Evolutionary Stable Strategies in Interacting Communities*

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ABSTRACT

In many social networks or biological systems, several populations may be influenced by interactions with not only individuals of their own communities but also with other communities in the network. Moreover, the inter-community interactions can affect the outcome of a community in addition to the strategies followed inside the community itself. However, throughout the variety of research works in this topic, classical model of evolutionary games rather assumes individuals are equally likely to interact with any other member of the population and the success of any individual depends only on the frequency with which all strategies are represented in the population. Instead, we devote in this paper a novel mathematical tool to model the evolution of a population composed of several communities where the interaction between them is non-uniform.

The main objective of our work is to model these interactions using evolutionary games and to find the evolutionary stable strategies (ESS). Hence, we show through the study of two communities and pairwise interactions that when taking into account the non-uniform interaction, any mixed Nash equilibrium is not an ESS wherein all communities are robust against a deviation of a fraction of the population from both communities which may wish to deviate. Moreover, the same mixed equilibrium is an ESS under some conditions when we consider the fitness of the whole population instead of the fitness of each community. We also demonstrate that this equilibrium is globally asymptotically stable for the replicator dynamic which describes the dynamic of the evolution of communities as well as the impact of the interaction between communities.

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1. INTRODUCTION

In recent years, *Evolutionary Game Theory* has become a powerful tool for predicting and even designing evolutions in many fields. Its origin comes from Biology where it was introduced by [9, 10] to model competitions among animals. It differs from classical Game Theory by: (i) its focusing on the evolution dynamics of the fraction of members of the population that use a given strategy, and (ii) in the notion of Evolutionary Stable Strategy (ESS, [10]) which includes robustness against a deviation of a whole (possibly small) fraction of the population who may wish to deviate; this is in contrast with the standard *Nash Equilibrium* that only incorporates robustness against deviation of a single player.

Throughout the variety of research works in this topic, the *Evolutionary Game Theory* has become perhaps the most important mathematical tool for describing and modeling evolution since *Darwin*. Indeed, on the importance of the ESS for understanding the evolution of species, *Dawkins* writes in his book entitled "The Selfish Gene" [14]: "We may come to look back on the invention of the ESS concept as one of the most important advances in evolutionary theory since Darwin". He further wrote: "Maynard Smith's concept of the ESS will enable us, for the first time, to see clearly how a collection of independent selfish entities can come to resemble a single organized whole". Despite its origin and original purpose, Evolutionary Game Theory has become of increasing interest to economists [2], sociologists, anthropologists, and philosophers. In computer science, Evolutionary Game Theory has attracted considerable interest among numerous research efforts ranging from multiple access protocols [11] to multihoming [8], and from resources competition in Internet [15] to Radio Resource Management in wireless networks.

For a wide variety of issues and important questions in economical, social, and biological sciences, a very successful formulation is that of the Evolutionary Game Theory, where

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interaction among agents is often modeled as a game with different possible strategies, and strategies are spread over the population. Unfortunately, the theory developed in this field has focused on the homogeneous population in which a given player is equally likely to interact with any other member of the population and that the success of any individual depends on the frequency of all other strategies represented in the population. In contrast, in many examples in social networks, the population is composed of several communities or groups and communities can be seen as clusters of the graph derived from a social network. In this scenario, a community is therefore influenced by interactions inside this community and with other communities. In fact, interactions between individuals are inherently non-uniform because they are separated by spatial locations, languages or cultures, and individuals are more likely to interact with some agents than others. In Biology, some animal species are strongly territorial, and territories vary in quality. Hence an animal may fight animals from different species and the payoff is strongly dependent on the nature of species. For example the probability that an animal being hurt or killed is higher if he meets a larger animal than smaller animal. It is also assumed that the payoff of an interaction is the same with different opponents. Embedding non uniform interaction dimensions into evolutionary games, our goal is to express the evolution of each community in which agents maintain different levels of cooperation and interactions with different ones.

We follow the definition of ESS and extend it to n interacting communities by introducing several levels of stability. The main focus of this work is on extending the evolutionary game theory to communities of individuals in which each group has its own set of strategies, payoff matrix and a resulting outcome. In addition, each group is interacting with another group with different payoff matrix and resulting outcome with a changing interacting probability (see figure 1). The main contributions of the paper are as follows

- Any mixed Nash equilibrium is not an ESS wherein all communities using this equilibrium cannot be invaded by a small group from all communities with a mutant strategy.
- Under some assumptions on the payoff and interaction probabilities, this mixed equilibrium is an ESS when we consider the global fitness of all the population instead of the fitness of each community. Further, this equilibrium is globally asymptotically stable for the replicator dynamic.
- Finally, we analyze one of the most studied examples in evolutionary games, that of the Prisoner's Dilemma which is a model for determining the degree of cooperation in the population.

Paper Structure. This paper is organized as follows: In Section 2, we introduce a mathematical model of evolutionary games in case of non-uniform interactions between multiple communities and we present different ESS characterizations which differ in the level of stability. The existence of ESS for two communities is studied in Section 3. In Section 4, we focus on the dynamical stability of the ESSs under the replicator dynamic. We study in Section 5 numerical examples inspired from the classical Prisoner's Dilemma. Finally, we conclude this paper in Section 6

2. MULTIPLE COMMUNITIES WITH NON-UNIFORM INTERACTION

We consider a large population of players or individuals divided into N communities. Let Γ be the set of communities. Each individual may interact either with an individual from its own community or an individual from another community. Let p_{ij} be the probability of an individual in class i interacts with an individual from class j with $\sum_j p_{ij} = 1$ (see figure 1). These individuals compete

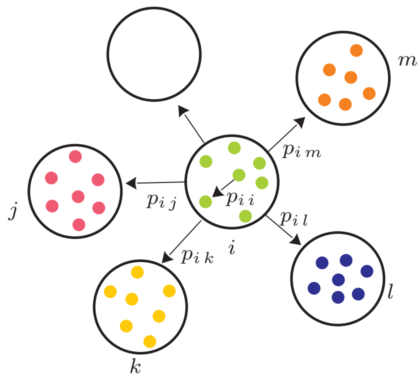


Figure 1: Graph of interacting communities.

through a large number of random pairwise interactions. We assume that there are n_i pure strategies for each community $i \in \Gamma$ and a strategy of an individual is a probability distribution over the pure strategies. We denote by $A_{ij} = (a^{ij}_{kl})_{k=1..n_i, l=1..n_j}$ the payoff matrix where $A_{ij} = A_{ji}$. If a player of community i using pure strategy k, interacts with a player of community j using pure strategy l, its payoff is a^{ij}_{kl}.

