Finite-buffer bulk service queue under Markovian service process

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ABSTRACT

We consider a single-server finite-buffer queue with renewal input and Markovian service process where server serves customers in batches according to general bulk service rule. Queue length distributions at pre-arrival and arbitrary epochs have been obtained along with some important performance measures such as, probability of blocking or loss of customers, mean queue lengths, mean waiting times, etc.

Categories and Subject Descriptors

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Keywords

Markovian service process, general independent arrival, queue, finite-buffer, general bulk service rule

1. INTRODUCTION

Bulk service queues have received considerable attention due to their wide applications in several areas including computer-communication, telecommunication, transportation and manufacturing systems. For example, in ATM networks with multiple input links where each link may serve messages that consist of several packets. Besides applications in telecommunication systems, bulk service queues have a wide range of applications in several areas including transportation systems, automatic manufacturing systems, etc. Chaudhry and Templeton [11], Medhi [18, 19] and Dshalalow [12] provide an extensive discussion of bulk-service systems. In such queues customers are served by a single server in batches of maximum size 'b' with a minimum threshold size 'a'. Such type of service rule is referred to as the general bulk service rule. In recent years few authors have analyzed this model, see *e.g.* Gold and Tran-Gia [8], Chaudhry and Gupta [2]. For the more general model $GI/G^{(1,b)}/1/N$, Hébuterne and Rosenberg [9] have obtained relations among queue length distributions at various epochs. Chakravarthy [6, 10] analyzes $GI/PH^{(1,b)}/1/b$ and $MAP/G^{(1,b)}/1/b$ queueing systems, respectively. Later Gupta and Vijaya Laxmi [15, 22] carried out the analysis of $MAP/G^{(a,b)}/1/N$ and $GI/M^{(1,b)}/1/N$ queues, respectively.

Queueing models with non-renewal arrivals and service processes are often used to model networks of complex computer and communication systems. Markovian arrival process (MAP) and batch Markovian arrival process ($BMAP$) is used to capture correlation effect in the inter arrival times. $BMAP$ is a convenient representation of the versatile Markovian point process (N-process), was introduced and discussed by Neuts [20]. Later on it was more formalized by Lucantoni et al. [17], Lucantoni [16] and are referred to as MAP and $BMAP$, respectively. Considerable amount of literature is available on the queueing systems with MAP or $BMAP$ arrivals. Like these non-renewal arrival processes, Markovian service process (MSP) is a versatile service process which can capture the correlation among successive service times. For details of MSP readers are referred to Bocharov [4], Albores and Tajonar [1] and Gupta and Banik [14]. Whereas in [4], the analysis of finite and infinite-buffer $G/MSP/1/r$ ($r \leq \infty$) queue has been performed, in [1], $GI/MSP/c/r$ queue has been analyzed, and in [14], $GI/MSP/1$ queue with finite- and infinite-buffer is discussed along with its computational procedure. Recently, there are some studies have been done on departure process of $MAP/MSP/1$ type queueing systems, e.g., see Zhang et al. [23] and references therein. In [23] particularly they have studied departure process of $BMAP/MAP/1$ queue using ETAQA truncation.

In this paper, we carry out the analytic analysis of the $GI/MSP^{(a,b)}/1/N$ queue using the methods of supplementary variable and embedded Markov chain. As stated earlier, the service to the queueing system is provided in batches of minimum size a and maximum b ($1 \leq a \leq b \leq N$). We obtain the steady-state queue length distributions at pre-arrival and arbitrary epochs. The model discussed in this

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paper is more general in the sense that several models discussed earlier in the literature becomes special cases of the present one, see, e.g., $M/G^{(a,b)}/1/N$ [2], $GI/M^{[b]}/1/N$ [22], $GI/PH^{(a,N)}/1/N$ [6], $GI/PH^{(1,b)}/1/N$, $PH/PH^{(a,b)}/1/N$ etc. It may be remarked that recently the analysis of the finite-buffer queue with renewal input and batch Markovian service process ($GI/BMSP/1/N$) has been carried out by Banik. et. al [3]. The queueing model analyzed in this paper $GI/MSP^{(a,b)}/1/N$ refers to the general bulk service (a, b)-rule described above on the other hand $GI/BMSP/1/N$ queueing system [3] refers to variable batch-size service capacity rule. Therefore, the queueing model analyzed in this paper is quite different from [3] except only one special case of fixed-size batch service capacity under Markovian service process the queueing models $GI/MSP^{(a,b)}/1/N$ and $GI/BMSP/1/N$ will become similar.

