsymbol{t}, \boldsymbol{p})$ lie in S and hence are feasible solutions of Problem (P_{CD}).

To see why NE allocations are optimal, let us look at the form of the tax (12). The tax for user i consists of three types of terms. Type-1 is $\boldsymbol{l_i^T}(\boldsymbol{m})\hat{\boldsymbol{p}}(\boldsymbol{m})$ that depends on the power proposals of all the users, and the price proposals of users other than user i. Type-2 term is the one that depends on $\boldsymbol{p_i}$ as well as $\boldsymbol{\pi_i}$, and type-3 term is the one that depends only on the messages of users other than user i. Since $\boldsymbol{\pi_i}$ does not affect the power allocation and affects only the type-2 term in t_i, the NE strategy of user $i, i \in \mathcal{N}$, that minimizes its tax is to propose for each $j \in \mathcal{N}$, $\pi_{ij} = 0$ unless at the NE, $p_{ij} = p_{i+1j}$. Since all the users $i \in \mathcal{N}$ choose the aforementioned strategy at the NE, the type-2 and type-3 terms vanish from every users' tax $t_i, i \in \mathcal{N}$, at the NE. Thus, the tax that users pay at a NE $\boldsymbol{m}^*$ is of the form $\boldsymbol{l_i}^T(\boldsymbol{m}^*)\hat{\boldsymbol{p}}(\boldsymbol{m}^*)$, $i \in \mathcal{N}$. The NE price term $\boldsymbol{l_i}^T(\boldsymbol{m}^*) =: \boldsymbol{l_i^{*T}}$, $i \in \mathcal{N}$, can therefore be interpreted as the "personalized price" [9] of the NE power profile $\hat{\boldsymbol{p}}(\boldsymbol{m}^*) =: \hat{\boldsymbol{p}}^*$ (treated as a public good) for user i; at the NE this price for user i is not controlled by i's message. The above reduction of tax terms in terms of the allocated power profile implies that, at the NE, the utilities of the users $i \in \mathcal{N}$ effectively depend only on the allocated power profile. Since each user has the capability (by choosing appropriate $\boldsymbol{p_i} \in \mathbb{R}^N$) to shift the allocated power profile to its desired value given that the proposals of all other users are fixed, the NE strategy of each user is to propose a power profile that results in an allocation that maximizes its corresponding utility. Thus, each user maximizes its net utility at the NE, and this results in the maximization of the system objective function at the NE.

It is worth mentioning at this point the significance of type-2 and type-3 terms in the users' tax. As explained above, these terms vanish at NE. However, if these terms are not present in t_i, user $i, i \in \mathcal{N}$, can propose arbitrarily high price for other users in $\boldsymbol{\pi_i}$ as $\boldsymbol{\pi_i}$ would not affect user i's utility at all. [10] It is also important that the NE price $\boldsymbol{l_i}$ is not affected by $\boldsymbol{\pi_i}$, otherwise user i may influence its own price in an unfair manner. However, since $\boldsymbol{\pi_i}$ would affect other users' price, it is necessary to prevent user i from proposing unfair prices for other users. Type-2 and type-3 terms in t_i do the above job by imposing a penalty on user i at off-equilibrium messages if user i proposes a high value of $\boldsymbol{\pi_i}$ or if it deviates too much from other users in its power profile proposal.

9[9] In Economics literature, these personalized prices for the public goods are called "Lindahl" prices.

9[0] Note that $\boldsymbol{l_i}$ depends on $\boldsymbol{\pi_{i+1}}$ and $\boldsymbol{\pi_{i+2}}$ and not $\boldsymbol{\pi_i}$.

4. CONCLUSION

Power allocation problems in wireless cellular networks with interference are analogous to public good allocation problems. Thus, pricing mechanisms that are useful in developing decentralized optimal power allocation algorithms for networks without interference, which are analogous to private good economies, do not result in globally optimal power allocations for cellular networks with interference. In this paper we studied power allocation for a single cell wireless CDMA network with interference. We formulated the downlink power allocation problem as a public good allocation problem. We presented a game form the NE of which result in power allocation that is the optimal solution of the corresponding centralized downlink network.

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APPENDIX

A. PROOF OF THEOREM 1

In this section we present the proof of Theorems 1 and 2. We divide the proof into several claims to organize the presentation.