Let s_i, $i \in \Gamma$ be the n_i dimensional vector whose j-th element s_{ij} is the population share of strategy j in community i and $\mathbf{s} = (s_1, ..., s_N)$. Let $U_i(k, \mathbf{s}, p)$ be the expected payoff of strategy k for community i when the profile of all the population is $\mathbf{s}$. Then the payoff function U_i is given by

$$U_i(k, \mathbf{s}, p) = \sum_{j \in \Gamma} p_{ij} e_k A_{ij} s_j, \qquad (1)$$

where e_k $k-$th element of the canonical basis of $\mathbb{R}^{n_i}$. Also, the expected payoff to an individual from community i when the profile of all the population is $\mathbf{s}$, is written as

$$\bar{U}_i(s_i, \mathbf{s}, p) = \sum_{k=1}^{n_i} s_{ik} U_i(k, \mathbf{s}, p). \qquad (2)$$

In the following subsections, we present different ESS characterizations that turn out to be interesting in analyzing the dynamical stability under the replicator dynamic. These ESS characterizations differ mostly in the static stability condition.

2.1 Strong ESS

A strong ESS is a strategy that remains robust against invasion from a small group composed from all communities and using an alternative strategy. Hence, the following definition can be stated:

Definition 1. A state $\mathbf{s}^*$ is a strong ESS, if for all $\mathbf{s} \neq \mathbf{s}^*$, there exists an $\epsilon(\mathbf{s}) > 0$ such that for all $i = 1, ..., N$ and $\epsilon \leq \epsilon(\mathbf{s})$

$$\bar{U}_i(s_i, \epsilon\mathbf{s} + (1-\epsilon)\mathbf{s}^*, p) < \bar{U}_i(s_i^*, \epsilon\mathbf{s} + (1-\epsilon)\mathbf{s}^*, p). \quad (3)$$

This strong ESS must in fact have a uniform invasion barrier or threshold where any proportion of invaders using an alternative strategy is repelled. Equivalently, an alternative definition can be established as follows:

Definition 2. A state $\mathbf{s}^*$ is a strong ESS if and only if it meets two conditions for all i and for all $\mathbf{s} \neq \mathbf{s}^*$:

- $\circ$ $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, (4)
- $\circ$ if $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, then $\bar{U}_i(s_i, \mathbf{s}, p) < \bar{U}_i(s_i^*, \mathbf{s}, p)$. (5)

The condition (5) states that all the communities have an incentive to remain at their ESS components when an alternative best-reply is used. Indeed, this is an immunity against a deviation from the ESS. The stability of the strong ESS requires that all populations stick to the ESS when a whole fraction of mutants, i.e. a fraction of mutants in all communities, is introduced.

2.2 A weak ESS

In this subsection, we introduce an alternative ESS version with a weaker stability condition. We focus particularly on the stability against a single local fraction of mutants in a single community. The definition of the weak ESS is given by:

Definition 3. A state $\mathbf{s}^*$ is a weak ESS if for all $\mathbf{s} \neq \mathbf{s}^*$ and for all $i \in \Gamma$, there exists $\epsilon_i(\mathbf{s}) > 0$ such that for all $\epsilon_i \leq \epsilon_i(\mathbf{s})$

$$\bar{U}_i(s_i, \eta(i, \mathbf{s}^*, s_i, \epsilon_i), p) < \bar{U}_i(s_i^*, \eta(i, \mathbf{s}^*, s_i, \epsilon_i), p), \quad (6)$$

where $\eta(i, \mathbf{s}^*, s_i, \epsilon_i) = (s_1^*, .., s_{i-1}^*, \epsilon_i s_i + (1-\epsilon_i)s_i^*, .., s_N^*)$, $\bar{U}_i(s_i, \eta(i, \mathbf{s}^*, s_i, \epsilon_i), p)$ is the expected payoff to a mutant in community i using s_i, and $\bar{U}_i(s_i^*, \eta(i, \mathbf{s}^*, s_i, \epsilon_i), p)$ is the expected payoff to a non-mutant in community i using s_i^* when the profile of all populations is $\eta(i, \mathbf{s}^*, s_i, \epsilon_i)$.

An equivalent definition of a weak ESS can be stated as follows:

Proposition 1. A state $\mathbf{s}^*$ is a weak ESS if and only if, for all i and for all $\mathbf{s} \neq \mathbf{s}^*$,

- $\circ$ $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, (7)
- $\circ$ if $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s_i^*, \mathbf{s}^*, p)$,
  then $\bar{U}_i(s_i, (s_i, \mathbf{s}_{-i}^*), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}_{-i}^*), p)$, (8)

where $\bar{U}_i(s_i, \mathbf{s}^*, p)$ is the expected payoff of a mutant in community i using s_i, and $\bar{U}_i(s_i^*, \mathbf{s}^*, p)$ is the expected payoff of a non-mutant in community i using s_i^*, both when the profile of all populations is $\mathbf{s}^*$.

PROOF. see appendix □

This ESS definition is quite different from that of Cressman, referred to as *Cressman ESS* or weak ESS in the literature [6, 1], which considers invasion of the communities by a fraction of mutants from all communities. For a state to be a *Cressman ESS*, it is enough that one incumbent community earn better than the correponding mutant group. In our definition, we consider invasion of a single community by a local group of mutants, i.e., a goup inside this community.

