Only the source's and sink's neighborhood matters: convergence results for unicast and multicast connections on random graphs and hypergraphs.

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ABSTRACT

We study the maximum flow on random weighted directed graphs and hypergraphs, that generalize Erdös-Rényi graphs. We show that, for a single unicast connection chosen at random, its capacity, determined by the max-flow between source and sink, converges in probability to the capacity around the source or sink. Using results from network coding, we generalize this result to different types multicast connections, whose capacity is given by the max-flow between the source(s) and sinks. Our convergence results indicate that the capacity of unicast and multicast connections using network coding are, with high probability, unaffected by network size in random networks. Our results generalize to networks with random erasures.

General Terms

Theory

Keywords

Flow, multicast, network coding, random graph, random hypergraph, unicast

1. INTRODUCTION

Network coding [1] has shown that the capacity of multicast connections is given by the min-cut max-flow upper bound between source(s) and sinks, thus generalizing the unicast results of Ford-Fulkerson to multicast connections. Indeed, network coding can be used to show the classical Ford-Fulkerson flow achievability results from an algebraic point of view [10]. Such cuts can be considered over hypergraphs [12], which provide a useful representation of the broadcast nature of wireless links. Moreover, the max-flow achievability of the network min-cut holds, for networks with ergodic random erasures, if we consider the mean of the flow and of the cut [12].

For a single source and sink, the problem of determining the behavior of the max-flow in random graphs was first envisaged in [5] and in [4]. These papers present results on undirected complete graphs with capacities that are randomly selected with a distribution that does not depend on node distance. Note that such results, obtained on undirected graphs, may not be illustrative of the behavior of directed graphs. For example, in directed graphs, the gain that network coding can obtain, in multicast connection capacity, over uncoded approaches, which are based upon convex combinations of Steiner trees, is arbitrarily large [8]. On the hand, the Kriesell conjecture [11] states that network coding in undirected graphs cannot more than double the capacity of a multicast connection, and [16] has shown the bound is bounded by 6.5. While results on max-flow in undirected graphs may not be readily applicable to directed graphs, there is little work, to the authors' knowledge, on max-flow in directed random graphs. Reference [15] provides, without proof, results for the problem of max-flow for a single source and sink in directed random graphs, with the restriction that arcs can only exist in a single direction between two nodes.

The capacity of random graphs using network coding was first considered in [14]. In that article, the first random graphs considered are directed random graphs built over complete graphs, where the existence of an edge from one node to another implies the existence of a reverse edge of equal capacity. Moreover, the probability of the edge's existence is constant. The second model presented in [14] is that of a geometric random graph.

Our work, after some corrections of a technical flaw in the original proof of [14], expands upon the ideas that [14] presented in the context of random graphs. Our results allow us to consider a rich set of random graph and hypergraph models, neither of which need to be geometric, and of types of connections, including different types of multicast connections, such as multi-source multicast, two-level multicast and disjoint multicast [10]. Moreover, vis-à-vis [14], we sharpen the types of convergence results that can be shown, by establishing convergence in probability, and are able to consider networks with ergodic random erasures, in a manner akin to [12]. Our approach is akin to the percolation results of [3] about the connectivity of random graphs, but

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we consider instead the dimension of connections, by characterizing the convergence, in probability, of the max-flow of our random graphs.

Our contribution is different from the scaling laws presented by Gupta and Kumar in [6] and from the extensive literature on scaling laws with network coding, see for instance [17]. This literature envisages a number of unicast connections that increases with the number of nodes. In such systems, bottlenecks arise at relay nodes in the interior of the networks, whereas our results establish that the bottlenecks are at the source or sink nodes.

The rest of this article is organized as follows. In Section 2, we define our random graph model and flows on graphs. In Section 3, we establish the convergence in probability of the max-flow of our random graphs. In Section 4, we define our random hypergraph model and flows on hypergraphs. In Section 5, we establish the convergence in probability of the max-flow of our random hypergraphs. Finally, we conclude in section 6.

2. MODEL OF WIRED NETWORK : RANDOM WEIGHTED DIRECTED GRAPH

2.1 Definitions and Notation

2.1.1 A Weighted Directed Graph

In this article, wired networks will be modeled by weighted directed graphs.

Definition 1. A *directed graph* $G = (N, E)$ is a pair of which the first element is the set of nodes N and the second is the set of edges E, a subset of $N \times N$.

A weight function is added to the model whereby, for each edge, a weight is assigned corresponding to the capacity of the link in the network.

Definition 2. A *weighted directed graph* $(G = (N, E), W)$ is a pair where the first element is a directed graph G and the second is a non-negative function from $N \times N$, with the constraint that $W_e = 0$ if $e \notin E$. The value W_e is called the weight of the edge e.

2.1.2 Flow in a Weighted Directed Graph

Our results will center on the maximum value of flows on the graph we consider. Our definition of flow is given below.

Definition 3. A *flow* from the source node i to the sink node j in a weighted directed graph is a function f on edges that satisfy these conditions:

1. the flow is less than the weight, i.e., for all nodes u,v,
$$f((u, v)) \leq W_{(u,v)}; \tag{1}$$
2. there is no incoming flow to i and outgoing flow from j, i.e., for all nodes u,
$$f((u, i)) = f((j, u)) = 0; \tag{2}$$
3. the outgoing flow from i is equal to the incoming flow to j and has value F:
$$\sum_{\substack{v \in N \\ (i,v) \in E}} f((i, v)) = \sum_{\substack{u \in N \\ (u,j) \in E}} f((u, j)) = F; \tag{3}$$

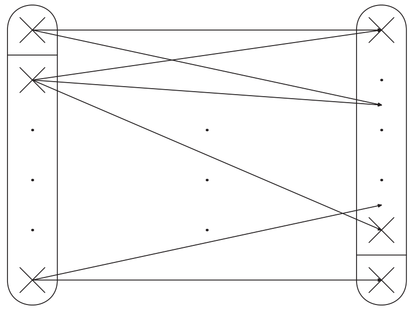

Figure 1: Min-cut from the set of the right nodes to the set of the left nodes.