CLAIM 1. *If $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.1 and the users' utility functions (1), then the allocation $(\hat{\boldsymbol{t}}(\boldsymbol{m}^*), \hat{\boldsymbol{p}}(\boldsymbol{m}^*)) =: (\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is a feasible solution of Problem (P_{CD}), i.e. $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*) \in S$.*

Proof:

By construction of the game form, the allocated tax (12) satisfies (4) which implies that the NE tax profile $\hat{\boldsymbol{t}}^*$ also satisfies (4). Therefore to prove the claim, we need to show that the NE power profile $\hat{\boldsymbol{p}}^* \in \cap_{i\in\mathcal{N}} S_i$ (where $S_i, i \in \mathcal{N}$, is defined by (2)). We will prove this by showing that, if $\hat{\boldsymbol{p}}^* \notin S_i$ for some $i \in \mathcal{N}$, then there exists a profitable unilateral deviation for user i.

Suppose $\hat{\boldsymbol{p}}^* \notin S_i$ for some $i \in \mathcal{N}$. Then, from (1), $u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*) = -\infty$. Consider $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}} = (\boldsymbol{\pi_i^*}, \widetilde{\boldsymbol{p}}_{\boldsymbol{i}})$ where $\boldsymbol{\pi_i^*}$ is the NE price profile and $\widetilde{\boldsymbol{p}}_{\boldsymbol{i}}$ ($\widetilde{\boldsymbol{p}}_{\boldsymbol{i}} \in \mathbb{R}^N$) is such that,

$$\hat{\boldsymbol{p}}(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i) = \frac{1}{N}\Big(\sum_{\substack{j\in\mathcal{N}\\ j\neq i}} \boldsymbol{p_j^*} + \widetilde{\boldsymbol{p}}_{\boldsymbol{i}}\Big) = \boldsymbol{0} \in S_i$$

Then,

$$u_i^A\big(\hat{t}_i(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i),\, \hat{\boldsymbol{p}}(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i)\big) = -\hat{t}_i(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i) + u_i(\boldsymbol{0}) > -\infty = u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*) \quad (14)$$

Thus user i will find it profitable to deviate to $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}$.

Inequality (14) implies that $\boldsymbol{m}^*$ cannot be a NE, which is a contradiction. Therefore we must have that, $\hat{\boldsymbol{p}}^* \in \cap_{i\in\mathcal{N}} S_i$ and hence, $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*) \in S$.

□

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Claim 2. *If $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.1 and the users' utility functions (1), then, the tax $\hat{t}_i(\boldsymbol{m}^*) =: \hat{t}_i^*$ paid by user i, $i \in \mathcal{N}$, at NE $\boldsymbol{m}^*$ is of the form, $\hat{t}_i^* = \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*$, where $\boldsymbol{l_i^*} := \boldsymbol{l_i}(\boldsymbol{m}^*)$.*

Proof:

Let $\boldsymbol{m}^*$ be a NE described in Claim 2. Then, for each $i \in \mathcal{N}$,

$$u_i^A\big(\hat{t}_i(\boldsymbol{m_i}, \boldsymbol{m}^*/i),\, \hat{\boldsymbol{p}}(\boldsymbol{m_i}, \boldsymbol{m}^*/i)\big) \leq u_i^A\big(\hat{t}_i^*, \hat{\boldsymbol{p}}^*\big), \quad \forall\, \boldsymbol{m_i} \in \mathcal{M}_i \tag{15}$$

Substituting $\boldsymbol{m_i} = (\boldsymbol{\pi_i}, \boldsymbol{p_i^*}),\ \boldsymbol{\pi_i} \in \mathbb{R}_+^N$, in (15) and using (11) implies that,

$$u_i^A\big(\hat{t}_i((\boldsymbol{\pi_i}, \boldsymbol{p_i^*}), \boldsymbol{m}^*/i), \hat{\boldsymbol{p}}^*\big) \leq u_i^A\big(\hat{t}_i^*, \hat{\boldsymbol{p}}^*\big), \ \forall\, \boldsymbol{\pi_i} \in \mathbb{R}_+^N \tag{16}$$

Since u_i^A decreases in t_i (see (1)), (16) implies that,

$$\hat{t}_i((\boldsymbol{\pi_i}, \boldsymbol{p_i^*}), \boldsymbol{m}^*/i) \geq \hat{t}_i^*, \quad \forall\, \boldsymbol{\pi_i} \in \mathbb{R}_+^N \tag{17}$$