2.3 Intermediate ESS

In the intermediate ESS version (Taylor, 1979) [12, 7, 13, 1], the main focus is the total payoff of the whole population instead of the fitness of each community. It guarantees that when the whole population adopts the strategy, for any small group of deviant strategies, the total fitness is worse than the intermediate ESS. This is an intermediate version because it does not require that the ESS be immune against mutant strategies for every community; and it needs more than stability against a small mutant group in a single community. The formal definition of an intermediate ESS is given by:

Definition 4. A state $\mathbf{s}^*$ is an intermediate ESS if for all $\mathbf{s} \neq \mathbf{s}^*$, there exists an $\epsilon(\mathbf{s}) > 0$ such that for all $\epsilon \leq \epsilon(\mathbf{s})$

$$\sum_{i \in \Gamma} \bar{U}_i(s_i, \epsilon\mathbf{s} + (1-\epsilon)\mathbf{s}^*, p) < \sum_{i \in \Gamma} \bar{U}_i(s_i^*, \epsilon\mathbf{s} + (1-\epsilon)\mathbf{s}^*, p). \quad (9)$$

Equivalently, we have the following definition:

Proposition 2. A state $\mathbf{s}^*$ is an intermediate ESS if and only if for all $\mathbf{s} \neq \mathbf{s}^*$

- $\circ$ $\sum_{i \in \Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \sum_{i \in \Gamma} \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, (10)
- $\circ$ if $\sum_{i \in \Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) = \sum_{i \in \Gamma} \bar{U}_i(s_i^*, \mathbf{s}^*, p)$, then

$$\sum_{i \in \Gamma} \bar{U}_i(s_i, \mathbf{s}, p) < \sum_{i \in \Gamma} \bar{U}_i(s_i^*, \mathbf{s}, p). \quad (11)$$

PROOF. see appendix □

Again, the condition (10) defines the best-reply requirement according to which a mutant strategy cannot yield a better total payoff than the ESS. When the comparison in this condition is an equality, i.e. in case of an alternative best-reply, the condition (11) guarantees that the population profile do not shift away from the ESS. It means that all population have a positive incentive to remain at the ESS when there is a mutant strategy. Intuitively, we expect the stability condition to guarantee the dynamic stability in the replicator dynamic as it focuses on the fitness (expected payoff) of the whole population when simultaneous perturbations arise in all communities [12].

2.4 A comparative view of the ESSs

It is fairly easy to check out that the strong ESS is an intermediate ESS, and an intermediate ESS is a weak ESS. The converse does not hold. Indeed, a strong ESS is robust against simultaneous perturbations in all communities, which includes the case of a single local perturbation, and hence a strong ESS is a weak ESS. In addition, each population cannot earn a higher payoff by deviating from the strong ESS, then the total payoff of all the populations, i.e. the sum of the payoffs, cannot be better and hence the requirement for an intermediate ESS is satisfied. Consequently, a strong ESS is an intermediate ESS. Also, when there is a single local mutation in any community, an intermediate ESS implies a weak ESS.

3. TWO COMMUNITIES - TWO STRATEGIES

In order to simplify the presentation of the paper, we focus on the case of two communities and two strategies, i.e., $N = 2$, $n_1 = 2$, and $n_2 = 2$. Suppose that each individual of a community only uses a pure strategy. Let p be the probability of having an interaction with an individual in the same community and consider four bi-matrices A, D, B, C that describe respectively the games played in the first community, in the second community and between the communities.

$$A = \begin{array}{c} \\ G_1 \\ H_1 \end{array} \begin{pmatrix} G_1 & H_1 \\ a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \; D = \begin{array}{c} \\ G_2 \\ H_2 \end{array} \begin{pmatrix} G_2 & H_2 \\ a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

$$B = \begin{array}{c} \\ G_1 \\ H_1 \end{array} \begin{pmatrix} G_2 & H_2 \\ a & b \\ c & d \end{pmatrix}, \; C = B' = \begin{array}{c} \\ G_2 \\ H_2 \end{array} \begin{pmatrix} G_1 & H_1 \\ a & c \\ b & d \end{pmatrix}.$$

We assume that each player acts the same when playing against an individual from either its own community or the other community. The matrices B and C are the transpose to each other. Let $\mathbf{s} = (s_1, s_2)$, with s_i denotes the population share of the strategy G_i in community i. The payoff obtained by an individual in community 1 when he uses G_1 (resp. H_1) is

$$U_1(G_1, \mathbf{s}, p) = p(s_1 a_1 + (1 - s_1)b_1) + (1 - p)(s_2 a + (1 - s_2)b),$$
$$U_1(H_1, \mathbf{s}, p) = p(s_1 c_1 + (1 - s_1)d_1) + (1 - p)(s_2 c + (1 - s_2)d),$$

and the payoff obtained by an individual in community 2 when he uses G_2 (resp. H_2) is

$$U_2(G_2, \mathbf{s}, p) = p(s_2 a_2 + (1 - s_2)b_2) + (1 - p)(s_1 a + (1 - s_1)c),$$
$$U_2(H_2, \mathbf{s}, p) = p(s_2 c_2 + (1 - s_2)d_2) + (1 - p)(s_1 b + (1 - s_1)d).$$

In addition, the expected payoff of any individual from community i, $i = 1, 2$, is given by:

$$\bar{U}_i(s_i, \mathbf{s}, p) = s_i U_i(G_i, \mathbf{s}, p) + (1 - s_i)U_i(H_i, \mathbf{s}, p).$$

3.1 Dominant strategies

A dominant strategy will eventually thrive and displace all dominated strategies. From the model above, the strategy G_i dominates the strategy H_i, $i = 1, 2$, if and only if

$$ps_i L_i + (1 - p)s_{-i}L + K_i \geq 0, \; \forall (s_i, s_{-i}) \in [0, 1]^2, \quad (12)$$

where $L_i = (a_i - b_i - c_i + d_i)$, $L = a - b - c + d$ and $K_i = p(b_i - d_i) + (1 - p)(x_i - d)$ with $x_1 = b$ and $x_2 = c$. We have the same kind of inequality for determining if H_i dominates G_i. Therefore, we can establish the following theorem:

THEOREM 1.
- *The strategy G_i dominates the strategy H_i if and only if $K_i \geq 0$, $pL_i + K_i \geq 0$, $(1 - p)L + K_i \geq 0$ and $pL_i + (1 - p)L + K_i \geq 0$.*

  Furthermore, if the strategy G_i is dominant, then it is a strong ESS.

