# Queueing networks and conditional product-forms

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## ABSTRACT

Product-forms are well-known in the community of performance evaluation because they allow the computation of the stationary state probabilities of large models that would otherwise be intractable. Roughly speaking, a product-form model consists of several interacting components. Under some conditions, the steady-state probabilities of these components can be derived in isolation *as if* the interactions with the remaining parts of the system are modelled by independent Poisson processes. The steady-state distribution of the joint model can be derived as the (normalised) product of the distributions of the isolated components. In the last few years some authors have introduced the idea of higher order product-forms or conditional product-forms that differ from ordinary product-forms because once the components of a model are isolated, the interactions with the rest of the system are not anymore seen as independent Poisson processes. However, to the best of our knowledge, up to now these methods have been applied only to approximate non-product-form models. In this paper we propose for the first time two classes of feed-forward queueing network models whose stationary distributions have conditional product-forms but *not* ordinary product-forms.

## Categories and Subject Descriptors

C.4 [**PERFORMANCE OF SYSTEMS**]: Modeling techniques

## General Terms

Performance

## Keywords

Queueing theory, Product-form solutions, Markov modulated models


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## 1. INTRODUCTION

Stochastic models are important for the performance evaluation of computer and communication systems. In particular, discrete state space Markovian models, i.e., those models whose underlying stochastic process is a Discrete or Continuous Time Markov Chain (DTMC/CTMC), are interesting thanks to their versatility and the wide availability of algorithms for automatic analysis (see e.g. [22]). Nevertheless, the monolithic specification of a Markov model is prone to errors due to the complexity of modern hardware and software architectures. To overcome this problem, several formalisms have been introduced in the literature to allow for a modular and hierarchical specification of the models (see, e.g. [21, 14, 13]). Unfortunately, a modular specification does not trivially lead to a modular analysis and in most of the practical cases the computation of the stochastic model underlying a composition of several sub-models (components) suffers the problem of the state space explosion. As a consequence the derivation of the performance indices becomes prohibitive if the brute force algorithms (e.g., those based on the solution of the system of global balance equations (GBE)) are applied. Roughly speaking, product-forms aim at allowing for a modular analysis of models defined in terms of cooperations of a set of (elementary) components. A well known example is represented by the BCMP queueing networks [4] where the components are the single queues. The key point of the product-form analysis is that the Markov chain underlying each component is studied in isolation, i.e., the interactions with the other components are taken into account to parametrise the isolated components. In the BCMP networks, this parametrisation is done by solving the system of traffic equations, but more sophisticated approaches may be required for more general models (in these cases we talk about the system of rate equations [10, 2, 12]). However, a common property of all standard product-form results is that the events due to the interactions with the remaining components occur according to homogeneous Poisson processes once the component is isolated. It is surprising that product-forms give exact results because, in general, in the joint models the occurrences of these events do not follow a Poisson process. For instance, in an open BCMP queueing network, once a queue is studied in isolation, all its parameters are known with the exception of the arrival rate for each class of customers. These rates are derived by solving the linear system of traffic equations that depends on the network routing characteristics. The same rates are then seen as the rates of independent homogeneous Poisson processes that feed the isolated queueing system even if it is well-known that in general the arrival streams at the same queue embedded in the network are not homogeneous Poisson processes.

### *Related work.*

Some recent papers have addressed the problem of finding a product-form characterisation in which once a component is isolated, its interactions with the rest of the system cannot be assumed to occur according to independent Poisson processes. In this line an important contribution is given in [5] where the author introduces the notion of *higher-order product-forms* with the aim of approximating the non-product-form cooperations of stochastic automata. The type of synchronisation that is considered is that defined by the Kronecker's algebra with the product operator $\otimes$. In this setting a product-form solution holds if the joint distribution is given by the Kronecker product of the vectors of probability distributions associated with each component. The author aims at deriving a product-form approximation of cooperating components by finding the probability vectors that minimise the norm of the residuals of the joint GBE system. It is noticed that the approximations can be improved if we associate a matrix $N_i \times O$ with the $i$-th component, where $N_i$ is the number of the $i$-th component states and $O$ is the product-form order. However, the author does not provide any model whose steady-state distribution has an exact higher-order product-form. In [6] the author proposes a technique for the exact and approximated analysis of Markov modulated models based on a conditional product-form expression. Let $s$ and $f$ be generic states of the modulated and the modulating processes, respectively. Then we can always write $\pi(f, s) = \pi_F(f)\pi^*(s|f)$, where $\pi$ is the joint stationary distribution, $\pi_F$ is the marginal distribution of the modulating process and $\pi^*$ denotes the stationary probability of observing state $s$ once it is known the state of the modulating process. In case of Markov modulated processes the approach may be efficient because the computation of $\pi_F$ is simple. Also in this paper the author does not show any model whose exact stationary distribution is such that $\pi^*(s|m) \neq \pi^*(s|m')$ if $m \neq m'$, i.e., for cases that do not have a standard product-form. In [1] we provide some algebraic conditions for conditional product-forms. Differently from [6], we deal with more general models than Markov modulated models and we avoid the computation of $\pi^*(s|m)$ whose complexity is in general the same required by the solution of the joint GBE system. The state space of one of the cooperating models is partitioned into a set of clusters (the smallest quantity possible) and the conditioning of $\pi^*$ is done on the cluster rather than on the state. As a consequence the complexity of the computation of the joint stationary distribution is lower than that required by [6]. Notice that when one can group all the states of a component into a single cluster, the quasi-reversibility [15] and Reversed Compound Agent Theorem (RCAT) [10] product-forms are obtained.

### *Contribution.*

All the previous papers on higher-order or conditional product-forms [6, 5, 1] did not show any model with a physical interpretation that enjoys a non-standard product-form. In the product-form literature we may recognise two principal lines of research: one that aims at finding general algebraic or structural properties that give a product-form solution (see, e.g., [15, 10, 19]) and the other that devotes efforts in defining new models with a physical interpretation that enjoy a product-form solution (see, e.g., [23, 17, 18]). Our work falls in the second line of research: we present a class of queueing network models whose stationary distribution can be expressed in a non-trivial conditional product-form. The first model consists of a tandem of two infinite buffer queueing systems. The former is modulated by a CTMC, i.e., its arrival and service rates depend on the state of the modulating chain. The second queue arrivals are synchronised with the departures at the first queue. The model is known to be not in (ordinary) product-form [24, 6, 3]. We give sufficient conditions for a conditional product-form. Moreover, we show by means of some examples that the result can be used to derive the conditional product-form of models with different behaviours such as with batch departures from the first queue or with negative arrivals at the second in the style of [8].