4. conservation: for each node except i and j, the incoming flow is equal to the outgoing flow, i.e., for all $u \neq i, j$,
$$\sum_{\substack{v \in N \\ (v,u) \in E}} f((v, u)) = \sum_{\substack{v' \in N \\ (u,v') \in E}} f\left((u, v')\right). \tag{4}$$

Definition 4. The *max-flow* from i to j is a flow with the maximal value. We will denote $F^G_{(i,j)}$ the value of this flow.

Our aim will be to evaluate this max-flow in large random graphs.

2.1.3 Cut in a Weighted Directed Graph

In order to characterize the max-flow of a graph, we shall study its min-cut. Below, we define a cut and the associated concept of min-cut.

Definition 5. A *cut* from the set of nodes N_0 to the set of nodes N_1 is a set S of edges such that if the edges in S are removed, then there is no directed path from u to v for any $u \in N_0$ and $v \in N_1$. The value of a cut is the sum of weights of its edges.

Definition 6. The *min-cut* from the set of nodes N_0 to the set of nodes N_1 is a cut whose value is minimum. We denote this value $C^G_{(N_0,N_1)}$.

The following theorem gives the value of the min-cut from a subset N_0 of nodes to its complementary N_0^c. It is illustrated by the figure 1.

THEOREM 1. *For any graph G and any subset N_0 of N, we have*
$$C^G_{(N_0,N_0^c)} = \sum_{u \in N_0} \sum_{v \in N_0^c} W_{(u,v)}. \tag{5}$$

The link between the max-flow and the min-cut of a graph is established by the min-cut max-flow theorem that was proven for the first time by Menger on unweighted undirected graphs. A proof for weighted directed graphs can be found in [13].

THEOREM 2 (MIN-CUT MAX-FLOW THEOREM). *For any weighted directed graph G, the max-flow from i to j is equal to the min-cut from $\{i\}$ to $\{j\}$, i.e.,*
$$F^G_{(i,j)} = C^G_{(\{i\},\{j\})}. \tag{6}$$

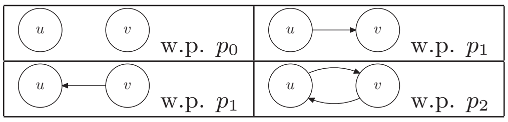

Figure 2: The distribution of the existence of the edges of two nodes u and v.

A corollary of this theorem links the max-flow from i to j to the min of the min-cuts between all 2-partitions of nodes where i and j are not in the same. This will be useful since it is easier to evaluate.

THEOREM 3. *For any weighted directed graph G, we have*

$$F^G_{(i,j)} = \min_{N_0 \subset N\setminus\{j\}, i \in N_0} C^G_{(N_0, N_0^c)}. \quad (7)$$

2.2 Studied Random Weighted Directed Graphs

As in many problems on random graphs, our results hold only for random graphs that satisfy some conditions. Therefore, in this article, results established will concern only this type of random graphs.

Definition 7. Random weighted directed graphs studied in this article satisfy these conditions:

1. an edge exists with probability p_l;
2. the weight of an edge is distributed as a random variable of density function f_W and of mean μ, i.e., for all nodes u,v,

$$P\left(W_{(u,v)} \geq w\right) = \begin{cases} 1 & \text{if } w = 0, \\ p_l \int_w^\infty f_W(x)\,\mathrm{d}x & \text{else;} \end{cases} \quad (8)$$

3. for each subset N_0 of nodes, the edges implied in the min-cut from N_0 to N_0^c are independent, i.e., for all N_0 subset of N,

$$\left(W_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent.} \quad (9)$$

In particular, we shall consider four types of such random graphs that have, already, been studied in the literature.

1. For the first type, for each pair of nodes $\{u, v\}$, we associate $p_{0,\{u,v\}}$ the probability for two nodes to be not linked (i.e., (u, v) and (v, u) do not exist), we denote $p_{1,\{u,v\}}$ the probability to have the edge (u, v) (resp. (v, u)) without (v, u) (resp. (u, v)) and $p_{2,\{u,v\}}$ to have the two edges (u, v) and (v, u) (as illustrated figure 2) such that $p_{1,\{u,v\}} + p_{2,\{u,v\}} = p_l$. Then, the capacities $W_{(u,v)}$ and $W_{(v,u)}$ can either be independent and distributed according to density function f_W, or map to the same random variable $W_{\{u,v\}}$, whose distribution is given by the density function f_W.
2. If, for all nodes u,v, $p_{1,\{u,v\}} = 0$ and $W_{(u,v)} = W_{(v,u)}$, then the model obtained is the one discussed in [14], where edges are two-way edges and each two-way edge has the same capacity on the two directions. This model could be extended to a weighted undirected graph.
3. If, for all nodes u,v, $p_{2,\{u,v\}} = 0$, then we obtain the model discussed in [15] where edges are one-sided. This can be seen like a random weighted undirected graph where sides of directed edges are chosen independently and uniformly.
4. If, for all nodes u,v, $p_{1,\{u,v\}} = p_l(1 - p_l)$, $p_{2,\{u,v\}} = p_l^2$ and $W_{(u,v)}$ and $W_{(v,u)}$ are independent, then we obtain an Erdös-Rényi weighted random graph, since all directed edges are generated independently in this case.

3. UNICAST AND MULTICAST CONNECTIONS ON RANDOM GRAPHS

3.1 Unicast

The unicast connection problem, to which we shall refer simply as the unicast problem, consists in characterizing the max-flow $F^G_{(i,j)}$ from a node i to a node j. The aim of this section is to evaluate the value of the max-flow $F^G_{(i,j)}$ in a large random graph G as defined in the previous section.

Some results about the unicast problem on random graph already exist. Grimett and Welsh, in [5], established results about particular type of random graphs when the probability p_l is fixed. Suen, in [15], provided, for random graphs where an edge between two nodes is unique and has a unique direction, convergence results when p_l can converge quickly to 0, but the results are given without proof. More recently, Ramamoorthy et al. in [14], established some results for random graphs where, for every edge between two nodes, there exists one in the opposite direction, and for graphs with a fixed p_l. This two kinds of random graphs will be two sub-classes of random graphs we study here.