- *The strategy H_i dominates the strategy G_i if and only if $K_i \leq 0$, $pL_i + K_i \leq 0$, $(1 - p)L + K_i \leq 0$ and $pL_i + (1 - p)L + K_i \leq 0$.*

  Furthermore, if the strategy H_i is dominant, then it is a strong ESS.

PROOF.
- The strategy G_i dominates the strategy H_i if and only if $\forall (s_i, s_{-i}) \in [0, 1]^2$, $U_i(G_i, (s_1, s_2), p) \geq U_i(H_i, (s_1, s_2), p)$ then as stated before, if and only if $ps_i L_i + (1 - p)s_{-i}L + K_i \geq 0$, which is $K_i \geq 0$, $pL_i + K_i \geq 0$, $(1-p)L + K_i \geq 0$ and $pL_i + (1-p)L + K_i \geq 0$.
- In the same way, strategy H_i dominates the strategy G_i if and only if $K_i \leq 0$, $pL_i + K_i \leq 0$, $(1-p)L + K_i \leq 0$ and $pL_i + (1 - p)L + K_i \leq 0$.

□

3.2 Mixed Nash equilibrium

In this subsection, we study the conditions for which the mixed Nash equilibrium, if it exists, is an ESS. At the mixed ESS, all strategies co-exist.

THEOREM 2. *Let $\Delta = p^2 L_1 L_2 - (1-p)^2 L^2$. When $\Delta > 0$, $(1-p)K_{-i}L - pK_i L_{-i} > 0$, and $\Delta > (1-p)K_{-i}L - pK_i L_{-i}$ or $\Delta < 0$, $(1 - p)K_{-i}L - pK_i L_{-i} < 0$, and $\Delta < (1 - p)K_{-i}L - pK_i L_{-i}$, for i=1,2 , then there exists a mixed Nash equilibrium (s_1^*, s_2^*) where*

$$s_i^* = \frac{(1 - p)K_{-i}L - pK_i L_{-i}}{\Delta}, i = 1, 2.$$

Furthermore, any mixed Nash equilibrium, i.e. $s_i > 0$ and $s_i < 1$, is not a strong ESS.

PROOF. There exists a mixed Nash equilibrium strategy (s_1^*, s_2^*) for which users from any group are indifferent from playing strategy G_i or H_i, i.e., for the two groups we have the following system of equations:

$$p[s_1^*(a_1 - c_1) + (1 - s_1^*)(b_1 - d_1)] + (1 - p)[s_2^*(a - c) +$$
$$+(1 - s_2^*)(b - d)] = 0$$
$$p[s_2^*(a_2 - c_2) + (1 - s_2^*)(b_2 - d_2)]$$
$$+(1 - p)[s_1^*(a - b) + (1 - s_1^*)(c - d)] = 0$$

Using the previous notations, we can rewrite the system as follow:

$$\left\{ \begin{array}{lcl} ps_1^* L_1 + (1 - p)s_2^* L + K_1 & = & 0 \text{ (a)} \\ ps_2^* L_2 + (1 - p)s_1^* L + K_2 & = & 0 \text{ (b)} \end{array} \right. \quad (13)$$

We multiply both sides of (13)(a) by pL_2 and (13)(b) by $(1 - p)L$, this yields

$$\left\{ \begin{array}{lcl} p^2 s_1^* L_1 L_2 + p(1 - p)s_2^* L L_2 + pK_1 L_2 & = & 0 \\ p(1 - p)s_2^* L_2 L + (1 - p)^2 s_1^* L^2 + (1 - p)K_2 L & = & 0 \end{array} \right.$$

then,

$$p^2 s_1^* L_1 L_2 + pK_1 L_2 = (1 - p)^2 s_1^* L^2 + (1 - p)K_2 L,$$

thus

$$s_1^* = \frac{(1 - p)K_2 L - pK_1 L_2}{\Delta}.$$

Doing the same inversely, we obtain :

$$s_2^* = \frac{(1 - p)K_1 L - pK_2 L_1}{\Delta},$$

with $\Delta = p^2 L_1 L_2 - (1 - p)^2 L^2$. Hence there exists a mixed strategy equilibrium if $0 < s_i^* < 1$ for $i = 1, 2$. Suppose $\Delta > 0$, then $0 < s_i^* < 1$ if only and if $0 <$

$(1-p)K_{-i}L - pK_iL_{-i} < \Delta$. If $\Delta < 0$, then $0 < s_i^* < 1$ if and only if $\Delta < (1-p)K_{-i}L - pK_iL_{-i} < 0$.