2. DESCRIPTION OF THE MODEL

Let us consider a single-server finite-buffer queueing system wherein inter-arrival times are independent and identically distributed (i.i.d.) random variables (r.vs.) with distribution function (DF) $A(x)$, probability density function $a(x)$, Laplace-Stieltjes transform (LST) $A^*(\theta)$ and mean inter-arrival time $1/\lambda$. The service process of the queueing system is Markovian and is governed by an underlying m-state Markov chain having transition rate $L_{ij},\ 1 \le i, j \le m$, with a transition from state i to j without service completion and having transition rate $M_{ij},\ 1 \le i, j \le m$, with a transition from state i to j with a service completion. The matrix $\mathbf{L} = [L_{ij}]$ has nonnegative off-diagonal and negative diagonal elements, and the matrix $\mathbf{M} = [M_{ij}]$ has nonnegative elements, and both have at least one positive entry. The fundamental service rate of the stationary MSP is given by $\mu^* = \overline{\boldsymbol{\pi}}\mathbf{Me}$, where $\mathbf{e}$ is the $m \times 1$ column vector with all its elements equal to 1. The case when server remains idle for a certain time interval then a customer enters, and the service process starts with the initial phase distribution given by $f_j,\ j = 1, 2, \ldots, m,\ \sum_{j=1}^{m} f_j = 1$, independently of the path followed by the previous service period. Therefore, MSP is characterized by the matrices $\mathbf{L},\ \mathbf{M}$ along with the vector $\mathbf{f} = (f_1, f_2, \ldots, f_m)$. For further illustration on MSP readers are referred to Bocharov [4] and Albores and Tajonar [1].

The customers are served in batches according to a Markovian service process. Let N be the capacity of the queue excluding those who are in service. The service takes place in batches of maximum size b with a minimum threshold equal to a $(1 \le a \le b \le N)$. However, if fewer than a (≥ 1) customers are present in the queue, the server waits till the number of customers in the queue reaches a and then initiates service for that group of customers. The offered load ρ is defined as $\rho = \lambda/b\mu^*$.

The state of the system at time t is described by the following random variables:

- $N_t = i$, where i $(0 \le i \le N)$ is number of customers present in the queue excluding those belonging to a batch currently in service,
- $J_t = \{i, j\}$ server is in the i-th $(1 \le i \le m)$ phase of service process where server may be idle $(j = 0)$ or busy $(j = 1)$,
- U_t= the remaining inter-arrival time for the next arrival.

We now define the joint probability densities of queue length N_t, state of the server J_t and the remaining inter-arrival time U_t, respectively, by

$$\begin{aligned} p_{i,j}(x,t)\Delta x &= P\{N_t = i,\ J_t = (j,0),\ x < U_t < x + \Delta x\}, \\ &\qquad 0 \le i \le a-1,\ 1 \le j \le m,\ x \ge 0, \\ \pi_{n,j}(x,t)\Delta x &= P\{N_t = n,\ J_t = (j,1),\ x < U_t < x + \Delta x\}, \\ &\qquad 0 \le n \le N,\ 1 \le j \le m,\ x \ge 0. \end{aligned}$$

As we discuss the model in steady state, i.e., when $t \to \infty$ the above probabilities will be denoted by
$p_{i,j}(x) = \lim_{t\to\infty} p_{i,j}(x,t)$ and $\pi_{n,j}(x) = \lim_{t\to\infty} \pi_{n,j}(x,t)$. Further, let $\mathbf{p}_i(x)$ be the row vectors of order $1 \times m$ whose j-th component $p_{i,j}(x)$ denotes the probability of i customers in the queue, server idle, and the service process in phase j while remaining inter-arrival time equal to x. Similarly, let $\boldsymbol{\pi}_n(x)$ be the row vectors of order $1 \times m$ whose j-th component $\pi_{n,j}(x)$ denotes the probability of n customers in the queue, server busy, and the service process in phase j while remaining inter-arrival time equal to x.