The second conditional product-form model consists of a tandem between two alternating queues and an exponential queue. The former queues alternate their working by means of mutual resets, i.e., once a queue transfers the control to the other the latter's state is chosen according to its stationary probability. The mechanism is inspired by Gelenbe and Fourneau's resets [9] although ours present some differences. First, the reset is mutual, i.e., one queue decides when to reset the other. Second, there is an alternation between the working periods of the queue that is not present in the model considered in [9]. Third, our model does not admit an ordinary product-form while the G-networks with reset does. However, we prove that a conditional product-form exists that allows for an efficient computation of the stationary distribution and the consequent derivation of the average performance indices.

### *Structure of the paper.*

Section 2 introduces the notation and reviews the main results about conditional product-forms. Section 3 studies the Markov modulated queueing network model. Section 4 considers the model with alternating queues and mutual resets. Finally, Section 5 concludes the paper.

## 2. THEORETICAL BACKGROUND

This paper deals with model synchronisations. Several formalisms have been developed to formally represent the notion of cooperating models but for the sake of maintaining a uniform framework with previous work on similar topics, we choose to express our results by means of cooperating automata and Kronecker's algebra [21, 5]. However, the results can be readily transferred to other domains such as the Markovian process algebra PEPA [14] with active/passive synchronisations. We study pairs of models $M_1$ and $M_2$ that cooperate on a finite set of labels $\mathcal{L} = \{t_1, \ldots, t_L\}$. Every label $\ell$ is associated with a pair of matrices $\mathbf{E}_{1\ell}$ and $\mathbf{E}_{2\ell}$. The entries in $\mathbf{E}_{1\ell}$ are non negative real numbers and represent the transition rates. Matrix $\mathbf{E}_{2\ell}$ is stochastic and $\mathbf{E}_{2\ell}(s_2 \rightarrow s_2')$ represents the probability that model $M_2$ synchronises on $\ell \in \mathcal{L}$ by changing its state from $s_2$ to $s_2'$, $s_2 \in \mathcal{S}_2$ and $\mathcal{S}_2$ is the denumerable set of states of component $M_2$. Matrices $\mathbf{E}_{1\tau}$ and $\mathbf{E}_{2\tau}$ model the internal, non synchronising, transitions of the components and the entries are non-negative real numbers. According to [21, 5] we assume that transitions occur after an exponentially distributed random time so that the underlying stochastic process is a CTMC whose transition rate matrix is given by the following expression:

$$\mathbf{Q} = \mathbf{E}_{1\tau} \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{E}_{2\tau} + \sum_{\ell \in \mathcal{L}} \mathbf{E}_{1\ell} \otimes \mathbf{E}_{2\ell}\,, \qquad (1)$$

where $\mathbf{I}_c$ denotes the identity matrix whose dimension is given by the cardinality of the state space of $M_c$, $c = 1, 2$. We may derive the infinitesimal generator in the standard way:

$$\overline{\mathbf{Q}} = \mathbf{Q} - \text{Diag}(\mathbf{Q}\mathbf{1}^\top)\,,$$

where $\mathbf{1}$ is a row-vector with dimension $|\mathcal{S}_1| \cdot |\mathcal{S}_2|$ and Diag$(\cdot)$ maps a vector into a diagonal matrix with appropriate dimension. Since $\mathbf{E}_{2\ell}$ is stochastic, we say that the cooperation between $M_1$ and $M_2$ is *feed-forward* and *non-blocking* (FFNB). Basically in FFNB cooperations, automaton $M_2$ can never prevent automaton $M_1$ to perform a transition. We call $M_1$ active and $M_2$ passive. Notice that in these cases we can compute the marginal stationary distribution (when it exists) of $M_1$ by considering it in isolation. Henceforth we assume that the CTMCs underlying $M_1$ and the cooperation between $M_1$ and $M_2$ are ergodic.

Definition 1 (Timed-reversed automata). *Given the active automaton $M_1$ synchronising on label set $\mathcal{L}$, we define the timed-reversed automaton $M_1^R$ as follows:*

$$\mathbf{E}^R_{1\ell}(s_1 \to s_1') = \frac{\pi_1(s_1')}{\pi_1(s_1)} \mathbf{E}_{1\ell}(s_1' \to s_1) \qquad (2)$$

*for all $s_1, s_1' \in \mathcal{S}_1$ and $\ell \in \mathcal{L} \cup \{\tau\}$, where $\pi_1$ is the marginal stationary distribution of $M_1$.*

Definition 2 (Time-reversible automata). *An active automaton $M_1$ is time-reversible if the following condition holds for all the pairs $s_1, s_1' \in \mathcal{S}_1$:*

$$\pi_1(s_1)\mathbf{E}_{1\ell}(s_1 \to s_1') = \pi_1(s_1')\mathbf{E}_{1\ell}(s_1' \to s_1)\,, \qquad (3)$$

*for every $\ell \in \mathcal{L} \cup \{\tau\}$ and where $\pi_1$ denotes the marginal stationary distribution of $M_1$.*

Notice that although every time-reversible automaton has an underlying reversible CTMC [15], the opposite is not true, since in Definition 2 the local balance equation must be satisfied for each label.

Proposition 1. *If an automaton $M_1$ is time-reversible then $M_1^R = M_1$.*

Proof. Let us apply Definition 1 to define the time-reversed automaton $M_1^R$. We notice that the right-hand-side of Equation (2) that specifies the rate $\mathbf{E}^R_{1\ell}(s_1 \to s_1')$ for $u \in \mathcal{L} \cup \{\tau\}$ and for every pair $s_1, s_1' \in \mathcal{S}_1$ is identical to the expression of $\mathbf{E}_{1\ell}(s_1 \to s_1')$ required by Equation (3). □

Before stating the main theorem, we introduce Definition 3 that extends the notion of lumping to automata.

Definition 3 (Exact lumped automata). *Given automaton $M_1$, a set of transition labels $\mathcal{L}$, and a partition of the states of $M_1$ into $\tilde{N}_1$ clusters $\tilde{\mathcal{S}} = \{\tilde{1}, \tilde{2}, \dots, \tilde{N}_1\}$, we say that $\tilde{\mathcal{S}}$ is an exact lumping for $M_1$ if it is possible to define a set of functions $\tilde{\varphi}^\ell_1 : \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \to \mathbb{R}^+$ such that:*

1. *$\forall \tilde{s}_1, \tilde{s}_1' \in \tilde{\mathcal{S}}$, $\tilde{s}_1' \neq \tilde{s}_1$, $\forall s_1 \in \tilde{s}_1$ $\sum_{s_1' \in \tilde{s}_1'} \mathbf{E}_{1\tau}(s_1 \to s_1') = \tilde{\varphi}^\tau_1(\tilde{s}_1, \tilde{s}_1')$*
2. *$\forall \ell \in \mathcal{L}, \forall \tilde{s}_1, \tilde{s}_1' \in \mathcal{S}$, $\forall s_1 \in \tilde{s}_1$ $\sum_{s_1' \in \tilde{s}_1'} \mathbf{E}_{1\ell}(s_1 \to s_1') = \tilde{\varphi}^\ell_1(\tilde{s}_1, \tilde{s}_1')$.*