Now, let us check for which condition, the equilibrium $\mathbf{s}^* = (s_1^*, s_2^*)$ is a strong ESS. Assume that there is a "mutant" that uses another strategy $\mathbf{s} = (s_1, s_2)$. By definition of the expected utility, we have

$$\bar{U}_1(s_1, (\bar{s}_1, \bar{s}_2), p) = (1-p)(s_1\bar{s}_2 a + s_1(1-\bar{s}_2)b + (1-s_1)\bar{s}_2 c + (1-s_1)(1-\bar{s}_2)d).$$

It follows that:

$$\begin{aligned} & \bar{U}_1(s_1^*, \mathbf{s}^*, p) - \bar{U}_1(s_1, \mathbf{s}^*, p) \\ = \ & (s_1^* - s_1)(ps_1^*L_1 + (1-p)s_2^*L + K_1) = 0 \end{aligned}$$

Following the same procedure for group 2, we obtain

$$\bar{U}_2(s_2^*, \mathbf{s}^*, p) - \bar{U}_2(s_2, \mathbf{s}^*, p) = 0$$

From (5), $\mathbf{s}^*$ is a strong ESS if only and if $\bar{U}_i(s_i^*, \mathbf{s}, p) - \bar{U}_i(s_i, \mathbf{s}, p) > 0$ for $i = 1, 2$. But

$$\begin{aligned} & \bar{U}_i(s_i^*, \mathbf{s}, p) - \bar{U}_i(s_i, \mathbf{s}, p) \\ = \ & p(s_i^* - s_i)\Big(s_i a_i + (1-s_i)b_i - s_i c_i - (1-s_i^*)d_i\Big) \\ & +(1-p)(s_i^* - s_i)\Big(s_{-i}a + (1-s_{-i})b_i - s_{-i}c_-i - (1-s_{-i}^*)d\Big) \\ = \ & (s_i^* - s_i)\Big(ps_iL_i + (1-p)s_{-i}L + K_i\Big). \end{aligned}$$

We define f_1 and f_2 as follows:

$$\begin{aligned} f_1(s_1, s_2) &= (s_1^* - s_1)\Big(ps_1L_1 + (1-p)s_2L + K_1\Big), \\ f_2(s_1, s_2) &= (s_2^* - s_2)\Big(ps_2L_2 + (1-p)s_1L + K_2\Big). \end{aligned}$$

We have $\nabla f_1{}^T = \Big[2pL_1(s_1^* - s_1) + (1-p)L(s_2^* - s_2), (1-p)L(s_1^* - s_1)\Big]$,
$\nabla f_1(s_1^*, s_2^*) = [0\ 0]$,
$\frac{\partial^2 f_1}{\partial s_1{}^2} = -2pL_1$, $\frac{\partial^2 f_1}{\partial s_2{}^2} = 0$,
$\frac{\partial^2 f_1}{\partial s_1 \partial s_2} = -(1-p)L$, $\frac{\partial^2 f_1}{\partial s_2 \partial s_1} = -(1-p)L$.

Hence $\frac{\partial^2 f_1}{\partial s_1{}^2}\frac{\partial^2 f_1}{\partial s_2{}^2} - \frac{\partial^2 f_1}{\partial s_1 \partial s_2}\frac{\partial^2 f_1}{\partial s_2 \partial s_1} = -(1-p)^2L^2 < 0$ at (s_1^*, s_2^*). Consequently, (s_1^*, s_2^*) is a saddle point. Since $f_1(s_1^*, s_2^*) = 0$, f_1 changes the sign around s^* (grows in some directions and declines in others). Following the same procedure with f_2, we find that (s_1^*, s_2^*) is a saddle point. Therefore, the condition of stability (5) does not hold and consequently s^* is not a strong ESS. □

A necessary and sufficient condition for the mixed Nash equilibrium to be an intermediate ESS is given in the following theorem:

Theorem 3. *A mixed Nash equilibrium is an intermediate ESS if and only if $L_1 < 0$ and $\Delta = p^2L_1L_2 - (1-p)^2L^2 > 0$.*

Proof. Assume there is a mutant that uses a strategy $\mathbf{s} = (s_1, s_2)$. We have $\bar{U}_1(s_1^*, \mathbf{s}^*, p) = \bar{U}_1(s_1, \mathbf{s}^*, p)$ and $\bar{U}_2(s_2^*, \mathbf{s}^*, p) = \bar{U}_2(s_2, \mathbf{s}^*, p)$. Therefore, $\mathbf{s}^*$ is an intermediate ESS if and only if

$$\bar{U}_1(s_1, \mathbf{s}, p) + \bar{U}_2(s_2, \mathbf{s}, p) < \bar{U}_1(s_1^*, \mathbf{s}, p) + \bar{U}_2(s_2^*, \mathbf{s}, p).$$

If we define g as follows
$g(s_1, s_2) = \bar{U}_1(s_1^*, \mathbf{s}, p) - \bar{U}_1(s_1, \mathbf{s}, p) + \bar{U}_2(s_2^*, \mathbf{s}, p) - \bar{U}_2(s_2, \mathbf{s}, p)$, we get

$$\begin{aligned} g(s_1, s_2) = \ & (s_1^* - s_1)\big[ps_1L_1 + (1-p)s_2L + K_1\big] \\ & +(s_2^* - s_2)\big[ps_2L_2 + (1-p)s_1L + K_2\big]. \end{aligned}$$

Moreover, we have
$\nabla g(s_1, s_2)^T = [(1-p)L(s_2^* - s_2) + pL_1(s_1^* - s_1) - ps_1L_1 - (1-p)s_2L - K_1 \quad (1-p)L(s_1^* - s_1) + (s_2^* - s_2)pL_2 - ps_2L_2 - (1-p)s_1L - K_2]$, $g(s_1^*, s_2^*) = 0$, and $\nabla g(s_1^*, s_2^*) = 0$.
The determinant of the Hessian matrix of g is given by

$$\Delta = p^2L_1L_2 - (1-p)^2L^2,$$

and $\frac{\partial^2 g}{\partial s_1^2} = -2pL_1$ at $(s_1^*, s_2^*) = 0$. Hence, g is strictly positive for all s_1, s_2, if and only if $L_1 < 0$ and $\Delta > 0$, which proves the theorem. □

Theorem 4. *A mixed Nash equilibrium is a weak ESS if and only if $L_1 < 0$ and $L_2 < 0$.*

Proof. Let s_i denote a mutant strategy in population i. From the proof of theorem 2, we have

$$\bar{U}_i(s_i^*, \mathbf{s}^*, p) = \bar{U}_i(s_i, \mathbf{s}^*, p), \text{for } i = 1, 2.$$

Therefore, from the definition, $\mathbf{s}^*$ is a weak ESS if and only if $\bar{U}_1(s_1^*, (s_1, s_2^*), p) > \bar{U}_1(s_1, (s_1, s_2^*), p)$ and $\bar{U}_2(s_2^*, (s_1^*, s_2), p) > \bar{U}_2(s_2, (s_1^*, s_2), p)$.