3. ANALYSIS OF THE MODEL

3.1 Queue length distribution at arbitrary epoch

To obtain queue length distribution at arbitrary epoch we use the supplementary variable method and relate the state of the system at two consecutive time epochs t and $t + \Delta t$. Using probabilistic arguments, we get a set of differential-difference equations. Taking limit as $t \to \infty$ and using matrices and vector notations, we get

$$-\frac{d}{dx}\mathbf{p}_0(x) = \boldsymbol{\pi}_0(x)\mathbf{M}, \tag{1}$$

$$-\frac{d}{dx}\mathbf{p}_i(x) = \boldsymbol{\pi}_i(x)\mathbf{M} + \mathbf{p}_{i-1}(0)a(x), \quad 1 \le i \le a-1, \tag{2}$$

$$-\frac{d}{dx}\boldsymbol{\pi}_0(x) = \boldsymbol{\pi}_0(x)\mathbf{L} + \sum_{j=a}^{b} \boldsymbol{\pi}_j(x)\mathbf{M} + \mathbf{p}_{a-1}(0)a(x) \tag{3}$$

$$-\frac{d}{dx}\boldsymbol{\pi}_n(x) = \boldsymbol{\pi}_n(x)\mathbf{L} + \Upsilon(n+b \le N)\boldsymbol{\pi}_{n+b}(x)\mathbf{M} + \boldsymbol{\pi}_{n-1}(0)a(x), \quad 1 \le n \le N-1, \tag{4}$$

$$-\frac{d}{dx}\boldsymbol{\pi}_N(x) = \boldsymbol{\pi}_N(x)\mathbf{L} + \Big(\boldsymbol{\pi}_{N-1}(0) + \boldsymbol{\pi}_N(0)\Big)a(x), \tag{5}$$

where $\Upsilon(n+b \le N)$ is an indicator function and will be equal to 1 when the inequality $n + b \le N$ is satisfied and will be equal to 0 when the inequality violated. The terms $\mathbf{p}_i(0)$ and $\boldsymbol{\pi}_n(0)$ denote $1 \times m$ vectors whose i-th component are the respective rates of entering to that state with remaining inter-arrival time equal to zero. Let us define the Laplace transforms of $\mathbf{p}_i(x)$ and $\boldsymbol{\pi}_n(x)$ as

$$\mathbf{p}_n^*(\theta) = \int_0^\infty e^{-\theta x}\mathbf{p}_n(x)\ dx, \quad \boldsymbol{\pi}_n^*(\theta) = \int_0^\infty e^{-\theta x}\boldsymbol{\pi}_n(x)\ dx,$$

so that

$$\mathbf{p}_n \equiv \mathbf{p}_n^*(0) = \int_0^\infty \mathbf{p}_n(x)\ dx, \quad \boldsymbol{\pi}_n \equiv \boldsymbol{\pi}_n^*(0) = \int_0^\infty \boldsymbol{\pi}_n(x)\ dx. \tag{6}$$