*If $M_1$ is lumpable with respect to $\mathcal{S}$, we define the automaton $\tilde{M}_1$ with $\tilde{N}_1$ states as follows:*

$$\tilde{\mathbf{E}}_{1\tau}(\tilde{s}_1 \to \tilde{s}_1') = \begin{cases} \tilde{\varphi}^\tau_1(\tilde{s}_1, \tilde{s}_1') & \textit{if } \tilde{s}_1 \neq \tilde{s}_2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\tilde{\mathbf{E}}_{1\ell}(\tilde{s}_1 \to \tilde{s}_1') = \tilde{\varphi}^\ell_1(\tilde{s}_1, \tilde{s}_1') \qquad \ell \in \mathcal{L}.$$

Observe that there is not a trivial correspondence between the notion of lumping for discrete or continuous time Markov chains [16] and that given by Definition 3. Indeed, in case of lumping of Markov chains, the transition rates (probabilities) among states belonging to the same cluster can be ignored. In case of lumped automata this is not true for the matrices corresponding to synchronising transitions (see point *2* in Definition 3). Therefore, although a lumpable automata has always an underlying exact lumpable CTMC, the opposite is not true.

For brevity, we denote by $M_1 \otimes M_2$ the model given by the cooperation of $M_1$ and $M_2$ whose underlying CTMC transition rate matrix is that given by Equation (1).

Theorem 1. *Given the model $M_1 \otimes M_2$, in a feed-forward and non-blocking synchronisation. Let $M_1^R$ be the reversed automaton of $M_1$ and let $\tilde{M}_1^R$ be an exact lumping of $M_1^R$ whose clusters are $\mathcal{S} = \{\tilde{1}, \dots, \tilde{N}_1\}$ and let $\tilde{M}_1^R$ be time-reversible. Then, under ergodicity assumption, the following conditional product-form expression holds:*

$$\pi(s_1, s_2) = \tilde{\pi}^R_{M_2|\tilde{M}_1^R}(s_2|\tilde{s}_1)\pi_1(s_1)\,, \qquad (4)$$

*where $\pi$ is the steady-state distribution of $M_1 \otimes M_2$ and $\tilde{\pi}^R$ that of $\tilde{M}_1^R \otimes M_2$, $\pi_1$ that of $M_1$ and:*

$$\tilde{\pi}^R_{M_2|\tilde{M}_1^R}(s_2|\tilde{s}_1) = \frac{\tilde{\pi}^R(\tilde{s}_1, s_2)}{\tilde{\pi}_1(\tilde{s}_1)}\,,$$

*where, since the stochastic process underlying $\tilde{M}_1^R$ is a lumping of that underlying $M_1^R$, we have $\tilde{\pi}_1(\tilde{s}_1) = \sum_{s_1 \in \tilde{s}_1} \pi_1(s_1)$.*

## 2.1 Differences with ordinary product-forms

The result stated in Theorem 1 is different from those well-known in product-form theory such as the reversibility or the quasi-reversibility [15] or the Reversed Compound Agent Theorem [10]. In particular, we may review these results in case of pairwise, feed-forward cooperations by requiring that $M_1$ must be lumpable into a single state (and hence is trivially reversible). This stricter condition with respect to that required by Theorem 1 allows us to rewrite Equation (4) and obtain the standard product-form result.

## 2.2 Differences with Markovian process algebra congruences

In the field of Markovian process algebra, the notion of lumping is strictly connected to that of congruences or bisimilarities (see, e.g., [14]). The idea is that in the cooperation of two components (our automata) we may replace them with simpler equivalent ones, i.e., with components that although they have a lower number of states, they still behave in an equivalent way with respect to the originals. However, differently from what stated by Theorem 1, this only allows one to derive the *marginal* stationary distributions of the components. Theorem 1 gives a way to compute both the marginal and the *joint* stationary distributions of the components.

# 3. CONDITIONAL PRODUCT-FORM IN MARKOV MODULATED TANDEM QUEUES

In this section we consider a tandem of two queues as shown in Figure 1. The queues have infinite capacity and customers arrive from the outside at the former queue according to a Markov modulated Poisson process. Let $F$ be the modulating process. The service time at the former queue is exponentially distributed and the rate is also modulated by process $F$. Let $\mathcal{F}$ be the denumerable state space of the CTMC underlying $F$, and $f \in \mathcal{F}$ a generic state. Then, we denote the arrival and the service rates at $Q_1$ by $\lambda_1(f)$ and $\mu_1(f)$, respectively. After being served, customers leave $Q_1$ and immediately enter $Q_2$ where they are served in an exponentially distributed time with rate $\mu_2$. Both the queues have a First Come First Served discipline (FCFS). Modulating process $F$ determines the transition rates of $Q_1$ but its own behaviour is never controlled neither by the transitions nor by the current state of $Q_1$. This means that if the CTMC underlying $F$ is ergodic, its stationary probabilities can be readily derived by its analysis in isolation.

Let us consider the Markov modulated queue $Q_1$, it is well-known from the literature (see [7] and the similar results recently considered in [3]) that a product-form between the modulated (the exponential queue) and the modulating processes exists if and only if the following condition holds:

$$\exists \rho_1 \in \mathbb{R}^+ \ s.t. \ \forall f \in \mathcal{F}, \ \frac{\lambda_1(f)}{\mu_1(f)} = \rho_1 \, . \tag{5}$$

If the modulating process is ergodic, the model is stable if $\rho_1 < 1$. Let $\pi_F(f)$, with $f \in \mathcal{F}$ be the marginal stationary probability of state $f$, i.e.,

$$\pi_F(f) = \sum_{n_1=0}^{\infty} \pi_1(f, n_1) \, ,$$

where $\pi_1(f, n_1)$ denotes the joint probability of $F$ being in state $f$ and $Q_1$ having $n_1$ customers (we are not considering the interaction with $Q_2$ yet). Then, the following result holds [24, 7, 3]:

$$\pi_1(f, n_1) = \pi_F(f)(1-\rho_1)\rho_1^{n_1} \, . \tag{6}$$

We now consider the interaction between $Q_1$ (modulated by $F$) and $Q_2$. Theorem 2 is a consequence of Theorem 1 and gives sufficient conditions for the cooperation of $Q_1$ and $Q_2$ to be in conditional product-form. Later on, we will also show that previous results on product-forms cannot be applied for the computation of the steady-state distribution.