We have

$$\begin{aligned} & \bar{U}_1(s_1^*, (s_1, s_2^*), p) - \bar{U}_1(s_1, (s_1, s_2^*), p) \\ = \ & (s_1^* - s_1)\big[U_1(G_1, (s_1, s_2^*), p) - U_1(H_1, (s_1, s_2^*), p)\big] \\ = \ & (s_1^* - s_1)\big[ps_1L_1 + (1-p)s_2^*L + K_1\big] \\ = \ & -pL_1(s_1^* - s_1)^2. \end{aligned}$$

which is strictly positive if and only if $L_1 < 0$.

Following the same procedure with the second population, we get:

$$\bar{U}_2(s_2^*, (s_1^*, s_2), p) - \bar{U}_2(s_2, (s_1^*, s_2), p) = -pL_2(s_2^* - s_2)^2.$$

which is strictly positive if and only if $L_2 < 0$. □

We conclude that the mixed (interior) Nash equilibrium cannot be a strong ESS (theorem 2). For some conditions on the payoff matrices, i.e. L_1 and $L_2 < 0$ (theorem 4), the mixed equilibrium is a weak ESS. If, in addition, the community interaction probability p satisfies $\Delta > 0$, then the mixed equilibrium is an intermediate ESS (theorem 3).

4. THE REPLICATOR DYNAMIC

The replicator dynamic is a commonly used tool to observe the asymptotic dynamic of strategy changes in an evolutionary process. It will converge to the attractors of the process and in particular to the ESS when it exists as a stable equilibrium point of the dynamic. In this section we apply this technique to evaluate the proportion of the population from each community selecting to play strategy G_i or H_i.

Under the replicator dynamic, the evolution of $s_i(t)$, i.e. population share of action G_i in class i, is proportional to the difference between the expected payoff of G_i and the average payoff of community i [4, 5, 13]. We assume that the community interaction probability p is stationary, and any individual can only choose the strategy to use. The replicator dynamic equation writes, for $i = 1, 2$,

$$\dot{s_i}(t) = s_i(t)\Big[U_i(G_i, \mathbf{s}(t), p) - \bar{U_i}(s_i(t), \mathbf{s}(t), p)\Big], \quad (14)$$

with $\mathbf{s}(t) = (s_1(t), s_2(t))$, which yields

$$\dot{s}_i(t) = s_i(t)(1 - s_i(t))\Big[ps_i(t)L_i + (1-p)s_{-i}(t)L + K_i\Big]. \quad (15)$$

Thus, we have the following pair of non-linear ordinary differential equations:

$$\begin{cases} \dot{s}_1(t) = s_1(t)(1 - s_1(t))\Big[ps_1(t)L_1 + (1-p)s_2(t)L + K_1\Big], \\ \dot{s}_2(t) = s_2(t)(1 - s_2(t))\Big[ps_2(t)L_2 + (1-p)s_1(t)L + K_2\Big]. \end{cases} \quad (16)$$

There are nine stationary points $(0,0)$, $(1,1)$,$(0,1)$, $(1,0)$, $(0, -\frac{K_2}{pL_2})$, $(-\frac{K_1}{pL_1}, 0)$,$(1, -\frac{K_2 + (1-p)L}{pL_2})$, $(-\frac{K_1 + (1-p)L}{pL_1}, 1)$, and $\mathbf{s}^* = (s_1^*, s_2^*)$, where

$$s_1^* = \frac{(1-p)K_2L - pK_1L_2}{p^2L_1L_2 - (1-p)^2L^2},$$

and

$$s_2^* = \frac{(1-p)K_1L - pK_2L_1}{p^2L_1L_2 - (1-p)^2L^2}.$$

The interior stationary point corresponds to the mixed Nash equilibrium given by theorem 2 and it is the only stationary point at which all strategies co-exist. Assuming that the state space is the unit square and that $\mathbf{s}^*$ exists, the dynamic properties of this equilibrium point are brought out in the next theorem:

THEOREM 5. *When $L_1 < 0$ and $\Delta = p^2L_1L_2 - (1-p)^2L^2 > 0$, then $\mathbf{s}^*$ is globally asymptotically stable for the replicator dynamic.*

PROOF. To study the stability of $\mathbf{s}^*$, we need to apply a translation to obtain an equivalent system with the equilibrium at the origin. By defining $x_1(t) = s_1(t) - s_1^*$ and $x_2(t) = s_2(t) - s_2^*$, we obtain the following system:

$$\begin{cases} \dot{x}_1(t) = (x_1(t) + s_1^*)(1 - x_1(t) - s_1^*)(pL_1x_1(t) + (1-p)Lx_2(t)), \\ \dot{x}_2(t) = (x_2(t) + s_2^*)(1 - x_2(t) - s_2^*)(pL_2x_2(t) + (1-p)Lx_1(t)). \end{cases}$$