Multiplying equations (1)-(5) by $e^{-\theta x}$ and integrating w.r.t. x over 0 to ∞, we obtain

$$\begin{aligned}
-\theta \mathbf{p}_0^*(\theta)+\mathbf{p}_0(0) &= \boldsymbol{\pi}_0^*(\theta)\mathbf{M}, & (7)\\
-\theta \mathbf{p}_i^*(\theta)+\mathbf{p}_i(0) &= \boldsymbol{\pi}_i^*(\theta)\mathbf{M}+\mathbf{p}_{i-1}(0)A^*(\theta),\\
&\qquad 1\le i\le a-1, & (8)\\
-\theta \boldsymbol{\pi}_0^*(\theta)+\boldsymbol{\pi}_0(0) &= \boldsymbol{\pi}_0^*(\theta)\mathbf{L}+\sum_{j=a}^{b}\boldsymbol{\pi}_j^*(\theta)\mathbf{M}\\
&\quad +\mathbf{p}_{a-1}(0)A^*(\theta), & (9)\\
-\theta \boldsymbol{\pi}_n^*(\theta)+\boldsymbol{\pi}_n(0) &= \boldsymbol{\pi}_n^*(\theta)\mathbf{L}+\Upsilon(n+b\le N)\boldsymbol{\pi}_{n+b}^*(\theta)\mathbf{M}\\
&\quad +\boldsymbol{\pi}_{n-1}(0)A^*(\theta),\ 1\le n\le N-1, & (10)\\
-\theta \boldsymbol{\pi}_N^*(\theta)+\boldsymbol{\pi}_N(0) &= \boldsymbol{\pi}_N^*(\theta)\mathbf{L}+\Big(\boldsymbol{\pi}_{N-1}(0)\\
&\quad +\boldsymbol{\pi}_N(0)\Big)A^*(\theta). & (11)
\end{aligned}$$

One important result is listed below in the form of a lemma using equations (7)-(11).

LEMMA 1.

$$\sum_{i=0}^{a-1}\mathbf{p}_i(0)\mathbf{e}+\sum_{n=0}^{N}\boldsymbol{\pi}_n(0)\mathbf{e} = \lambda. \qquad (12)$$

The left hand side denotes the mean number of entrances into the system per unit of time and is obviously equal to the mean arrival rate λ.

Proof. *Post-multiplying equations (7)-(11) by the vector* $\mathbf{e}$, *adding them, using* $(\mathbf{L}+\mathbf{M})\mathbf{e}=\mathbf{0}$, *we obtain after simplification*

$$\sum_{i=0}^{a-1}\mathbf{p}_i^*(\theta)\mathbf{e}+\sum_{n=0}^{N}\boldsymbol{\pi}_n^*(\theta)\mathbf{e} = \frac{1-A^*(\theta)}{\theta}\Big(\sum_{i=0}^{a-1}\mathbf{p}_i(0)+\sum_{n=0}^{N}\boldsymbol{\pi}_n(0)\Big)\mathbf{e}. \qquad (13)$$

Taking the limit as $\theta\to 0$ and applying normalizing condition, i.e., $\sum_{i=0}^{a-1}\mathbf{p}_i\mathbf{e}+\sum_{n=0}^{N}\boldsymbol{\pi}_n\mathbf{e}=1$, *we obtain the desired result.*

3.2 Queue length distribution at pre-arrival epoch

Consider the system just before arrival epochs which are taken as embedded points. Let $t_0, t_1, t_2, ...$ be the time epochs at which arrivals occur and t_n^- the time epochs just before the arrival instant t_n. The inter-arrival times $T_{n+1} = t_{n+1}-t_n,\ n=0,1,2,\dots$ are i.i.d.r.vs. with common distribution function $A(x)$. The state of the system at t_i^- is defined as $\{N_{t_i^-}, J_{t_i^-}\}$ where $N_{t_i^-}$ and $J_{t_i^-}$ are the same as defined in Section 2. In the limiting case, we define the following probability densities:

$$\begin{aligned}
p_{i,j}^- &= \lim_{i\to\infty} P\{N_{t_i^-}=i\ J_{t_i^-}=(j,0)\},\\
&\qquad 0\le i\le a-1,\ 1\le j\le m,\\
\pi_{n,j}^- &= \lim_{i\to\infty} P\{N_{t_i^-}=n\ J_{t_i^-}=(j,1)\},\\
&\qquad 0\le n\le N,\ 1\le j\le m,
\end{aligned}$$