Theorem 2. *Given the tandem of queues depicted in Figure 1, where $Q_1$ is modulated by $F$. Let $f \in \mathcal{F}$ denote a state of the modulating process, $n_1, n_2 \in \mathbb{N}$ denote the population at queue $Q_1$ and $Q_2$, respectively. If the following conditions hold:*

1. *$F$ and $Q_1$ are in product-form*
2. *$F$ is reversible*

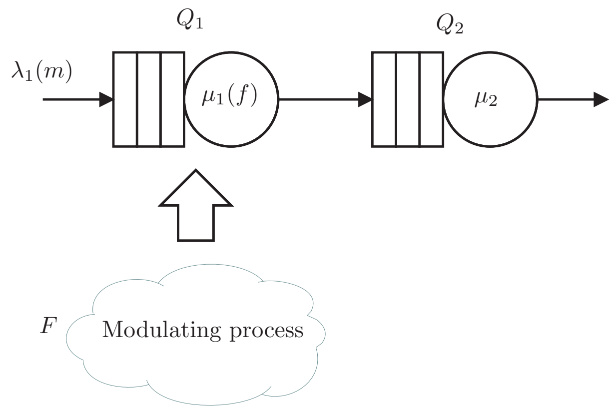


**Figure 1: Tandem of exponential queues with the first modulated by another process.**

*then, in stability, the steady-state distribution $\pi(f, n_1, n_2)$ is given by the following expression:*

$$\pi(f, n_1, n_2) = (1-\rho_1)\rho_1^{n_1}\pi_2^*(f, n_2) \, , \tag{7}$$

*where $\pi_2^*$ is the steady-state distribution of $Q_2$ under a Markov Modulated Poisson Process (MMPP) with the same transition rates of $F$ and whose intensity of the arrivals at state $f$ is $\lambda_1(f)$.*

Remark 1. *It is worth of notice that one may try to obtain the same result in a simpler way by proving that the output process of the Markov Modulated queueing network is a Markov Modulated Poisson Process. Therefore, it is not surprising that the second queue behaves as an exponential queue under MMPP arrival. This approach would not require the assumption of the reversibility of the modulating process. However, the statement of Theorem 2 is stronger than what one would obtain following this approach. Indeed, by considering the output process as specified, one could prove a result only about the marginal stationary distributions of the models and not about the joint model, as the conditional product-form does. Theorem 2 states that a sufficient condition for the stationary distribution to be the product of the marginals is that the modulating process is reversible.*

Before giving the proof of the theorem let us define the automata $M_1$ and $M_2$, underlying the modulated queue $Q_1$ and $Q_2$, respectively. The two automata synchronise on a single label $a$ that corresponds to the departure of a customer from $Q_1$ and a consequent arrival at $Q_2$. Let $q_F(f \to f')$ be a non-negative real number denoting the transition rate of the CTMC underlying the modulating process from state $f$ to $f'$. Therefore, we have:

$$\mathbf{E}_{1\tau}(f, n_1 \to f', n_1') = \begin{cases} \lambda_1(f) & \text{if } f' = f, n_1' = n_1 + 1 \\ q_F(f \to f') & \text{if } n_1' = n_1, f' \neq f \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{E}_{1a}(f, n_1 \to f', n_1')$

$$= \begin{cases} \mu_1(f) & \text{if } f' = f, n_1' = n_1 - 1 \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 1. *The automaton $M_1^R$, inverse of $M_1$ is defined as follows:*

$\mathbf{E}_{1\tau}^R(f, n_1 \to f', n_1')$

$$= \begin{cases} \mu_1(f) & \textit{if } f' = f, n_1' + 1 = n_1 \\ q_F(f \to f') & \textit{if } n_1' = n_1, f' \neq f \\ 0 & \textit{otherwise} \end{cases}$$

$\mathbf{E}_{1a}(f, n_1 \to f', n_1')$

$$= \begin{cases} \lambda_1(m) & \textit{if } f' = f, n_1' = n_1 + 1 \\ 0 & \textit{otherwise} \end{cases}$$

PROOF. Since by hypothesis $F$ and $Q_1$ are in product-form, the steady-state distribution is given by Equation (6) and we know that $\lambda(f)/\mu(f) = \rho_1$ for all $f$. Let us consider an internal transition between two states of the modulating process $\mathbf{E}_{1\tau}(f, n_1 \to f', n_1)$ with rate $q(f \to f')$ and $f \neq f'$, then:

$$\mathbf{E}_{1\tau}^R(f, n_1 \to f', n_1) = \frac{\pi_1(f', n_1)}{\pi_1(f, n_1)} q_F(f' \to f)$$
$$= \frac{\pi_F(f')(1-\rho_1)\rho_1^{n_1}}{\pi_F(f)(1-\rho_1)\rho_1^{n_1}} q_M(f' \to f) = q_F(f \to f'),$$

where the last equality follows from the hypothesis of reversibility of $F$. Let us now consider the internal transitions modelling an arrival event, i.e., $\mathbf{E}_{1\tau}(f, n_1 \to f, n_1 + 1)$, its inverse has the following rate:

$$\mathbf{E}_{1\tau}^R(f, n_1 + 1 \to f, n_1) = \frac{\pi_1(f, n_1)}{\pi_1(f, n_1 + 1)} \lambda_1(f)$$
$$= \frac{\pi_F(f)(1-\rho_1)\rho_1^{n_1}}{\pi_F(f)(1-\rho_1)\rho_1^{n_1+1}} \lambda_1(f)$$
$$= \rho_1^{-1}\lambda_1(f) = \mu_1(f),$$

where the last equality follows from Zhu's and Economou's [24, 6] results about the necessary and sufficient condition (5) for the product-form. We finally consider the synchronising transition $\mathbf{E}_{1a}(f, n_1 + 1 \to f, n_1)$ whose rate is $\mu_1(f)$. It's inverse is:

$$\mathbf{E}_{1a}^R(f, n_1 \to f, n_1 + 1) = \frac{\pi_1(f, n_1 + 1)}{\pi_1(f, n_1)} \mu_1(f)$$
$$= \frac{\pi_F(f)(1-\rho_1)\rho_1^{n_1+1}}{\pi_F(f)(1-\rho_1)\rho_1^{n_1}} \mu_1(f)$$
$$= \rho_1\mu_1(f) = \lambda_1(f),$$

where we applied again Equation (5). □

LEMMA 2. *Model $M_1^R$ is exactly lumpable into an automaton $\tilde{M}_1^R$ with a number of states equal to the cardinality of the modulating process's state space and whose transitions are defined as follows:*

$$\tilde{\mathbf{E}}_{1\tau}^R(f \to f') = q_F(f \to f')$$

*and*

$$\tilde{\mathbf{E}}_{1a}^R(f \to f') = \begin{cases} \lambda_1(f) & \textit{if } f = f' \\ 0 & \textit{otherwise} \end{cases}$$

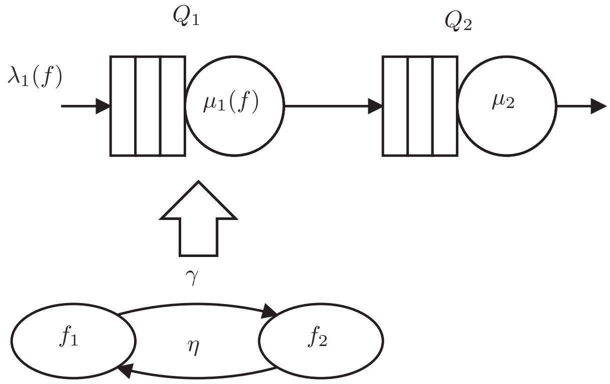


**Figure 2: Example of tandem queueing network with modulating process.**

PROOF. The proof immediately follows from Definition 3 and Lemma 1. □

We can now prove Theorem 2.