The new system is equivalent to the first one, and $s_1 \to s_1^*$ and $s_2 \to s_2^*$ when $x_1 \to 0$ and $x_2 \to 0$. We consider the candidate Lyapunov function:

$$V(x_1(t), x_2(t)) = -pL_1x_1(t)^2 - pL_2x_2(t)^2 - 2(1-p)Lx_1(t)x_2(t).$$

We have $V(0,0) = 0$, and remembering that

$$\dot{V}(t) = \frac{\partial V}{\partial x_1}(t)\dot{x}_1(t) + \frac{\partial V}{\partial x_2}(t)\dot{x}_2(t),$$

we obtain $\dot{V}(t) = -2(x_1(t) + s_1^*)(1 - x_1(t) - s_1^*)(pL_1x_1(t) + (1-p)Lx_2(t))^2 - 2(x_2(t) + s_2^*)(1 - x_2(t) - s_2^*)(pL_2x_2(t) + (1-p)Lx_1(t))^2$ which is strictly negative when $(x_1(t), x_2(t)) \in]-s_1^*, 1-s_1^*[\times]-s_2^*, 1-s_2^*[$ ($(s_1(t), s_2(t)) \in]0,1[^2$.) Moreover, V should meet the positivity requirement. Let us study the sign of V. We have $\nabla V(x)^T = \Big[-2pL_1x_1(t) - 2(1-p)Lx_2(t) \quad -2pL_2x_2(t) - 2(1-p)Lx_1(t)\Big]$, $\nabla V = 0$ at $(0,0)$.

The Hessian matrix of V is as follows

$$\mathcal{H}(V) = \begin{pmatrix} -2pL_1 & -2(1-p)L \\ -2(1-p)L & -2pL_2 \end{pmatrix}.$$

The determinant of $\mathcal{H}(V)$ is

$$\Delta = p^2L_1L_2 - (1-p)^2L^2. \quad (17)$$

When $L_1 < 0$ and $\Delta > 0$, V is convex. In this case, V is definite positive and meets all the requirements to be a Lyapunov function on $D =]-s_1^*, 1-s_1^*[\times]-s_2^*, 1-s_2^*[$. Hence, the origin is locally asymptotically stable over D, that is for every initial position in D, the solutions x_1 and x_2 converge to the origin. Equivalently, if we come back to the original sytem defined in (16), we obtain the following sufficient condition for asymptotic stability of s^*: when $L_1 < 0$ and $\Delta > 0$, then for any strictly interior position in the unit square, s_1 and s_2 converge respectively to s_1^* and s_2^*. □

Joining this result with the theorem 3, we conclude that an intermediate ESS is asymptotically stable. A weak ESS could be also asymptotically stable under some condition. The replicator dynamic converges to the mixed Nash equilibrium when the latter is optimal for the total payoff of interacting communities. This result is in coherence with the single-population case, when the replicator dynamic converges to the strong ESS (which is equivalent to the intermediate ESS in this case). Besides, this result fits with Taylor [12] in its ESS definition with homogeneous (uniform) interactions between two types of player, when he showed that the intermediate ESS version is asymptotically stable.

5. NUMERICAL EXAMPLES FROM PRISONER'S DILEMMA

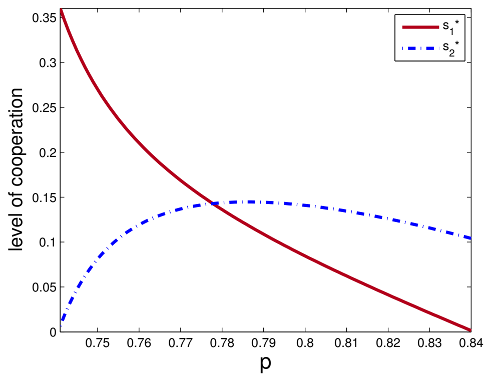
In the classical Prisoner's Dilemma [3], defecting strategy (D) is dominant over cooperating (C). In a single-population case, all cooperating strategies will eventually get extinct and displaced by defecting strategies. We propose to evaluate the evolution of the frequency of cooperators and defectors in case of non-uniform interactions between two communities. In the first example, A_1 and D_1 denote the intra-group interaction matrices for group 1 and group 2, and C_1 is the inter-group interaction matrix. In the second example, A_1 and D_1 denote the intra-group interaction matrices for group 1 and group 2, and C_2 is the inter-group interaction matrix.

$$A_1 = \begin{matrix} & C & D \\ C & 5 & 2 \\ D & 13 & 4 \end{matrix}, \quad D_1 = \begin{matrix} & C & D \\ C & 9 & 0 \\ D & 15 & 1 \end{matrix},$$

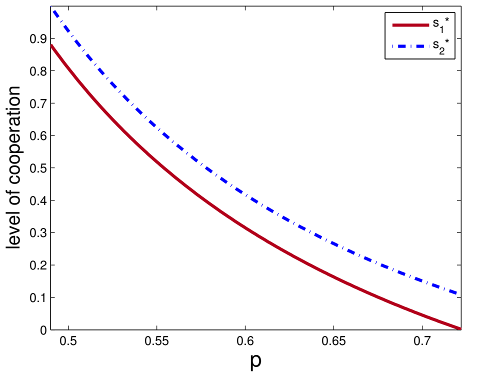
$$B_1 = C_1 = \begin{matrix} & C & D \\ C & 7 & 13 \\ D & 9 & 1 \end{matrix}.$$

$$B_2 = C_2 = \begin{matrix} & C & D \\ C & 11 & 5 \\ D & 4 & 0 \end{matrix}.$$

For the first example, the inter-group interaction matrix has a mixed Nash equilibrium $(\frac{6}{7}, \frac{1}{7})$. In the second example, the inter-group interaction matrix has cooperating as a dominant strategy. Note that we kept the same intra-group interaction matrices for both examples to study the impact of increasing cooperation in the inter-group interaction matrix.



(a) Level of cooperation as a function of p for example 1



(b) Level of cooperation as a function of p for example 2

Figure 2: Mixed intermediate ESS.

In Figure 2, we plot the mixed intermediate ESS (and so the mixed weak ESS) as a function of the probability of the intra-community interaction p, for two examples. As we can clearly observe, the inter-group interaction between the two groups allows the co-existence of cooperators and defectors over some range of p ($[0.74, 0.84]$ for example 1), for which the conditions of theorem 3 are satisfied. Besides, by using an inter-group interaction matrix that favors cooperation (example 2), the range of p over which the mixed-intermediate ESS exists is larger ([0.49, 0.72] for example2).