Substituting (12) in (17) implies that,

$$\begin{aligned} & \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^* + (\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) \\ & -(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T \mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*}) \\ \geq\ & \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^* + (\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) \\ & -(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T \mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*}), \\ & \qquad \forall\, \boldsymbol{\pi_i} \in \mathbb{R}_+^N \end{aligned} \tag{18}$$

Canceling the common terms in (18) implies,

$$(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i} - \boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) \ \geq\ 0, \quad \forall\, \boldsymbol{\pi_i} \in \mathbb{R}_+^N \tag{19}$$

Since (19) must hold for all $\boldsymbol{\pi_i} \geq \boldsymbol{0}$, it implies that,

$$\text{for each } j \in \mathcal{N}, \quad \text{either} \quad p_{i\,j}^* = p_{i+1\,j}^* \quad \text{or} \quad \pi_{i\,j}^* = 0 \tag{20}$$

It follows from (20) that at any NE $\boldsymbol{m}^*$,

$$(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) = 0, \ \forall\, i \in \mathcal{N} \tag{21}$$

Using (21) in (12) we obtain that any NE tax profile must be of the form,

$$\hat{t}_i^* = \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \quad \forall\, i \in \mathcal{N} \tag{22}$$

□

Claim 3. *The game form given in Section 3.1 is individually rational, i.e. for every NE $\boldsymbol{m}^*$ corresponding to it, the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is weakly preferred by all the users to the initial allocation $(\boldsymbol{0}, \boldsymbol{0})$.*

Proof:

Suppose $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.1 and the users' utility functions (1). From Claim 2 we know the form of the tax at $\boldsymbol{m}^*$. Substituting that from (22) into (15) we obtain that, for each $i \in \mathcal{N}$,

$$\begin{aligned} & u_i^A(\hat{t}_i((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i),\, \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)) \\ & \leq\ u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*,\, \hat{\boldsymbol{p}}^*), \quad \forall\, \boldsymbol{m_i} = (\boldsymbol{\pi_i}, \boldsymbol{p_i}) \in \mathcal{M}_i \end{aligned} \tag{23}$$

Substituting for $\hat{t}_i$ in (23) from (12) and using equality (21) we obtain,

$$\begin{aligned} & u_i^A\Big(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i) + (\boldsymbol{p_i} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i}) \\ & (\boldsymbol{p_i} - \boldsymbol{p_{i+1}^*}),\ \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)\Big) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \\ & \qquad \forall\, \boldsymbol{\pi_i} \in \mathbb{R}_+^N,\ \forall\, \boldsymbol{p_i} \in \mathbb{R}^N \end{aligned} \tag{24}$$

In particular, $\boldsymbol{\pi_i} = \boldsymbol{0}$ in (24) implies that,

$$\begin{aligned} & u_i^A\big(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}((\boldsymbol{0}, \boldsymbol{p_i}), \boldsymbol{m}^*/i),\ \hat{\boldsymbol{p}}((\boldsymbol{0}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)\big) \\ & \leq\ u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \qquad \forall\, \boldsymbol{p_i} \in \mathbb{R}^N \end{aligned} \tag{25}$$

Since (25) holds for all $\boldsymbol{p_i} \in \mathbb{R}^N$, substituting $1/N\big(\boldsymbol{p_i} + \sum_{j \in \mathcal{N}\setminus\{i\}} \boldsymbol{p_j^*}\big) = \overline{\boldsymbol{p}}$ in (25) implies that,

$$u_i^A\left(\boldsymbol{l_i^*}^T \overline{\boldsymbol{p}},\ \overline{\boldsymbol{p}}\right) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall\, \overline{\boldsymbol{p}} \in \mathbb{R}^N \tag{26}$$

For $\overline{\boldsymbol{p}} = \boldsymbol{0}$, (26) implies that,

$$u_i^A(0, \boldsymbol{0}) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall\, i \in \mathcal{N} \tag{27}$$

This proves that the NE allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is weakly preferred to $(\boldsymbol{0}, \boldsymbol{0})$ by all the users $i \in \mathcal{N}$.