PROOF. We aim at studying the cooperation between the modulated queue $Q_1$ and $Q_2$. Since $Q_2$ has an infinite buffer it never blocks $Q_1$ (as required by Theorem 1), therefore we can focus on the process underlying $Q_1$ modulated by $F$. Observe that $\tilde{M}_1^R$ is trivially reversible and hence the conditional product-form given by Theorem 1 holds. Notice that $\tilde{M}_1^R$ behaves like a Markov Modulated Poisson Process for $Q_2$, and we have:

$$\pi(f, n_1, n_2) = \frac{\pi_2^*(f, n_2)}{\pi_F(f)} \pi_1(f, n_1)$$
$$= \frac{\pi_2^*(f, n_2)}{\pi_F(f)} \pi_F(f)(1-\rho_1)\rho_1^{n_1},$$

which simplifies to Equation (7) as required. □

## 3.1 Example

Let us consider the model depicted in Figure 2. Let

$$\lambda_1(f) = \begin{cases} \lambda_{11} & \text{if } f = f_1 \\ \lambda_{12} & \text{if } f = f_2 \end{cases}$$

and

$$\mu_1(f) = \begin{cases} \mu_{11} & \text{if } f = f_1 \\ \mu_{12} & \text{if } f = f_2 \end{cases}.$$

Observe that the modulating process, consisting of only states $f_1$ and $f_2$, is trivially reversible and therefore Theorem 2 can be applied given that $\lambda_{11}/\mu_{11} = \lambda_{12}/\mu_{12} = \rho_1$. In this case the stationary state distribution is given by Equation (7), where $\pi_2^*$ can be numerically computed as the solution of a quasi-birth-and-death process (QBD) using the matrix geometrics method [20]. Figure 3 shows $M_1$, $M_1^R$ and $\tilde{M}_1^R$ and Figure 4 shows $M_2$. We now discuss the impossibility of deriving a standard product-form solution for the model of Figure 2 but in the trivial case of $\lambda_{11} = \lambda_{12}$ and $\mu_{11} = \mu_{12}$. Indeed according to the product-form results presented in [15], the quasi-reversibility, and in [10], the Reversed Compound Agent Theorem, the rate associated with the transitions labelled by $a$ in $M_1^R$ should be always the same, which is in general not required by Theorem 2.

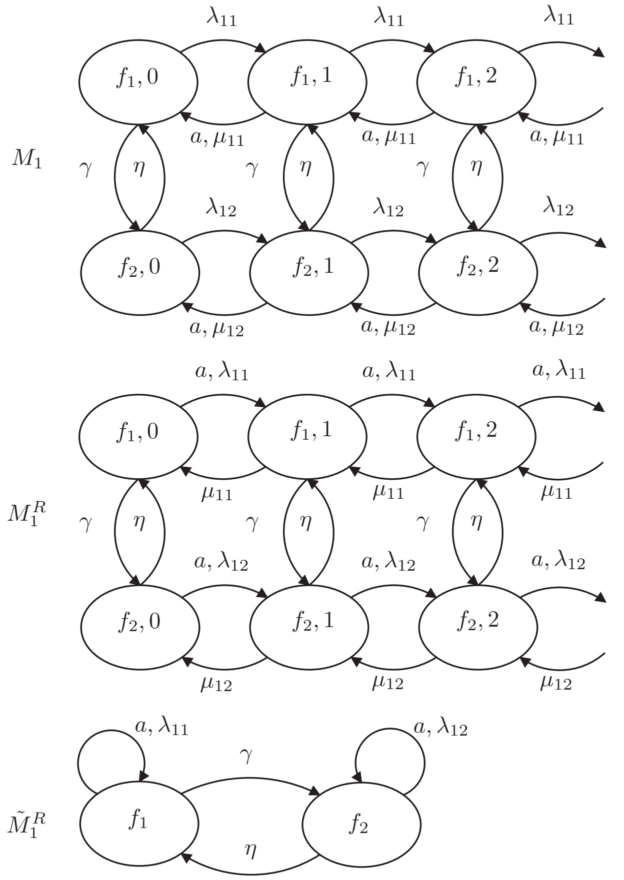


**Figure 3: Automata defined for the analysis of the model of Figure 2.**

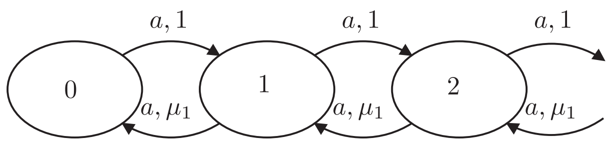


**Figure 4: Automaton underlying queue $Q_2$ of Figure 1.**

## 3.2 Extensions

Theorem 2 may be applied for studying the joint steady-state distributions of models that are behaviourally different from that shown in the previous section but whose underlying automata are similar.

### *Tandem queues with batch departures*

We consider here a tandem of two queues, $Q_1$ and $Q_2$. Customers arrive at $Q_1$ according to a Poisson process with rate $\lambda_1$ and are served in an exponentially distributed time with rate $\mu_1$. A customer departure from $Q_1$ causes an instantaneous arrival at $Q_2$ of batch of customers. The batch size is decided by the state of a reversible CTMC such as a truncated birth-and-death process whose state represents the batch size.

The analysis of such a model exploits Theorem 2 because it is sufficient to use different labels to synchronise $M_1$ and $M_2$ according to the state of the modulating process. It is easy to prove that $M_1^R$ is still lumpable and the lumped automaton $\tilde{M}_1^R$ is time reversible. The stationary distribution is given by Equation (7) where $\pi_2^*$ is the stationary distribution of the QBD process with Markov modulated batch arrivals associated with $\tilde{M}_1^R \otimes M_2$.