THEOREM 4. *Consider a random weighted directed graph with $n+1$ nodes. Let i and j be two nodes, with i the source and j the sink. If*

$$\frac{np_l}{\ln n} \to \infty \quad (10)$$

and, for all subset of nodes N_0 of N such that $i \in N_0$ and $j \notin N_0$,

$$\left(W_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent} \quad (11)$$

then

$$\frac{F^G_{(i,j)}}{np_l\mu} \xrightarrow{p} 1. \quad (12)$$

In particular, the min-cut is around the source i or the sink j. Therefore, in a random network, the capacity is limited by what happens locally around the source and the sink and not in the rest of the network.

In certain cases, the condition (10) can be relaxed. In particular, for a constant weight, this condition becomes: there exist $c > 1$ such that $\frac{np_l}{\ln n} \to 32c$, as we can see in the following proof.

3.2 Proof

The proof generalizes the approach of [5] and [14].

3.2.1 $P\left(\frac{F^G_{(i,j)}}{np_l\mu} \leq 1 - \epsilon\right) \to 0$

We shall carry out a proof by steps. First, we shall prove the result when f_W is a Dirac delta function (i.e., the random variable for the weight is a Bernouilli of parameter p_l). Then, we generalize for f_W that is a finite sum of Dirac delta functions. Finally, we conclude the proof by approximating a general f_W by a sum of Dirac delta functions.

For f_W a Dirac Function.

We consider, first, a distribution $f_W = \delta_\mu$ that is a Dirac delta function (i.e., if the edge (u,v) exists then its weight is the value of this Dirac delta function μ). We assume that $\mu = 1$ (we can do that since μ is independent of n and multiplying all the edges by μ multiplies the flow by μ).

First, we establish a lemma about the probability that the min-cut $C^G_{(N_0,N_0^c)}$ is less than $np_l\mu$ when $N_0 \neq \{i\}$ and $N_0^c \neq \{j\}$.

LEMMA 1. *For any N_0 subset of N such that $i \in N_0$, but $N_0 \neq \{i\}$, and $j \in N_0^c$, but $N_0^c \neq \{j\}$, with $|N_0| = k+1$ (note that $1 \leq k \leq n-2$), we have*

$$P\left(C^G_{(N_0,N_0^c)} \leq np_l\mu\right) \leq \exp\left(-\frac{k(n-k-1)}{8}p_l\right). \quad (13)$$

PROOF. We have

$$C^G_{(N_0,N_0^c)} = \sum_{u\in N_0}\sum_{v\in N_0^c} W_{(u,v)}. \quad (14)$$

Moreover, since $\left(W_{(u,v)}\right)_{u\in N_0, v\in N_0^c}$ are independent and identically distributed Bernoulli random variables, we have that $C^G_{(N_0,N_0^c)}$ has a binomial distribution of mean $|N_0|(n+1-|N_0|)E\left[W_{(u,v)}\right] = (k+1)(n-k)p_l$. Hence,

$$\begin{aligned}
&P\left(C^G_{(N_0,N_0^c)} \leq np_l\right)\\
&\quad= P\left(C^G_{(N_0,N_0^c)} \leq E\left[C^G_{(N_0,N_0^c)}\right] -k(n-k-1)p_l\right)\\
&\quad\leq \exp\left(-\tfrac{(k(n-k-1)p_l)^2}{2(k+1)(n-k)p_l}\right) \quad \text{(see [2],p.12 or [9],p.26)}\\
&\quad\leq \exp\left(-\tfrac{k(n-k-1)}{8}p_l\right).
\end{aligned}$$

□

Then, we continue the proof by looking what happens for the min-cut not around the source.

$$\begin{aligned}
&P\left(\min_{N_0\subset N\backslash\{j\}, i\in N_0, N_0\neq\{i\}, N_0^c\neq\{j\}} C^G_{(N_0,N_0^c)} \leq np_l\right)\\
&\quad= P\left(\exists N_0 \subset N\backslash\{j\}, i\in N_0, N_0\neq\{i\}, N_0^c\neq\{j\}, C^G_{(N_0,N_0^c)} \leq np_l\right)\\
&\quad\leq \textstyle\sum_{k=1}^{n-2}\binom{n-1}{k}\exp\left(-\frac{k(n-k-1)}{8}p_l\right)\\
&\quad= \textstyle\sum_{k=0}^{n-1}\binom{n-1}{k}\beta^{(n-1)\frac{k}{n-1}(1-\frac{k}{n-1})} - 2\\
&\quad\leq 2(1+\sqrt{\beta})^{n-1} - 2 \quad \text{(see [14])}\\
&\qquad \text{where } \beta = \exp\left(-\tfrac{(n-1)p_l}{8}\right).
\end{aligned}$$

Then, since $\frac{(n-1)p_l}{\ln(n-1)} \to \infty$, there exists N such that for all $n \geq N$,

$$(n-1)p_l \geq 32\ln(n-1). \quad (15)$$

Moreover,

$$(1+\sqrt{\beta})^{n-1} \sim \exp\left((n-1)\exp\left(-\frac{(n-1)p_l}{16}\right)\right). \quad (16)$$

Since for all $n \geq N$,

$$\begin{aligned}
&(n-1)\exp\left(-\tfrac{(n-1)p_l}{16}\right)\\
&\quad\leq (n-1)\exp\left(-\tfrac{32\ln(n-1)}{16}\right)\\
&\quad= \tfrac{n-1}{(n-1)^2}\\
&\quad= \tfrac{1}{n-1} \to 0.
\end{aligned}$$

Therefore,

$$2(1+\sqrt{\beta})^{n-1} - 2 \to 2-2 = 0. \quad (17)$$

Finally, we obtain

$$P\left(\min_{N_0\subset N\backslash\{j\}, i\in N_0, N_0\neq\{i\}, N_0^c\neq\{j\}} C^G_{(N_0,N_0^c)} \leq np_l\right) \to 0. \quad (18)$$