□

Claim 4. *A NE allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is an optimal solution of the centralized problem (P_{CD}).*

Proof:

For each $i \in \mathcal{N}$, (26) can be equivalently written as,

$$\begin{aligned} \hat{\boldsymbol{p}}^* &= \arg\max_{\overline{\boldsymbol{p}} \in \mathbb{R}^N} u_i^A\left(\boldsymbol{l_i^*}^T \overline{\boldsymbol{p}},\ \overline{\boldsymbol{p}}\right) \\ &= \arg\max_{\overline{\boldsymbol{p}} \in \mathbb{R}^N} -\boldsymbol{l_i^*}^T \overline{\boldsymbol{p}} + u_i(\overline{\boldsymbol{p}}) - \left[\frac{1 - I_{S_i}(\overline{\boldsymbol{p}})}{I_{S_i}(\overline{\boldsymbol{p}})}\right] \\ &= \arg\max_{\overline{\boldsymbol{p}} \in S_i} -\boldsymbol{l_i^*}^T \overline{\boldsymbol{p}} + u_i(\overline{\boldsymbol{p}}) \end{aligned} \tag{28}$$

Since for each $i \in \mathcal{N}$, $u_i(\overline{\boldsymbol{p}})$ is assumed to be strictly concave in $\overline{\boldsymbol{p}}$ over S_i and the set S_i is convex, Karush Kuhn Tucker (KKT) conditions [3, Chapter 11] are necessary and sufficient for $\hat{\boldsymbol{p}}^*$ to be the maximizer in (28). Thus, for each $i \in \mathcal{N}$, $\exists\ \boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$ such that, $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i}$ and $\boldsymbol{\lambda_2^i}$ satisfy the KKT conditions given below:

$$\boldsymbol{l_i^*} - \nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i} = 0 \tag{29}$$

$${\boldsymbol{\lambda_1^i}}^T \hat{\boldsymbol{p}}^* = 0 \tag{30}$$

$${\boldsymbol{\lambda_2^i}}^T (\hat{\boldsymbol{p}}^* - P_0^{max} \boldsymbol{1}) = 0 \tag{31}$$

$$\text{where,} \quad \boldsymbol{1} = (\underbrace{1, 1, \ldots, 1}_{N \text{ times}}) \ \in\ \mathbb{R}^{N \times 1}$$

Combining the KKT conditions of all the users, i.e. summing (29) for all $i \in \mathcal{N}$, and using the fact that $\sum_{i \in \mathcal{N}} \boldsymbol{l_i^*} = 0$ (see (13)), we obtain,

$$\sum_{i \in \mathcal{N}} \Big(-\nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i}\Big) = 0 \tag{32}$$

Eq. (32) along with (30) and (31) for all i, and the non-negativity of $\boldsymbol{\lambda_1^i}, \boldsymbol{\lambda_2^i}$, $i \in \mathcal{N}$, specify the KKT conditions (for variable $\boldsymbol{p}$) for (6). Since (6) is a concave optimization problem, the KKT conditions are necessary and sufficient for its optimum. Since $\hat{\boldsymbol{p}}^*$ satisfies these KKT conditions, it is the maximizer of the objective function in (6). Therefore, as described in Section 2.1 an optimal solution of Problem (P_{CD}) is of the form $(\boldsymbol{t}, \hat{\boldsymbol{p}}^*)$ where $\boldsymbol{t} \in \mathbb{R}^N$ is any tax profile that

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satisfies (4). Since by construction of the tax the NE allocation $\hat{\boldsymbol{t}}^*$ satisfies (4), we conclude that $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is an optimal solution of (P_{CD}).

□

Since the NE $\boldsymbol{m}^*$ we analyzed in Claims 1–4 was arbitrarily chosen, the results of Claims 1–4 hold true for all NE corresponding to the game form of Section 3.1. Therefore, all NE corresponding to the aforementioned game form result in an optimal solution of Problem (P_{CD}). Since Problem (P_{CD}) has a unique solution in $\boldsymbol{p}$, all NE result in the same optimal power allocation, i.e. if $\overline{\boldsymbol{m}}$ is a NE other than $\boldsymbol{m}^*$, then, $\hat{\boldsymbol{p}}(\overline{\boldsymbol{m}}) = \hat{\boldsymbol{p}}(\boldsymbol{m}^*)$. This completes the proof of Theorem 1.