### *3.2.1 State-dependent service rate*

We can consider the model of Figure 1 in which the service time of $Q_1$ depends both on the state of the modulating process and the population at $Q_1$, i.e., $\mu(f, n) > 0$. The conditional product-form may be derived under the same assumptions required by Theorem 2. In this case a necessary and sufficient condition for the (standard) product-form between the modulating process and $Q_1$ is the following [24, 6, 3]:

$$\forall f \in \mathcal{F} \ \forall n \in \mathbb{N} \ \exists \rho_n > 0 \ \text{ s.t. } \ \rho_n = \frac{\lambda(f)}{\mu_1(f, n)} \,.$$

In this case the key-point to prove the conditional product-form is to notice that the population-dependent transitions in $M_1^R$ become internal and hence, according to Definition 3 do not change the derivation of $\tilde{M}_1^R$. Then, the conditional product-form expression becomes:

$$\pi(f, n_1, n_2) = (1 - \rho_1) \left( \prod_{n=0}^{\infty} \rho_n \right) \pi_2^*(f, n_2) \,.$$

# 4. CONDITIONAL PRODUCT-FORM IN QUEUES WITH MUTUAL RESETS

Queues with resets have been introduced by Gelenbe and Fourneau in [9] in the context of product-form G-networks. A reset is a type of signal that once it arrives at an empty queue it restores a non-empty state. The destination non-empty state is chosen according to its stationary probability. In [11] the notion of reset is extended to include the possibility of the queue to remain empty after the arrival of a reset signal. These models are proved to be in product-form when the external arrivals follow independent Poisson processes, with state-independent probabilistic routing and negative exponential service time distributions. In this section we study a similar reset mechanism that we call *mutual reset*. We consider two queues, $Q_1$ and $Q_2$, with exponential service time distributions with intensities $\mu_1$ and $\mu_2$ and independent Poisson arrival streams with rates $\lambda_1$ and $\lambda_2$,

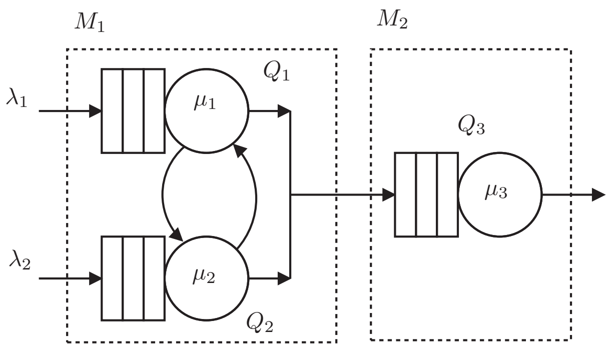


**Figure 5: System studied in Section 4. The curved lines denote the mutual exclusion and reset control mechanism.**

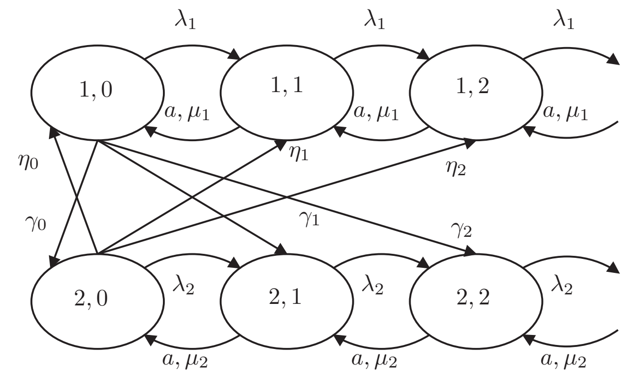


**Figure 6: Automaton underlying the system consisting of two exponential queues with mutual resets.**

respectively. The queues work according to a mutual exclusion policy. When $Q_1$ ($Q_2$) is in the empty state it may reset $Q_2$ ($Q_1$) by taking its state to a population which is chosen with a rate $\gamma$ and a probability that is proportional to the stationary probability of the arriving state. Then the just reset queue starts working and the other stops. Notice that, despite the analogies with the reset policy proposed in [11], the mechanism proposed here is novel since the queues work in a mutual exclusive way and the reset occurs when one queue is empty but on the other queue. This system consisting of queues $Q_1$ and $Q_2$ feeds a third queue $Q_3$ whose service rate is exponentially distributed with rate $\mu_3$. The interactions among queues $Q_1$, $Q_2$ and $Q_3$ is sketched in Figure 5.

If the queue alternation is slow enough we can give the following interpretation of the model. $Q_1$ and $Q_2$ are two queueing systems whose arrivals and services are independent. A controller chooses if feeding $Q_3$ with the output of $Q_1$ or that of $Q_2$. In order to maximise the system throughput, when the selected queue is empty, the controller may choose with a certain probability to change the operating queue. If the operating time of the new queue is long enough, at the switching epoch the other queue will be in a certain state with a probability that is approximately equal to its stationary probability.

Figure 6 depicts the transition diagram of the automaton $M_1$ underlying the pair of queues with mutual reset. According to the system definition, we define rate $\eta_i$ and $\gamma_i$ as follows:

$$\gamma_i = \gamma \frac{\pi_1(2,i)}{\sum_{j=0}^{\infty} \pi_1(2,j)} \quad \eta_i = \eta \frac{\pi_1(1,i)}{\sum_{j=0}^{\infty} \pi_1(1,j)} \tag{8}$$

where $\pi_1$ denotes the stationary distribution function of $M_1$ assuming stability. The transition diagram of the automaton underlying $Q_3$, namely $M_2$, is shown in Figure 4. Formally, automaton $M_1$ is defined as follows:

$$\mathbf{E}_{1\tau}(q, n \to q', n') = \begin{cases} \lambda_1 & \text{if } q = q' = 1, n' = n+1 \\ \lambda_2 & \text{if } q = q' = 2, n' = n+1 \\ \gamma_i & \text{if } q = 1, n = 0, n' = i, q' = 2 \\ \eta_i & \text{if } q = 2, n = 0, n' = i, q' = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{E}_{1a}(q, n \to q', n') = \begin{cases} \mu_1 & \text{if } q = q' = 1, n' = n-1 \\ \mu_2 & \text{if } q = q' = 2, n' = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

The following theorem gives the conditional product-form between $M_1$ and $M_2$.

THEOREM 3. *Given the queueing network depicted in Figure 5, under the stability conditions:* $0 < \rho_1 < 1$, $0 < \rho_2 < 1$, $[\eta(1-\rho_2)\rho_1\mu_1 + \gamma(1-\rho_1)\rho_2\mu_2][\gamma(1-\rho_1)+\eta(1-\rho_2)]^{-1} < \mu_3$, *the following conditional product-form holds:*

$$\pi(q,i,j) = (1-\rho_1)\rho_1^i \pi_3^*(1,j)\delta_{q=1} + (1-\rho_2)\rho_2^i \pi_3^*(2,j)\delta_{q=2}\,, \tag{9}$$

*where $\delta$ denotes the Kronecker delta function, $\rho_1 = \lambda_1/(\mu_1 - \gamma)$, $\rho_2 = \lambda_2/(\mu_2 - \eta)$ and $\pi_3^*$ is the stationary distribution of the MMPP/M/1 queue with exponential service time distribution with rate $\mu_3$ and a MMPP arrival process with two states, $\{1,2\}$, where state $i$ imposes a Poisson arrival stream with rate $\rho_i\mu_i$ and the transition rate from state 1 to 2 is $\gamma(1-\rho_1)$ and from state 2 to 1 is $\eta(1-\rho_2)$.*

Before proving the theorem, we introduce some useful lemmas. The first gives the closed form expression of the stationary distribution of $M_1$ that we will use to derive its reversed $M_1^R$.