Now, around the source i and the sink j, we have, by the law of large numbers

$$\begin{aligned}
&P\left(C^G_{(\{i\},\{i\}^c)} \leq (1-\epsilon)np_l\right) \to 0,\\
&P\left(C^G_{(\{j\}^c,\{j\})} \leq (1-\epsilon)np_l\right) \to 0.
\end{aligned}$$

Then, by lemma 4 in the appendix, we obtain

$$P\left(F^G_{(i,j)} \leq (1-\epsilon)np_l\right) \to 0. \quad (19)$$

The case $\mu \neq 1$ is obtained by observing that $F^G_{(i,j)} = \mu F^{G'}_{(i,j)}$, where G' is the same graph as G, with every link of capacity 1 instead of μ. Thus,

$$P\left(\frac{F^G_{(i,j)}}{\mu} = F^{G'}_{(i,j)} \leq (1-\epsilon)np_l\right) \to 0. \quad (20)$$

For f_W a sum of Dirac delta functions.

We suppose that $f_W = \sum_{k=1}^m q_k\delta_{\mu_k}$ is a sum of Dirac delta functions. We can assume that $\mu_1 < \cdots < \mu_m$. Clearly,

$$\mu = \sum_{k=1}^m q_k\mu_k. \quad (21)$$

We split the graph into m subgraphs G^k where the edge (u,v) exists and have weight μ_k if it is the case in the original graph G. The subgraphs G^k are all random graphs with $p_l^k = p_l q_k$ and with $f_W^k = \delta_{\mu_k}$, hence, the previous result, for a simple Dirac delta function, can be applied. This split implies

$$F^G_{(i,j)} \geq \sum_{k=1}^m F^{G^k}_{(i,j)}. \quad (22)$$

Indeed, we can take the union of the edges given by the right term, this is a flow for the original graph. Hence,

$$\begin{aligned}
&P\left(F^G_{(i,j)} < (1-\epsilon)np_l\mu\right)\\
&\quad\leq P\left(\textstyle\sum_{k=1}^m F^{G^k}_{(i,j)} < (1-\epsilon)np_l\mu\right)\\
&\quad= P\left(\textstyle\sum_{k=1}^m F^{G^k}_{(i,j)} < \sum_{k=1}^m (1-\epsilon)np_l q_k\mu_k\right)\\
&\quad\leq P\left(\exists k\ F^{G^k}_{(i,j)} < (1-\epsilon)np_l q_k\mu_k\right)\\
&\quad\leq \textstyle\sum_{k=1}^m P\left(F^{G^k}_{(i,j)} < (1-\epsilon)np_l q_k\mu_k\right)\\
&\quad\to \textstyle\sum_{k=1}^m 0 = 0.
\end{aligned}$$

For a general f_W.

We approximate a general f_W by a finite sum of Dirac delta functions. We have a first lemma about the approximation of the infinite tail of the distribution.

LEMMA 2. *There exists M such that, for all $x \geq M$,*

$$\int_x^{\infty} (t-x) f_W(t)\, dt < \epsilon. \tag{23}$$

Thus, we approximate the function f_W by a sum of Dirac delta functions $\tilde{f_W}$ as follows

$$\tilde{f_W} = \sum_{k=0}^{m=\lceil M/\epsilon \rceil -1} \left(\int_{n\epsilon}^{(n+1)\epsilon} f_W(x)\, \mathrm{d}x \right) \delta_{n\epsilon} + \left(\int_{(m+1)\epsilon}^{\infty} f_W(x)\, \mathrm{d}x \right) \delta_{(m+1)\epsilon}.$$

We have

$$\mu - 2\epsilon \leq \tilde{\mu} = \int_0^{\infty} \tilde{f_w}(x)\, \mathrm{d}x \leq \mu. \tag{24}$$

Using this approximation, we can conclude the proof in the general case. Let $\frac{3}{4} > \epsilon > 0$. We denote $\eta = \frac{\epsilon}{4(1-\epsilon)}\mu$ and $\epsilon' = 1 - \frac{\mu}{\mu - \eta}(1-\epsilon) = 1 - \frac{1-\epsilon}{1-\frac{\epsilon}{4(1-\epsilon)}} > 0$ since $0 < \epsilon < \frac{3}{4}$. We denote $\tilde{G}$ the η-approximation of G. We have $F^{\tilde{G}}_{(i,j)} \leq F^{G}_{(i,j)}$ since $\tilde{G}$ is the same graph with less capacity. Hence,

$$P\left(\frac{F^G_{(i,j)}}{np_l\tilde{\mu}} < 1 - \epsilon' \right) \leq P\left(\frac{F^{\tilde{G}}_{(i,j)}}{np_l\tilde{\mu}} < 1 - \epsilon' \right) \to 0. \tag{25}$$

Now,

$$\begin{aligned} \frac{\tilde{\mu}}{\mu}(1-\epsilon') &\geq \frac{\mu - \eta}{\mu}(1-\epsilon') \\ &\geq 1 - \epsilon. \end{aligned}$$

Therefore,

$$P\left(\frac{F^G_{(i,j)}}{np_l\mu} < 1 - \epsilon \right) \to 0. \tag{26}$$

Hence, the probability for the min-cut to be less that the cut around the source or the sink goes to 0 as n goes to infinity.

3.2.2 $P\left(\frac{F^G_{(i,j)}}{np_l\mu} \geq 1 + \epsilon \right) \to 0$

To finish establishing convergence in probability, we must prove the other inequality. For that, we consider directly any function f_W. We have $F^G_{(i,j)} \leq C^G_{(\{i\},\{i\}^c)}$, thus

$$P\left(\frac{F^G_{(i,j)}}{np_l\mu} \geq 1 + \epsilon \right) \leq P\left(\frac{C^G_{(\{i\},\{i\}^c)}}{np_l\mu} \geq 1 + \epsilon \right). \tag{27}$$

However, $\frac{C^G_{(\{i\},\{i\}^c)}}{p_l}$ is the sum of n independent random variables whose mean is μ. Then we obtain, by the law of large numbers,

$$P\left(\frac{C^G_{(\{i\},\{i\}^c)}}{np_l\mu} \geq 1 + \epsilon \right) \to 0. \tag{28}$$

That concludes the proof.