where $p_{i,j}^-$ represents the probability that there are i customers in the queue just prior to an arrival epoch of a customer when the server is idle and the phase of the service process j. Similarly, $\pi_{n,j}^-$ denotes the probability that there are n customers in the queue just prior to an arrival epoch of a customer when the server is busy and phase of the service process j. Let $\mathbf{p}_i^-$ and $\boldsymbol{\pi}_n^-$ be the row vectors of order $1\times m$ whose j-th components are $p_{i,j}^-$ and $\pi_{n,j}^-$, respectively.

Let $\mathbf{S}_k$ $(k\ge 0)$ denote an $m\times m$ matrix whose (i,j)-th element represents the conditional probability that k batches have been served during an inter-arrival time and the underlying Markov chain of the service process is in phase j just before the arrival given that the underlying Markov chain was in phase i at the previous pre-arrival epoch.

Now observing the state of the system at two consecutive embedded points, we have an embedded Markov chain whose state space is $\Omega=\{(i,j,0)\cup(n,j,1),\ 0\le i\le a-1,\ 0\le n\le N,\ 1\le j\le m\}$. We construct the one step transition probability matrix (TPM) $\mathcal{P}$ for an idealistic case, that is, when N is an integer multiple of b. Let us assume that $N=rb$, where $r\ge 1$ is an integer. The TPM for this case is given by

states	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\cdots$	$\underline{a-1}$	0	1	2	$\cdots$	a	$\cdots$	b	$\cdots$	$a+b$	$\cdots$	$2b$	$\cdots$	N
$\underline{0}$	$\mathbf{0}$	$\mathbf{I}_m$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\underline{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{I}_m$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\underline{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{a-2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{I}_m$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\underline{a-1}$	$\mathbf{B}_0$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_0$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
0	$\mathbf{0}$	$\mathbf{B}_0$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_0$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{B}_0$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a-1$	$\mathbf{B}_1$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_1$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$b-1$	$\mathbf{B}_1$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_1$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
b	$\mathbf{0}$	$\mathbf{B}_1$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_1$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a+b-1$	$\mathbf{B}_2$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_2$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_1$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2b-1$	$\mathbf{B}_2$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_2$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_1$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$
$2b$	$\mathbf{0}$	$\mathbf{B}_2$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_2$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N-1$	$\mathbf{B}_r$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_r$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_{r-1}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_{r-2}$	$\cdots$	$\mathbf{S}_0$
N	$\mathbf{B}_r$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_r$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_{r-1}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_{r-2}$	$\cdots$	$\mathbf{S}_0$

where $\underline{\mathbf{i}}$ denotes the states $\{(i, j, 0),\ 0 \le i \le a-1,\ 1 \le j \le m\}$ and $\mathbf{n}$ represents the states $\{(n, j, 1),\ 0 \le n \le N,\ 1 \le j \le m\}$. $\mathbf{B}_k$'s are obtained by

$$\mathbf{B}_k = \Big(\mathbf{I}_m - \sum_{i=0}^{k} \mathbf{S}_i\Big)\mathbf{ef} = \mathbf{B}^{00} - \sum_{i=0}^{k} \mathbf{S}_i \mathbf{B}^{00},\ k \ge 0,$$

where $\mathbf{B}^{00} = \mathbf{ef}$ is stochastic and has the invariant vector $\mathbf{f}$. The TPM for other cases, that is, when N is not an integral multiple of b can be constructed similarly the only thing one has to remember is that the elements of the $(N-1)$-th row will be equal to those of the N-th row due to the restriction of finite-buffer space. The other two special cases of the TPM when N is not an integral multiple of b can be constructed in a similar manner.

The matrices $\mathbf{S}_k$, in general, for arbitrary inter-arrival time distribution require numerical integration and can be carried out along the lines proposed by Lucantoni [16]. However, when the inter-arrival time distributions are of phase type (PH-distribution), these matrices can be evaluated without any numerical integration, Neuts ([21], *pg.* 67-70). It may be noted here that various inter-arrival time distributions arising in practical applications can be approximated by PH-distributions. The following theorem gives a procedure for the computation of the matrices $\mathbf{S}_k$.