B. PROOF OF THEOREM 2

Theorem 1 shows that if there exists a NE of the game corresponding to the game form of Section 3.1, then the allocation at the NE is an optimal solution of the centralized problem (P_{CD}). However, Theorem 1 does not guarantee the existence of a NE; in other words, it does not guarantee that the centralized optimum power profile is attainable through NE. This is guaranteed by Theorem 2 which is proved next.

We prove Theorem 2 in two steps. In the first step we show that if the centralized problem (P_{CD}) has a solution, there exist a set of personalized prices, one for each user $i \in \mathcal{N}$, such that when every user individually maximizes its own utility taking the above prices as given, then each of them obtains the same optimal power profile which is also the optimal solution of Problem (P_{CD}). In the second step we show that the above set of personalized prices and the optimal power profile for (P_{CD}) can be used to construct message profiles that are NE of the game corresponding to the game form of Section 3.1. We also show that the set of NE obtained from such a construction is exactly the set of all NE that result in the optimal power and tax profiles of Problem (P_{CD}) that we start with.

Claim 5. *If Problem (P_{CD}) has a solution and the optimum power profile is $\hat{\boldsymbol{p}}^*$, there exist a set of personalized prices $\boldsymbol{l}_i^*$, $i \in \mathcal{N}$, such that,*

$$\arg\max_{\boldsymbol{p}\in S_i} \; -\boldsymbol{l}_i^{*T}\boldsymbol{p} + u_i(\boldsymbol{p}) \;=\; \hat{\boldsymbol{p}}^*, \quad \forall\, i \in \mathcal{N} \tag{33}$$

Proof:

Suppose $\hat{\boldsymbol{p}}^*$ is the optimal power profile corresponding to Problem (P_{CD}). Problem (P_{CD}) does have a solution since it involves maximization of a continuous function in $\boldsymbol{p}$ over a compact set in $\boldsymbol{p}$ (The solution in $\boldsymbol{t}$ trivially exists). Furthermore, (P_{CD}) has a unique solution in $\boldsymbol{p}$ since it is a concave optimization problem in $\boldsymbol{p}$. Writing the optimization problem (P_{CD}) for $\boldsymbol{p}$ we have,

$$\begin{aligned} \hat{\boldsymbol{p}}^* \;&=\; \arg\max_{\boldsymbol{p}} \; \sum_{i\in\mathcal{N}} u_i(\boldsymbol{p}) \\ \text{s.t.} \quad & \boldsymbol{p} \in S_i, \;\; \forall\, i \in \mathcal{N} \end{aligned}$$

Since the above problem is a concave optimization problem, its optimal solution must satisfy the KKT conditions. Therefore there exist $\boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$, $i \in \mathcal{N}$, such that $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i}$ and $\boldsymbol{\lambda_2^i}, i \in \mathcal{N}$, satisfy,

$$\sum_{i\in\mathcal{N}} \left(-\nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i}\right) \;=\; 0 \tag{34}$$

$$\text{and} \qquad \boldsymbol{\lambda_1^i}^T \hat{\boldsymbol{p}}^* \;=\; 0, \quad \forall\, i \in \mathcal{N} \tag{35}$$

$$\boldsymbol{\lambda_2^i}^T(\hat{\boldsymbol{p}}^* - P_0^{max}\boldsymbol{1}) \;=\; 0, \quad \forall\, i \in \mathcal{N} \tag{36}$$

We define for each $i \in \mathcal{N}$,

$$\boldsymbol{l}_i^* \;:=\; \nabla u_i(\hat{\boldsymbol{p}}^*) + \boldsymbol{\lambda_1^i} - \boldsymbol{\lambda_2^i} \tag{37}$$

Then,

$$\boldsymbol{l}_i^* - \nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i} = 0, \quad \forall\, i \in \mathcal{N} \tag{38}$$

Equations (38), (35) and (36) together imply that for each $i \in \mathcal{N}$, $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$ satisfy the KKT conditions for the following maximization problem,

$$\max_{\boldsymbol{p}\,\in\, S_i} \; -\boldsymbol{l}_i^{*T}\boldsymbol{p} + u_i(\boldsymbol{p}) \tag{39}$$

Since (39) is a concave optimization problem, KKT conditions are necessary and sufficient for its optimum. Therefore, from (35), (36) and (38) we conclude that,

$$\hat{\boldsymbol{p}}^* \;=\; \arg\max_{\boldsymbol{p}\,\in\, S_i} \; -\boldsymbol{l}_i^{*T}\boldsymbol{p} + u_i(\boldsymbol{p}) \tag{40}$$