3.3 Multicast Connections

We obtained results for the unicast problem in random graphs. Thanks to network coding, we may extend them to different types of multicast connections. We refer to [10] for the definitions of different types of multicast that we recall below.

3.3.1 Multicast

First, we look the usual multicast that is between one source node i and r sink nodes $J = \{j_k\}_{k=1,\dots,r}$ that want all the information. We denote by $F^M_{(i,J)}$ the max-flow between the source and all these sink nodes. The result is

THEOREM 5.

$$\frac{F^M_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{29}$$

This theorem tells us that the max-flow is only dependent, for the multicast, on the capacities around the source and the sinks.

PROOF. For each sink node j_k, we have

$$\frac{F^G_{(i,j_k)}}{np_l\mu} \xrightarrow{p} 1. \tag{30}$$

Then, by lemma 4 in the appendix,

$$\frac{F^M_{(i,J)}}{np_l\mu} = \min_{k=1,\dots,r} \frac{F^G_{(i,j_k)}}{np_l\mu} \xrightarrow{p} 1. \tag{31}$$

□

3.3.2 Two-layer Multicast

Definition 8. In the *two-layer multicast* problem, a source node has all the information and there are two types of sink nodes. The first type wants just a part of the information, whereas the second type wants all the information.

In the two-layer multicast case, there is always one source node i and r sink nodes $J = \{j_k\}$ but one of them, say j_1, does not want all the information but instead just a fraction ϵ of it. We denote by $F^M_{(i,J\setminus\{j_1\})}$ the maximal flow for the sink nodes $j_2, \dots, j_r$. We have

PROPOSITION 1.

$$\frac{F^M_{(i,J\setminus\{j_1\})}}{np_l\mu} \xrightarrow{p} 1 \tag{32}$$

and ϵ can take any value between 0 *and* 1.

PROOF. The proof is the same that in the simple multicast case. □

3.3.3 Disjoint Multicast

Definition 9. In the *disjoint multicast* problem, one source node has all the information, but each sink node just wants a portion of information, that is disjoint from the information needed by each other sink.

In the disjoint multicast case, we have, always, one source and r sink nodes $J = \{j_k\}$, but each node j_k just wants a disjoint portion ϵ_k of the total information sent by the source node. We denote $F^D_{(i,J)}$ the maximal flow that the source can send. We have

PROPOSITION 2.

$$\frac{F^D_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{33}$$

PROOF. For all I, subset of $\{1,\ldots,r\}$, we want

$$\max_{i\in I} C^G_{(\{i\},\{j_k|k\in I\})} \geq \sum_{i\in I} \epsilon_i F^D_{(i,J)}. \quad (34)$$

Dividing by $np_l\mu$ and taking the limit in probability, we obtain

$$1 \geq \left(\sum_{i\in I}\epsilon_i\right) \lim_p \frac{F^D_{(i,J)}}{np_l\mu}. \quad (35)$$

Then

$$\frac{F^D_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \quad (36)$$

□

3.3.4 Multisource-Multicast

Definition 10. In the *multisource-multicast* problem, the information is split among several source nodes and each sink node wants all the information of each source node.

In the multisource-multicast problem, we have t independent source nodes $I = \{i_k\}_{k=1,\ldots,t}$ and r sink nodes $J = \{j_{k'}\}_{k'=1,\ldots,r}$. Each sink node wants all the information sent by the source nodes. We denote by $F^M_{(I,J)}$ the maximal flow transmitted by all the sources (i.e., received by each node). We have

PROPOSITION 3.

$$\frac{F^M_{(I,J)}}{np_l\mu} \xrightarrow{p} 1. \quad (37)$$

PROOF. The multisource problem with one sink node is the same problem as the disjoint multicast problem if the edges are inverted. Hence, we have, if we denote $F^D_{(I,j_{k'})}$ the maximal flow between all the source nodes I and the sink node $j_{k'}$

$$\frac{F^D_{(I,j_{k'})}}{np_l\mu} \xrightarrow{p} 1. \quad (38)$$

By $F^M_{(I,J)} = \min_{k'} F^D_{(I,j_{k'})}$ and lemma 4,

$$\frac{F^M_{(I,J)}}{np_l\mu} \xrightarrow{p} 1. \quad (39)$$

□

This section concludes the results about wired networks modeled by random graphs. We have seen that, for our class of random graphs, the min-cut is around the source or the sink. Therefore, in random graphs, the max-flow is local and independent from the rest of the graph. By using network coding, in the case of multicast, it is only necessary to examine cuts around the source and the sink (i.e., local conditions) to determine the maximum amount of information that can be sent, for instance by using random linear network coding developed in [7]. On the contrary, if routing is used, we consider the whole random network (i.e., global conditions) to determine how many Steiner spanning trees can be built.

Now, we shall study the flow in random hypergraphs. To our knowledge, this work is the first proposal to extend the results from random graphs to hypergraphs. In the first section, we present the model of random hypergraphs and, in the second section, we establish asymptotic flows in some random hypergraphs.

4. MODEL OF RANDOM WIRELESS NETWORK : RANDOM WEIGHTED DIRECTED HYPERGRAPH

Wired networks can be studied using random weighted directed graphs, since a user in a wired network can send different information on his links. However, in a wireless network, a node broadcasts information to its neighbors. To model this, hypergraphs can be used to model wireless networks.

4.1 Definitions and Notation

4.1.1 Weighted Directed Hypergraph

In this section, we present a general definition of directed hypergraphs and weighted directed hypergraphs. However, the hypergraphs we shall study are more specific and their properties are given later in this section.

Definition 11. A *directed hypergraph* $H = (N, E)$ is a pair where the first element N is a set of nodes and the second element E is a set of edges. An edge is a pair (U, V), where U and V are subsets of N.