LEMMA 3. *The stationary distribution of $M_1$ is:*

$$\pi_1(q,n) = k_q \rho_q^n \tag{10}$$

*for $q \in \{1,2\}$ and $n \geq 0$, where $\rho_1$, $\rho_2$ are defined in Theorem 3 and $k_1 = \eta G$, $k_2 = \gamma G$ with*

$$G = (\lambda_1 + \gamma - \mu_1)(\lambda_2 + \eta - \mu_2)\big[\gamma\eta(\lambda_1 + \lambda_2 + \gamma + \eta) + \mu_1\mu_2(\gamma+\eta) - \eta\mu_1(\lambda_2 + \gamma + \eta) - \gamma\mu_2(\lambda_1 + \gamma + \eta)\big]^{-1}.$$

*The stability condition for this model is $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ and, under this condition, the normalising constant $G$ that appears in the definition of $k_1$ and $k_2$ is a positive finite real number.*

PROOF. First of all, observe that given Equation (10) the reset rates of Equation (8) can be rewritten as:

$$\begin{aligned} \gamma_i &= \gamma\pi_1(2,i)\frac{1-\rho_2}{k_2} = \gamma(1-\rho_2)\rho_2^i \quad \text{and} \\ \eta_i &= \eta\pi_1(1,i)\frac{1-\rho_1}{k_1} = \eta(1-\rho_1)\rho_1^i \end{aligned} \tag{11}$$

Therefore, the GBE for the system consisting of queues $Q_1$ and $Q_2$ are:

$$\begin{cases} \pi_1(1,0)(\gamma+\lambda_1) = \pi_1(1,1)\mu_1 + \pi_1(2,0)\eta(1-\rho_1) \\ \pi_1(1,i)(\lambda_1+\mu_1) = \pi_1(1,i+1)\mu_1 + \pi_1(1,i-1)\lambda_1 \\ \quad +\pi_1(2,0)\eta(1-\rho_1)\rho_1^i & i>0 \\ \pi_1(2,0)(\eta+\lambda_2) = \pi_1(2,1)\mu_2 + \pi_1(1,0)\gamma(1-\rho_2) \\ \pi_1(2,i)(\lambda_2+\mu_2) = \pi_1(2,i+1)\mu_2 + \pi_1(2,i-1)\lambda_2 \\ \quad +\pi_1(1,0)\gamma(1-\rho_2)\rho_2^i & i>0 \end{cases}$$

The proof can be carried out by substitution and by proving that the normalising condition:

$$\sum_{q\in\{1,2\}} \sum_{j=0}^{\infty} \pi_1(q,j) = 1\,,$$

is satisfied if and only if $0<\rho_1<1$ and $0<\rho_2<1$. $G$ is well-defined because, given the stability condition, is strictly positive and finite. □

Lemma 4. *The automaton $M_1^R$, inverse of $M_1$, is defined as follows:*

$$\mathbf{E}_{1\tau}^R(q,i\to q',i') = \begin{cases} \mu_1-\gamma & \text{if } q'=q=1, i=i'+1 \\ \mu_2-\eta & \text{if } q'=q=2, i=i'+1 \\ \eta(1-\lambda_2/(\mu_2-\eta)) & \text{if } q=2, q'=1, i'=0 \\ \gamma(1-\lambda_1/(\mu_1-\gamma)) & \text{if } q=1, q'=2, i'=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{E}_{1a}^R(q,i\to q',i') = \begin{cases} \lambda_1\mu_1/(\mu_1-\gamma) & \text{if } q'=q=1, i'=i+1 \\ \lambda_2\mu_2/(\mu_2-\eta) & \text{if } q'=q=2, i'=i+1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let us consider the transition in $M_1^R$ going from state $(1,i)$ to $(1,i+1)$, $i\geq 0$, labelled $a$, its rate is:

$$\mathbf{E}_{1a}^R(1,i\to 1,i+1) = \frac{\pi_1(1,i+1)}{\pi_1(1,i)}\mathbf{E}_{1a}(1,i+1\to 1,i) = \rho_1\mu_1 = \frac{\lambda_1}{\mu_1-\gamma}\mu_1\,.$$

Analogously, we prove $\mathbf{E}_{1a}^R(2,i\to 1,i+1) = \lambda_2\mu_2/(\mu_2-\eta)$. Moreover, for $i\geq 0$, we have:

$$\mathbf{E}_{1\tau}^R(1,i+1\to 1,i) = \frac{\pi_1(1,i)}{\pi_1(1,i+1)}\mathbf{E}_{1\tau}\lambda_1 = \frac{\lambda_1}{\rho_1} = \mu_1-\gamma\,,$$

and symmetrically $\mathbf{E}_{1\tau}^R(2,i+1\to 2,i) = \mu_2-\eta$. Let us compute the reversed rate of the transitions going from state $1,0$ to state $2,i$, with $i\geq 0$.

$$\mathbf{E}_{1\tau}^R(2,i\to 1,0) = \frac{\pi_1(1,0)}{\pi_1(2,i)}\mathbf{E}_{1\tau}(1,0\to 2,i)$$
$$= \frac{\pi_1(1,0)}{\pi_1(2,i)}\gamma\frac{\pi_1(2,i)(1-\rho_2)}{k_2} = \frac{k_1}{k_2}\gamma(1-\rho_2) = \eta(1-\rho_2)\,.$$

By symmetry, we obtain $\mathbf{E}_{1\tau}(1,i\to 2,0) = \gamma(1-\rho_1)$. □

We can now prove Theorem 3.

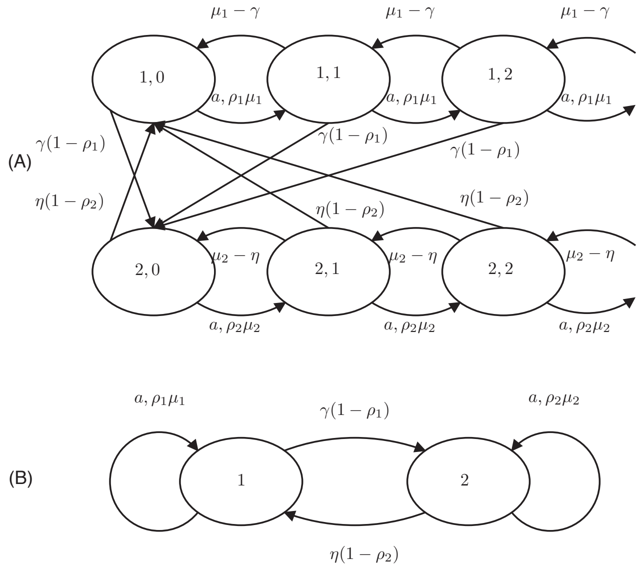


**Figure 7: $M_1^R$ and $\tilde{M}_1^R$ associated with automaton $M_1$ shown in Figure 6.**

Proof. According to Definition 3, $M_1^R$ as given by Lemma 4 and shown in Figure 7-(A) is lumpable in the automaton shown in Figure 7-(B). whose formal definition is:

$$\tilde{\mathbf{E}}_{1\tau}(q\to q') = \begin{cases} \gamma(1-\rho_1) & \text{if } q=1, q'=2 \\ \eta(1-\rho_2) & \text{if } q=2, q'=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{\mathbf{E}}_{1a}(q\to q') = \begin{cases} \rho_1\mu_1 & \text{if } q=1, q'=1 \\ \rho_2\mu_2 & \text{if } q=2, q'=2 \\ 0 & \text{otherwise} \end{cases}$$