THEOREM 3.1. *Let $A(x)$ follow a PH-distribution with irreducible representation $(\boldsymbol{\alpha},\ \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension γ, then the matrices $\mathbf{S}_n$ are given by*

$$\begin{aligned} \mathbf{S}_n &= \mathbf{U}_n(\mathbf{I}_m \otimes \mathbf{T}^0), \quad 0 \le n \le N-1, \qquad (14)\\ \textit{where } \mathbf{U}_0 &= -(\mathbf{I}_m \otimes \boldsymbol{\alpha})[\mathbf{L} \otimes \mathbf{I}_\gamma + \mathbf{I}_m \otimes \mathbf{T}]^{-1},\\ \mathbf{U}_n &= -\mathbf{U}_{n-1}(\mathbf{M} \otimes \mathbf{I}_\gamma)[\mathbf{L} \otimes \mathbf{I}_\gamma + \mathbf{I}_m \otimes \mathbf{T}]^{-1},\\ & \quad 1 \le n \le N-1, \end{aligned}$$

with $\mathbf{T}^0 = -\mathbf{Te}$ and the symbol $\otimes$ denotes the Kronecker product of two matrices.

Proof: *See Neuts [21] and Gupta and Vijaya Laxmi [15] for PH-type service.*

Note that the pre-arrival epoch probabilities $\mathbf{p}_i^-$ $\boldsymbol{\pi}_n^-$, $(0 \le n \le N)$ can be evaluated by solving the system of equations: $(\boldsymbol{\pi}_0^-, \boldsymbol{\pi}_1^-, \ldots, \boldsymbol{\pi}_N^-) = (\boldsymbol{\pi}_0^-, \boldsymbol{\pi}_1^-, \ldots, \boldsymbol{\pi}_N^-)\mathcal{P}$. Algorithms such as, the GTH algorithm (Grassmann, Taksar and Heyman [13]) can be used to solve the system of equations under consideration as it works well even for large number of states.

3.2.1 Relations between queue length distributions at arbitrary and pre-arrival epochs

We first relate the pre-arrival epoch probabilities with the probabilities $\mathbf{p}_n(0)$ and $\boldsymbol{\pi}_n(0)$ which are given by

$$\begin{aligned} \mathbf{p}_i^- &= \lambda^{-1}\mathbf{p}_i(0), \quad 0 \le i \le a-1. \qquad (15)\\ \boldsymbol{\pi}_n^- &= \lambda^{-1}\boldsymbol{\pi}_n(0), \quad 0 \le n \le N. \qquad (16) \end{aligned}$$

Now we are in a position to express arbitrary epoch probabilities in terms of pre-arrival epoch probabilities. Setting $\theta = 0$ in equations (11) and (10), and using (16), we obtain

$$\begin{aligned} \boldsymbol{\pi}_N &= \lambda \boldsymbol{\pi}_{N-1}^-(-\mathbf{L})^{-1}, \qquad (17)\\ \boldsymbol{\pi}_n &= \Big(\Upsilon(n+b \le N)\boldsymbol{\pi}_{n+b}\mathbf{M} + \lambda(\boldsymbol{\pi}_{n-1}^- - \boldsymbol{\pi}_n^-)\Big)(-\mathbf{L})^{-1},\\ & \qquad n = N-1, N-2, \ldots, 1. \qquad (18)\\ \boldsymbol{\pi}_0 &= \Big(\sum_{j=a}^{b} \boldsymbol{\pi}_j\mathbf{M} + \lambda(\mathbf{p}_{a-1}^- - \boldsymbol{\pi}_0^-)\Big)(-\mathbf{L})^{-1}. \qquad (19) \end{aligned}$$