Definition 12. A *weighted directed hypergraph* (denoted by $H = (N, E), W$) is a pair where the first element is a directed hypergraph H and the second element is a non-negative function from $P(N) \times P(N) \times P(N)$ (where $P(N)$ is the set of all the subsets of N) with the constraint that $W_{(U,V,V')} = 0$ if $(U, V) \notin E$ or V' is not a subset of V.

In this work, we focus on the following sub-class of weighted directed hypergraphs to model wireless network.

Definition 13. Weighted directed hypergraphs have the following properties

1. the edge has only one node u as sender, i.e., for all edge (U, V),
$$|U| = 1 \text{ (i.e., } U = \{u\}); \quad (40)$$
2. a sender u can send to only one set of receiver nodes U, i.e., for all node u,
$$\left.\begin{array}{l}(\{u\}, U) \in E\\ (\{u\}, U') \in E\end{array}\right\} \Rightarrow U = U'; \quad (41)$$
3. a weight $w_{(u,v)} \leq 1$ is associated to each pair (u, v) of nodes (this weight represents the probability for the node u to transmit well to the node v). Then we obtain the weight of the sub-edge $(\{u\}, V, V')$ through the expression
$$W_{(\{u\},V,V')} = \prod_{v'\in V'} \prod_{v\in V\setminus V'} w_{(u,v')}\left(1 - w_{(u,v)}\right). \quad (42)$$
The weight $W_{(\{u\},V,V')}$ is the probability that the node u transmits in a lossless fashion only to the nodes in the subset V' of V.

This model of hypergraphs corresponds to a network of wireless broadcast channels without interference and with independent packet erasures for the receiver nodes. The two notions of flow and cut are applicable to hypergraphs as shown below.

4.1.2 Flow

Definition 14. The *flow* from i to j in an hypergraph is a function f on edges such that

1. it cannot send more than one bit per edge:
$$f \leq 1; \tag{43}$$
2. j does not send information:
$$f\left((\{j\}, J)\right) = 0; \tag{44}$$
3. for all node v except i, we have that the outgoing flow is less than the incoming flow:
$$f\left((\{v\}, V)\right) \leq \sum_{u \in N \setminus \{v\}} w_{(u,v)} f\left((u, U)\right). \tag{45}$$

The value F of the flow is the value of the incoming flow in j:

$$F = \sum_{u \in N \setminus \{j\}} w_{(u,j)} f\left((\{u\}, U)\right). \tag{46}$$

Definition 15. The *max-flow* is a flow with a maximal value in the hypergraph. We denote this value $F^H_{(i,j)}$.

The max-flow as before corresponds to the maximum information that can be sent from the source to the sink.

4.1.3 Cut

Definition 16. A *cut* from the set of nodes N_0 to the set of nodes N_1 is a set of sub-edges S such that if we delete these sub-edges, there is no directed path from a node in N_0 to a node in N_1. The value of the min-cut is the sum of the weights of the sub-edges in S.

Definition 17. The *min-cut* from N_0 to N_1 is a cut from N_0 to N_1 with the minimal value. We denote this value $C^H_{(N_0,N_1)}$.

As before, the min-cut max-flow theorem connects the notion of cut and flow in an hypergraph, since the max-flow from i to j is equal to the min-cut from $\{i\}$ to $\{j\}$ (i.e., $F^H_{(i,j)} = C^H_{(\{i\},\{j\})}$). As before, we have two theorems about the min-cut in hypergraphs that mirror theorems 1 and 3, which hold for graphs.

THEOREM 6. *For any hypergraph H and any subset N_0 of N, we have*

$$C^H_{(N_0,N_0^c)} = \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right)\right). \tag{47}$$

THEOREM 7. *For any weighted directed hypergraph H, we have*

$$F^H_{(i,j)} = \min_{N_0 \subset N \setminus \{j\}, i \in N_0} C^H_{(N_0,N_0^c)}. \tag{48}$$

PROOF OF THEOREM 6. The proof holds as in the graph case, but we consider the subedges

$$\left((\{u\}, \{v \in N_0^c | v \in U\}\right)_{u \in N_0}$$

where $(\{u\}, U) \in E$ that correspond to the edges

$$\left((u, v)\right)_{u \in N_0, v \in N_0^c}$$

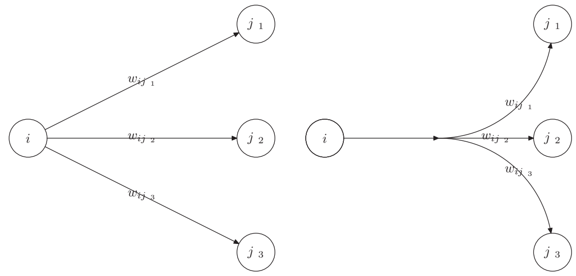

Figure 3: On the left, there is a node with its outgoing links for the graph whose weights are less than 1. On the right, the corresponding hyperedge for the hypergraph where weights are the probability to the receiver node to get the information without error.

in the graph case. These sub-edges are a cut from N_0 to N_0^c for the hypergraph, so

$$C^H_{(N_0,N_0^c)} \leq \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right)\right). \tag{49}$$

For our lower bound, we need to remove the directed edges (u, v) for all $u \in N_0$ and $v \in N_0^c$ and the minimum weight to remove all of that is to consider the sub-edges

$$\left((u, \{v \in N_0^c | v \in U\}\right)\right)_{u \in N_0},$$

and, so

$$C^H_{(N_0,N_0^c)} \geq \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right)\right). \tag{50}$$

□

4.2 Random Weighted Directed Hypergraph

Definition 18. We can associate a graph to the hypergraph in the following way. For every node u, we create the edges $\left((u, v)\right)_{v \in U, (\{u\}, U) \in E}$ and the weight for the edge (u, v) is the weight $w_{(u,v)}$ as in figure 3.

Then we have a bijection between the set of graphs with weights less than 1 and the set of the hypergraphs studied.