Since automaton $\tilde{M}_1^R$ is reversible, then by Theorem 1 the conditional product-form expression of Equation (9) follows from Lemma 3. The stability conditions for $M_1$ are proved in Lemma 3 while that for $M_2$ follows from queueing theory considerations on the comparison of the expected interarrival and service time. □

The following propositions follow immediately from the results just given and they give closed-form expressions for the performance indices of the system consisting of $Q_1$ and $Q_2$. The performance indices of $Q_3$ may be derived according to the matrix-geometric approach as shown for instance in [22].

Proposition 2. *In stability, the expected number of customers $N$ in the system modelled by $M_1$ is:*

$$N = k_1\frac{\rho_1}{(1-\rho_1)^2} + k_2\frac{\rho_2}{(1-\rho_2)^2}\,,$$

*where $k_1$, $k_2$ are defined in Lemma 3 and $\rho_1$, $\rho_2$ are defined in Theorem 3.*

Proof. By definition of $N$, we have:

$$N = \sum_{i=0}^{\infty} i(\pi_1(1,i)+\pi_1(2,i))\,,$$

where $\pi_1$ is given by Lemma 3. Therefore, we have:

$$N = k_1 \sum_{i=1}^{\infty} i\rho_1^i + k_2 \sum_{i=1}^{\infty} i\rho_2^i = k_1\rho_1 \sum_{i=1}^{\infty} \frac{\partial \rho_1^i}{\partial \rho_1} + k_2\rho_2 \sum_{i=1}^{\infty} \frac{\partial \rho_2^i}{\partial \rho_2}$$

$$= k_1\rho_1 \frac{\partial \sum_{i=1}^{\infty} \rho_1^i}{\partial \rho_1} + k_2\rho_2 \frac{\partial \sum_{i=1}^{\infty} \rho_2^i}{\partial \rho_2} = k_1 \frac{\rho_1}{(1-\rho_1)^2} + k_2 \frac{\rho_2}{(1-\rho_2)^2} ,$$

as required. □

Proposition 3. *In stability, the throughput $X$ of model $M_1$ is:*

$$X = \frac{\eta\mu_1\rho_1(1-\rho_2) + \gamma\mu_2\rho_2(1-\rho_1)}{\eta(1-\rho_2) + \gamma(1-\rho 1)} , \qquad (12)$$

*or equivalently:*

$$X = \frac{k_1\rho_1\mu_1}{1-\rho_1} + \frac{k_2\rho_2\mu_2}{1-\rho_2} , \qquad (13)$$

*where $\rho_1$ and $\rho_2$ are defined in Theorem 3 and $k_1$, $k_2$ in Lemma 3.*

Observe that although the expression of $X$ given by Equation (13) is more compact than that of Equation (12), the former requires the computation of $k_1$ and $k_2$ while the latter does not.

Proof. By definition of throughput we have:

$$X = \sum_{i=1}^{\infty} (\pi_1(1,i)\mu_1 + \pi_1(2,i)\mu_2) .$$

According to Lemma 3, we have $\sum_{i=1}^{\infty} \pi_1(1,i) = k_1\rho_1/(1-\rho_1)$ and hence we can derive immediately Equation (13). In order to derive Equation (12) it is convenient to analyse $\tilde{M}_1^R$ (see Figure 7-(B)). Here the throughput of the model is the throughput of the transitions labelled $a$. This computation immediately gives Equation (12). □

Notice that since $Q_3$ does not destroy or create any customer, then by the conservation law $X$ is also the throughput of $Q_3$.

In Figure 8 we show the throughput $X$ of $M_1$, given different sets of parameters, for $0 < \gamma = \eta < 4$. Notice that $X$ can exceed the sum $\lambda_1 + \lambda_2$ and that increasing the service rate (here $\mu_2$) of a queue may indeed decrease the system's throughput, since the faster queue has a higher probability to be empty and thus to jump to the other one whose throughput may be lower.

Proposition 4. *In stability, the utilisation $U_i$ of the server of $Q_i$, for $i \in \{1, 2\}$ is given by:*

$$U_i = k_i \frac{\rho_i}{1-\rho_i} ,$$

*and the total utilisation (when one of the two servers is in use) is:*

$$U = 1 - k_1 - k_2 ,$$

*where $\rho_1$ and $\rho_2$ are defined in Theorem 3 and $k_1$, $k_2$ in Lemma 3.*

Proof. The proof is immediate given the steady-state distribution proved in Lemma 3 and the definition of utilisation. □

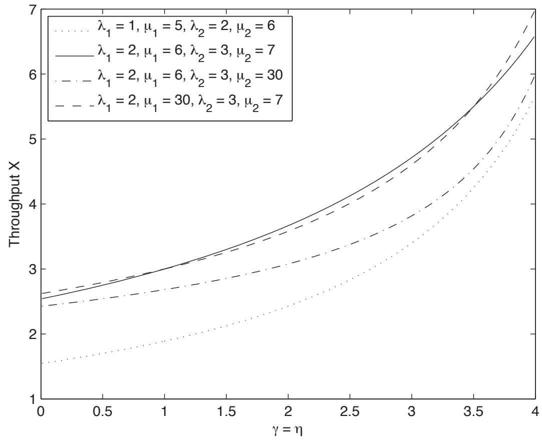


**Figure 8: Throughput of $M_1$ of Figure 5 for different sets of parameters.**

# 5. CONCLUSION

In this paper we have studied two classes of models with conditional product-forms that are not tractable with ordinary product-forms. To the best of our knowledge, this is the first time that these results are applied to queueing network models to obtain exact and closed-form expressions for the models' stationary distributions. The first model that we studied consisted in a Markov modulated queueing network feeding another exponential queue. We gave sufficient conditions for the separable solution and showed that the final station of the tandem sees a Markov Modulated Poisson Process rather than a Poisson process as would happen in standard product-form analysis. The second model is inspired by Gelenbe and Fourneau's resets [9] although the studied mechanism is rather different since two queues alternate their outgoing stream to another queue and can, under some conditions, reset the state of the other according to its stationary distribution. We have shown that despite to its complexity, the model admits a conditional product-form that allowed us to derive a closed form expression of the stationary distribution functions. Future research efforts will be devoted to extend Theorem 1 in order to encompass the feedback in the studied models and to provide new approximation methods in queueing networks based on the (approximate) lumping of the reversed processes rather than their forward.

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