□

Claim 6. *Let $\hat{\boldsymbol{p}}^*$ be the optimal power profile corresponding to Problem (P_{CD}), let $\boldsymbol{l}_i^*$, $i \in \mathcal{N}$, be the personalized prices defined in Claim 5, and let $\hat{t}_i^* := \boldsymbol{l}_i^{*T}\hat{\boldsymbol{p}}^*, i \in \mathcal{N}$. Let $\boldsymbol{m}_i^* := (\boldsymbol{\pi}_i^*, \boldsymbol{p}_i^*), i \in \mathcal{N}$, be a solution to the following set of relations:*

$$\frac{1}{N}\sum_{i\in\mathcal{N}} \boldsymbol{p}_i^* \;=\; \hat{\boldsymbol{p}}^*, \tag{41}$$

$$\boldsymbol{\pi}_{i+1}^* - \boldsymbol{\pi}_{i+2}^* \;=\; \boldsymbol{l}_i^*, \; i \in \mathcal{N}, \tag{42}$$

$$(\boldsymbol{p}_i^* - \boldsymbol{p}_{i+1}^*)^T \mathit{diag}(\boldsymbol{\pi}_i^*)(\boldsymbol{p}_i^* - \boldsymbol{p}_{i+1}^*) \;=\; 0, \; i \in \mathcal{N}, \tag{43}$$

$$\boldsymbol{\pi}_i^* \;\geq\; \boldsymbol{0}, \; i \in \mathcal{N} \tag{44}$$

Then, $\boldsymbol{m}^* := (\boldsymbol{m}_1^*, \boldsymbol{m}_2^*, \ldots, \boldsymbol{m}_N^*)$ is a NE of the game corresponding to the game form defined in Section 3.1. Furthermore, $\hat{\boldsymbol{p}}(\boldsymbol{m}^*) = \hat{\boldsymbol{p}}^*$, and for each $i \in \mathcal{N}$, $\boldsymbol{l_i}(\boldsymbol{m}^*) = \boldsymbol{l}_i^*$ and $\hat{t}_i(\boldsymbol{m}^*) = \hat{t}_i^*$.

Proof:

It should be noted that, (41)–(44) are necessary conditions for any NE $\boldsymbol{m}^*$ corresponding to the game form of Section 3.1 to result in the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ (This follows from (11), (13) and (21)). Therefore, the set of solutions of (41)–(44), if one exists, is a superset of the set of all NE that result in $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$. Below we show that the solution set of (41)–(44) is in fact exactly the set of the NE that result in $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$.

To prove this we first show that the set of relations (41)–(44) do have a solution. Notice that by setting $\boldsymbol{p}_i^* = \hat{\boldsymbol{p}}^* \;\; \forall\, i \in \mathcal{N}$, equations (41) and (43) are satisfied. Notice also that the right hand side of (42) sums to 0 by taking the sum over $i \in \mathcal{N}$. Therefore, (42) has a solution in $\boldsymbol{\pi}_i^*, i \in \mathcal{N}$. Furthermore, for any solution $\boldsymbol{\pi}_i^*, i \in \mathcal{N}$, of (42), $\boldsymbol{\pi}_i^* + c, i \in \mathcal{N}$, where c is some constant, is also a solution of (42). Therefore by appropriately choosing c, we can select a solution of (42) such that (44) is satisfied.

It is clear from above that (41)–(44) have multiple solutions. We now show that the set of solutions $\boldsymbol{m}^*$ of (41)–(44)

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is the set of NE that result in the given centralized solution. From Claim 5, (33) can be equivalently written as,

$$\begin{aligned}\hat{\boldsymbol{p}}^* &= \arg\max_{\boldsymbol{p}\in\mathbb{R}^N} \; -\boldsymbol{l_i^*}^T\boldsymbol{p} + u_i(\boldsymbol{p}) - \left[\frac{1-I_{S_i}(\boldsymbol{p})}{I_{S_i}(\boldsymbol{p})}\right] \\ &= \arg\max_{\boldsymbol{p}\in\mathbb{R}^N} \; u_i^A(\boldsymbol{l_i^*}^T\boldsymbol{p},\, \boldsymbol{p}), \qquad \forall\, i\in\mathcal{N} \qquad (45)\end{aligned}$$