Definition 19. The random hypergraphs, studied here, are the hypergraphs associated to the random graphs defined in definition 7. Therefore, the random hypergraphs studied have these properties:

1. for each node u the directed hyperedge $(\{u\}, U)$ is distributed such that: for all node v,
$$P\left(v \in U\right) = p_l; \tag{51}$$
2. the weights of the edges are distributed such that: for all nodes u,v,
$$P\left(w_{(u,v)} \geq w\right) = \begin{cases} 1 & \text{if } w = 0, \\ p_l \int_w^\infty f_W(x)\,\mathrm{d}x & \text{else;} \end{cases} \tag{52}$$
3. for all N_0 subset of N,
$$\left(w_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent.} \tag{53}$$

5. UNICAST AND MULTICAST TYPES ON RANDOM HYPERGRAPHS

5.1 Unicast

In this section, we shall consider flows on random weighted directed hypergraphs. To the best of our knowledge, there is no mention of such hypergraphs in the prior literature.

THEOREM 8. *We take a random weighted directed hypergraph with $n+1$ nodes. We take i and j two nodes, i is the source node and j the sink node. If*

$$\frac{np_l}{\ln n} \to \infty \tag{54}$$

and, for all N_0 subset of N such that $i \in N_0$ and $j \notin N_0$,

$$\left(w_{(u,v)}\right)_{u\in N_0, v\in N_0^c} \text{ are independent} \tag{55}$$

then

$$F^H_{(i,j)} \xrightarrow{p} 1. \tag{56}$$

This shows a similar result as for random graphs, i.e., the capacity is limited by the capacity of the source (and only the source here) and not by the rest of the hypergraph.

5.2 Proof

We shall prove an upper and lower bound to show the probability convergence. First, we prove $P\left(F^H_{(i,j)} \leq 1-\epsilon\right) \to 0$ in two parts, since we need a technical trick to obtain the required result. In the second part, we prove the second convergence through the law of large numbers. The most important idea in this proof is that we compare graphs and hypergraphs and we show that, for two corresponding cuts, the difference between the value of this cut and the cut around the source is bigger for hypergraphs than for graphs.

5.2.1 $P\left(F^H_{(i,j)} \leq 1-\epsilon\right) \to 0$

To prove this statement, we need to consider, first, a restricted f_W that satisfies that there exists w_m such that, for all $w < w_m$,

$$f_W(w) = 0. \tag{57}$$

Then, we shall generalize to an arbitrary f_W.

For the restricted f_W.

The proof begins by a lemma that establishes that, if a cut around a node is less than $1-\epsilon$ in the random hypergraph, then it is also the case in the associated random graph.

LEMMA 3. *For all ϵ, there exists $n_{\max}$ such that for all $n \geq n_{\max}$, for all $k \leq n-1$ and for all sequences $(w_q)_{q=1,\ldots,k}$, we have*

$$1-\prod_{q=1}^{k}(1-w_q) \leq 1-\epsilon \Rightarrow \frac{\sum_{q=1}^{k} w_q}{np_l\mu} \leq 1-\epsilon. \tag{58}$$

PROOF. We prove, first, an easier result where k is fixed. For all ϵ, for all k and for all sequences $(w_q)_{q=1,\ldots,k}$, there exists $n_{\max,k}$ such that for all $n \geq n_{\max,k}$,

$$1-\prod_{q=1}^{k}(1-w_q) \leq 1-\epsilon \Rightarrow \frac{\sum_{q=1}^{k} w_q}{np_l\mu} \leq 1-\epsilon. \tag{59}$$

That is clear since

- on the one hand, we have $1-\prod_{q=1}^{k}(1-w_q)$ that is constant;
- on the other hand, $\frac{\sum_{q=1}^{k} w_q}{np_l\mu}$ converges to 0 as $n \to \infty$.

Now, we derive an upper-bound for k. To upper bound k, we need the special form of the distribution,

$$(1-w_m)^k \geq \prod_{q=1}^{k}(1-w_q) \geq \epsilon. \tag{60}$$

Thus,

$$k \leq \frac{\ln \epsilon}{\ln(1-w_m)}. \tag{61}$$

Therefore, now, we can switch $\forall k$ and $\exists n_{\max}$, by taking $n_{\max} = \max_{k \leq \frac{\ln\epsilon}{\ln(1-p_m)}} n_{\max,k}$. □

Now, we shall prove that the probability for a node u to have a cut around it less than $1-\epsilon$ is less probable in the hypergraph than in the graph divided by $np_l\mu$. For all $v \in N\backslash\{u\}$, we denote $l_{(u,v)}$ the random variable that is 1 if the edge (u,v) exists and 0 else. Then, for all $n \geq n_{\max}$, for all subset N_0 of N, $i \in N_0$ with $|N_0| = k+1$,

$$\begin{aligned}
&P\left(C^H_{(\{u\},N_0^c)} \leq 1-\epsilon\right)\\
&= \int_{\substack{\{0,1\}^{n-k}\\ [p_m,1]^{n-k}}} \mathbf{1}_{C^H_{(\{u\},N_0^c)} \leq 1-\epsilon} \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} (\mathrm{d}f_W)_{v\in N_0^c}\\
&= \int_{\substack{\{0,1\}^{n-k}\\ [p_m,1]^{n-k}}} \mathbf{1}_{1-\prod_{v\in compN_0}\left(1-l_{(u,v)}w_{(u,v)}\right)\leq 1-\epsilon} \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} (\mathrm{d}f_W)_{v\in N_0^c}\\
&\leq \int_{\substack{\{0,1\}^{n-k}\\ [p_m,1]^{n-k}}} \mathbf{1}_{\frac{\sum_{v\in N_0^c} l_{(u,v)}w_{(u,v)}}{np_l\mu}\leq 1-\epsilon} \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} (\mathrm{d}f_W)_{v\in N_0^c}\\
&= \int_{\substack{\{0,1\}^{n-k}\\ [p_m,1]^{n-k}}} \mathbf{1}_{C^G_{(\{u\},N_0^c)} \leq 1-\epsilon} \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} (\mathrm{d}f_W)_{v\in N_0^c}\\
&= P\left(C^G_{(\{u\},N_0^c)} \leq 1-\epsilon\right).
\end{aligned}$$

For a general f_W.