To obtain the probabilities $\mathbf{p}_i$ $(0 \le i \le a-1)$, we adopt a slightly different approach. For that we first differentiate equation (7) and (8) then set $\theta = 0$, and obtain

$$\begin{aligned} \mathbf{p}_0 &= -\boldsymbol{\pi}_0^{*(1)}(0)\mathbf{M}, \qquad (20)\\ \mathbf{p}_i &= \mathbf{p}_{i-1}^- - \boldsymbol{\pi}_i^{*(1)}(0)\mathbf{M}, \quad 1 \le i \le a-1. \qquad (21) \end{aligned}$$

As we see that we need to know the values $\boldsymbol{\pi}_n^{*(1)}(0)$ $(0 \le i \le a-1)$, and for that we differentiate (11), (10), (9) and set

$\theta = 0$. We thus obtain the following results:

$$\pi_N^{*(1)}(0) = \Big(\pi_N - \pi_{N-1}^- - \pi_N^-\Big)(-\mathbf{L})^{-1}, \quad (22)$$

$$\pi_n^{*(1)}(0) = \Big(\Upsilon(n+b \le N)\pi_{n+b}^{*(1)}(0)\mathbf{M} + \pi_n - \pi_{n-1}^-\Big)(-\mathbf{L})^{-1}, \quad n = N-1, N-2, \cdots, 1 \quad (23)$$

$$\pi_0^{*(1)}(0) = \Big(\sum_{j=a}^{b} \pi_j^{*(1)}(0)\mathbf{M} + \pi_0 - \mathbf{p}_{a-1}^-\Big)(-\mathbf{L})^{-1}. \quad (24)$$

From these equations it is possible to evaluate all $\pi_i^{*(1)}(0)$ $(0 \le i \le a-1)$ through recursive calculations, and then substituting these values in the above equations (20) and (21), one can obtain the unknown probabilities $\mathbf{p}_i$ $(0 \le i \le a-1)$.

As state probabilities at various epochs are known, performance measures of the queue can be easily obtained. The average number of customers in the queue at an arbitrary epoch $(L_q) = \sum_{n=0}^{N} n\pi_n\mathbf{e} + \sum_{i=0}^{a-1} i\mathbf{p}_i\mathbf{e}$, and the probability of loss or blocking $(P_{loss}) = \pi_N^-\mathbf{e}$. Applying Little's rule, mean waiting time in the queue (W_q) can be obtained and is equal to $L_q/\lambda^{'}$, where $\lambda^{'} = \lambda(1 - P_{loss})$ is the effective arrival rate of customers.

4. NUMERICAL RESULTS

To demonstrate the applicability of the results obtained in previous sections, some numerical experiments have been carried out.

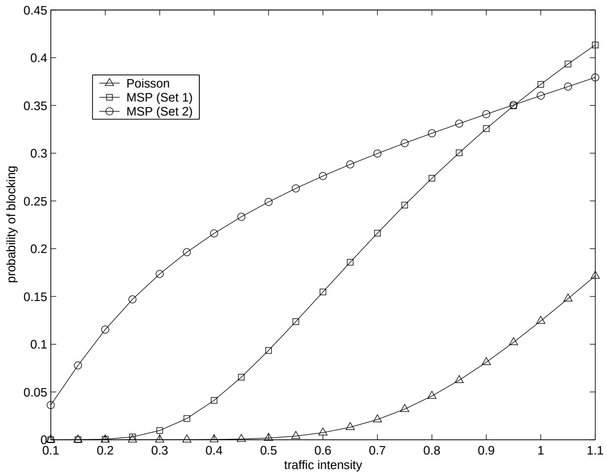

Figure 1: ρ **versus** P_{loss}

In Figure 1 and 2, we have compared the performance between MSP and Poisson process in a $E_2/MSP^{(4,8)}/1/30$ queue. For this we have taken two sets of MSP representation given by