A change of variable $N\boldsymbol{p} - \sum_{\substack{j\in\mathcal{N}\\ j\neq i}} \boldsymbol{p_j^*} = \boldsymbol{p_i}$ in (45) implies,

$$\boldsymbol{p_i^*} = \arg\max_{\boldsymbol{p_i}\in\mathbb{R}^N} \; u_i^A\left(\boldsymbol{l_i^*}^T\frac{1}{N}\Big(\boldsymbol{p_i}+\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\boldsymbol{p_j^*}\Big),\, \frac{1}{N}\Big(\boldsymbol{p_i}+\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\boldsymbol{p_j^*}\Big)\right) \quad (46)$$

Because of (43) Eq. (46) also implies the following,

$$\begin{aligned}(\boldsymbol{\pi_i^*},\, \boldsymbol{p_i^*}) = &\arg\max_{(\boldsymbol{\pi_i},\,\boldsymbol{p_i})\in\mathbb{R}_+^N\times\mathbb{R}^N} \; u_i^A\Big(\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i) - (\boldsymbol{p_{i+1}^*} \\ &-\boldsymbol{p_{i+2}^*})^T\mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*}-\boldsymbol{p_{i+2}^*}),\; \hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i)\Big) \quad (47)\end{aligned}$$

Furthermore, since u_i^A is strictly decreasing in the tax (see (1)), Eq. (47) also implies the following,

$$\begin{aligned}(\boldsymbol{\pi_i^*},\boldsymbol{p_i^*}) = &\arg\max_{(\boldsymbol{\pi_i},\boldsymbol{p_i})\in\mathbb{R}_+^N\times\mathbb{R}^N} \; u_i^A\Big(\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i)+(\boldsymbol{p_i}-\boldsymbol{p_{i+1}^*})^T \\ &\mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i}-\boldsymbol{p_{i+1}^*})-(\boldsymbol{p_{i+1}^*}-\boldsymbol{p_{i+2}^*})^T\mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*}-\boldsymbol{p_{i+2}^*}), \\ &\hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i)\Big), \quad i\in\mathcal{N} \quad (48)\end{aligned}$$

Eq. (48) implies that, if the message exchange and allocation is done according to the game form defined in Section 3.1, then user $i, i\in\mathcal{N}$, maximizes its utility at $\boldsymbol{m_i^*}$ given that all other users $j\in\mathcal{N}\backslash\{i\}$ use their respective messages $\boldsymbol{m_j^*}, j\in\mathcal{N}\backslash\{i\}$. This implies that a message profile $\boldsymbol{m}^*$ that is a solution to (41)–(44) is a NE corresponding to the aforementioned game form. Furthermore, it follows from (41)–(44) that the allocation at $\boldsymbol{m}^*$ is,

$$\hat{\boldsymbol{p}}(\boldsymbol{m}^*) = \frac{1}{N}\sum_{i\in\mathcal{N}}\boldsymbol{p_i^*} = \hat{\boldsymbol{p}}^*, \quad (49)$$

and for each $i\in\mathcal{N}$,

$$\boldsymbol{l_i}(\boldsymbol{m}^*) = \boldsymbol{\pi_{i+1}^*} - \boldsymbol{\pi_{i+2}^*} = \boldsymbol{l_i^*}, \quad (50)$$

$$\begin{aligned}\hat{t}_i(\boldsymbol{m}^*) = &\boldsymbol{l_i}^T(\boldsymbol{m}^*)\hat{\boldsymbol{p}}(\boldsymbol{m}^*)+(\boldsymbol{p_i^*}-\boldsymbol{p_{i+1}^*})^T\mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*}-\boldsymbol{p_{i+1}^*}) \\ &-(\boldsymbol{p_{i+1}^*}-\boldsymbol{p_{i+2}^*})^T\mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*}-\boldsymbol{p_{i+2}^*}) = \boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}^* = \hat{t}_i^* \quad (51)\end{aligned}$$

It follows from (48)–(51) that the set of solutions $\boldsymbol{m}^*$ of (41)–(44) is exactly the set of NE corresponding to the game form of Section 3.1 that result in the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$. This completes the proof of Claim 6 and hence the proof of Theorem 2.

□

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