We now provide an approximation for the general case. For that, we shall delete all the edges whose weight is less than a certain w_m (we can choose any $w_m < \mu$). The new hypergraph will be denoted by $\check{H}$ and each previous quantity x in the first hypergraph or graph associated will be denoted by $\check{x}$ when we delete the edges whose weight is less than w_m. We have that the new probability for two nodes to be linked is given by

$$\check{p_l} = \left(1-\int_{x=0}^{w_m} f_W(x)\,\mathrm{d}x\right)p_l. \tag{62}$$

However, we still have that $\frac{n\check{p_l}}{\ln n} \to \infty$. In this limiting regime, we have that

$$P\left(C^H_{(\{u\},N_0^c)} < 1-\epsilon\right) \leq P\left(C^{\check{H}}_{(\{u\},N_0^c)} < 1-\epsilon\right) \tag{63}$$

since any edge weight in $\check{H}$ is less than or equal to that of the corresponding edge in H.

We may now readily establish our result. Indeed, since

$$\begin{aligned} C^H_{(N_0,N_0^c)} &= \textstyle\sum_{u\in N_0} C^H_{(\{u\},N_0^c)}, \\ C^G_{(N_0,N_0^c)} &= \textstyle\sum_{u\in N_0} C^G_{(\{u\},N_0^c)}, \end{aligned}$$

we obtain that

$$P\left(C^H_{(N_0,N_0^c)} < 1-\epsilon\right) \le P\left(\frac{C^{\check{G}}_{(N_0,N_0^c)}}{np_l\mu} < 1-\epsilon\right). \quad (64)$$

By minimizing over all N_0 subset of N where $i \in N_0$ and $j \notin N_0$,

$$P\left(F^H_{(i,j)} < 1-\epsilon\right) \le P\left(\frac{F^{\check{G}}_{(i,j)}}{np_l\mu} < 1-\epsilon\right) \to 0 \quad (65)$$

5.2.2 $P\left(F^H_{(i,j)} \ge 1+\epsilon\right) \to 0$

Since the cut around the source i is a cut, we have

$$F^H_{(i,j)} \le C^H_{(\{i\},\{i\}^c)} \le 1. \quad (66)$$

Hence,

$$P\left(F^H(n) > 1+\epsilon\right) = 0. \quad (67)$$

This concludes the proof.

5.3 Multicast Types in Random Hypergraph

The proofs are the same as in the multicast types of random graphs. We state the results here without proof for brevity.

5.3.1 Multicast

THEOREM 9. *We denote $F^M_{(i,J)}$, the maximal flow through the network from one source node i to r sink nodes $J = \{j_k\}_{k=1,\dots,r}$ that want all information. We have*

$$F^M_{(i,J)} \xrightarrow{p} 1. \quad (68)$$

5.3.2 Two-layer Multicast

PROPOSITION 4. *We denote $F^M_{(i,J\setminus\{j_1\})}$ the maximal flow achievable for the multicast from the source node i to the $r-1$ sink nodes $J\setminus\{j_1\} = \{j_k\}_{k=2,\dots,r}$ when j_1 just wants a fraction ϵ of the total information. We have*

$$F^M_{(i,J\setminus\{j_1\})} \xrightarrow{p} 1 \quad (69)$$

and ϵ can take any value between 0 and 1.

5.3.3 Disjoint Multicast

PROPOSITION 5. *We denote $F^D_{(i,J)}$ the maximal flow that the source can send where the source node i sends information to r sink nodes $J = \{j_k\}_{k=1,\dots,r}$ that want, each, a fraction ϵ_k of information and all disjoint from each other. We have*

$$F^D_{(i,J)} \xrightarrow{p} 1. \quad (70)$$

Note that we cannot generalize the multisource-multicast results by the same method used in the graph case, since we cannot reverse the hyperedges as we have done in the graph case.

These results show that, if we use network coding in the random hypergraph case since, if only the capacity around the source is known, the maximal amount of information that can be sent through the random hypergraph for a multicast can be determined. Furthermore, it can be determined locally without prior knowledge of the whole hypergraph.

6. CONCLUSION

We have shown that, for a large class of random graphs and hypergraphs, the capacity of the network can be easily known by looking at the cut around the source, a local procedure. This result generalizes a large number of results previously obtained about random graphs in [5], [15] and [14]. Moreover, to the best of our knowledge, we provide the first result about max-flows in random hypergraphs.

We use simple *geometryless* models for random graphs and hypergraphs. In addition, our results are asymptotic. Therefore, our work opens up interesting questions and areas of research. Primary amongst them is the extension of our results to random geometric graphs. Another important question for future work involves the determination of the minimal graph size that guarantees the validity of our results.

7. ACKNOWLEDGMENTS

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APPENDIX

LEMMA 4. *Let $Y(n) = (Y_1(n), \ldots, Y_l(n))$ be a sequence of random vectors and $(y_1, \ldots, y_l)$ a vector of real numbers such that, for all i, $Y_i(n)$ converge in probability to y_i ($Y_i(n) \xrightarrow{p} y_i$). Then,*

$$\min_i Y_i(n) \xrightarrow{p} \min_i y_i. \tag{71}$$

PROOF. Assume that y_1 is the minimum of the y_is. Let $\epsilon > 0$. We have,

$$\begin{aligned} P\left(\min_i Y_i(n) - y_1 > \epsilon\right) &\leq P\left(Y_1(n) - y_1 > \epsilon\right) \\ &\to 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &P\left(\min_i Y_i(n) - y_1 < -\epsilon\right) \\ &\quad \leq \sum_{i=1}^{l} P\left(Y_i(n) - y_1 < -\epsilon\right) \\ &\quad \leq \sum_{i=1}^{l} P\left(Y_i(n) - y_i < -\epsilon\right) \\ &\quad \to 0 \end{aligned}$$

□