Set 1: $\mathbf{L} = \begin{bmatrix} -6.9375 & 0.9375 \\ 0.0625 & -0.1958 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 6.0 & 0.0 \\ 0.0 & 0.1333 \end{bmatrix}$,

and

Set 2: $\mathbf{L} = \begin{bmatrix} -0.542409519 & 0.0037279 & 0.00 \\ 0.004349217 & -0.02298872 & 0.000621317 \\ 0.00 & 0.001242633 & -2.2696700 \end{bmatrix}$,

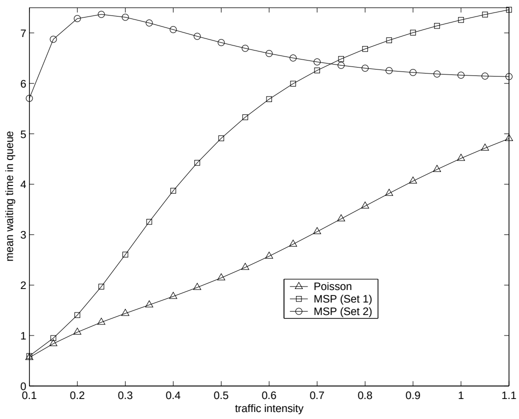

Figure 2: ρ **versus** W_q

$$\mathbf{M} = \begin{bmatrix} 0.020503453 & 0.00 & 0.518178166 \\ 0.00 & 0.017396869 & 0.000621317 \\ 2.259107688 & 0.004970534 & 0.004349217 \end{bmatrix}.$$

The above MSP's have the same service rate $(\mu^* = 0.5)$ and $\mathbf{f}$ is taken as $\begin{bmatrix}1.0 & 0.0\end{bmatrix}$ and $\begin{bmatrix}1.0 & 0.0 & 0.0\end{bmatrix}$, respectively. Set 1 and Set 2 have lag 2 correlation coefficient equal to 0.143181 and 0.271908, respectively. For E_2 inter-arrival time, PH-type representation is taken as $\boldsymbol{\alpha} = \begin{bmatrix}1.0 & 0.0\end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} -\gamma & \gamma \\ 0.0 & -\gamma \end{bmatrix}$ with $\lambda = \gamma/2.0$ and by suitably varying γ one can get various values of ρ. In Figure 1, we have plotted the blocking probabilities against traffic intensities in a $E_2/MSP^{(4,8)}/1/30$ queue with MSP taken as (i) Poisson with $\mathbf{L} = -0.5$, $\mathbf{M} = 0.5$ and $\mathbf{f} = 1.0$ so that $\mu^* = 0.5$, (ii) Set 1 MSP, and (iii) Set 2 MSP, respectively. It can be seen from the figure that as ρ increases, for case (i) and (ii) P_{loss} increases linearly; but for case (iii) initial P_{loss} is very high after that P_{loss} took a steady linear increase. Further, one may note that P_{loss} increases as correlation coefficient increases upto certain level of traffic intensity, say $\rho = 1.0$. Under the same conditions as stated above, in Figure 2 we have plotted mean waiting time in queue against traffic intensity. It can be seen from the figure that as ρ increases, for case (i) and (ii) W_q increases linearly; but for case (iii), initially W_q is very high after that it asymptotically approaches to a steady value. This is due to the fact that high correlation among service times is affecting the mean waiting time in queue. Also upto certain level of traffic intensity $(\rho = 0.7)$ mean waiting time for case (iii) is higher than the other two cases and after that it becomes lower than the results presented in case (ii). From the analysis presented in the figures, it can be concluded that performance measures are not only affected by the arrival and service patterns but also correlation among service times plays a crucial role.

5. CONCLUSIONS AND FUTURE SCOPE

In this paper, we have analyzed $GI/MSP^{(a,b)}/1$ queue with finite-buffer. Several other variations of batch service queueing models under Markovian service process can be analyzed in this direction. For example, the analysis of the corresponding infinite-buffer queue, or $GI^{[X]}/MSP^{(a,b)}/1/N$

queueing system, or $BMAP/MSP^{(a,b)}/1/N$ queue, etc.

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