In Figure 3, we plot the evolution of cooperators, $s_1(t)$ and $s_2(t)$ in group 1 and 2 respectively, over time. For the first example, when $p = 0.75$, the replicator dynamic converges to the intermediate ESS, $s_1^* = 0.27$ and $s_2^* = 0.08$, (condition of theorem 5 satisfied). When $p = 0.9$, defecting is a dominant strategy in both groups (condition of the second part of theorem 1 satisfied). For the second example, when $p = 0.3$, conditions of the first part of theorem 1 are satisfied and the replicator dynamic converges to the dominant strategies. Indeed, for this value of p, the inter-interaction rate is high, and the inter-interaction matrix favors cooperation, which leads to the dominance of cooperators. When $p = 0.57$, the replicator dynamic converges to the mixed intermediate ESS, $s_1^* = 0.42$ and $s_2^* = 0.53$, and both strategies survive. We clearly observe that for the high values of p, defecting is dominant in both communities (example 1, figure 3(b)). Indeed, in this case, intra-group interaction is dominant, the two groups are almost isolated, which leads to domination of defection. By introducing more inter-group interactions, no pure strategy is dominating and cooperators and defectors co-exist (example 1,figure 3(a)). In the second example, for the low values of p, i.e. inter-group interaction is dominant, cooperating is the dominant strategy in both groups, which is intuitively expected remembering that the inter-group interaction matrix favors cooperation (example 2, figure 3(c)). By allowing intra-group interaction to be more frequent, both strategies co-exist at the equilibrium(example 2, figure 3(d)).

6. CONCLUSIONS

In this paper, we suggested different ESS definitions in the context of non-uniform interactions between several communities. These interactions may happen for example, when an individual might be more likely to meet and interact with an individual from his own community than with an individual from another community. We defined several types of ESS depending on the level of robustness of the equilibrium against mutations. We showed that a mixed strong ESS cannot exist. The intermediate ESS, which considers the global payoff (fitness) of the communities, may exist for some range of the inter-group rate and is globally asymptotically stable for the replicator dynamic. The weak ESS may be stable or unstable. We illustrate our results by considering the prisoner's dilemma. We obtained that the mixed intermediate ESS exists for some range of the inter-group rate and it allows the co-existence of cooperators and defectors even if the defection strategy is dominant in each group.

As a further work, we propose to generalize our analysis on the conditions on the inter-group rate for the existence of the different ESSs, on more than two groups. We also plan to consider that this rate may depend on some characteristics of the groups like sizes, influences, etc.

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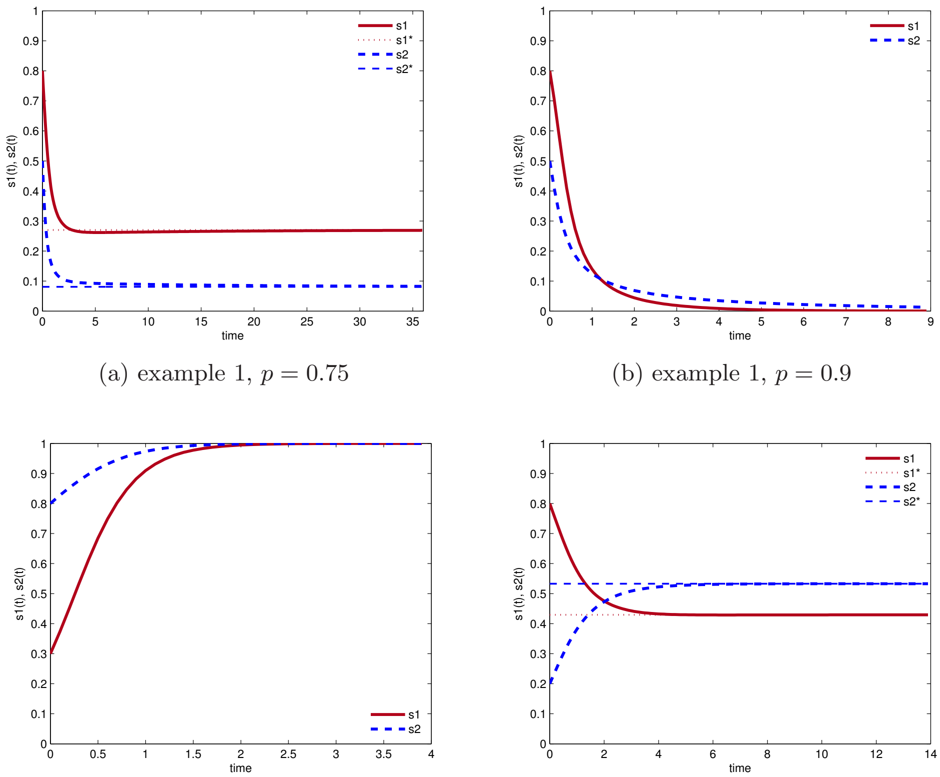

(a) example 1, $p = 0.75$ (b) example 1, $p = 0.9$

(c) example 2, $p = 0.3$ (d) example 2, $p = 0.57$

Figure 3: Replicator dynamic for different values of p.

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APPENDIX

A. PROOF OF PROPOSITION 1

First, let us prove that the definition 3 implies the proposition 1. If we take $\epsilon_i \to 0$ in definition 3, we get $\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p)$ for all i, and so condition (7). Now, to establish the condition (8) in proposition 1, we suppose there exists i such that $\bar{U}_i(s_i^*, \mathbf{s}^*, p) = \bar{U}_i(s_i, \mathbf{s}^*, p)$, we need to prove that $\bar{U}_i(s_i, (s_i, \mathbf{s}^*_{-i}), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}^*_{-i}), p)$. We can write condition (6) as follows:

$$\bar{U}_i(s_i, (\epsilon_i s_1^* + (1-\epsilon_i)s_1^*, .., \epsilon_i s_i + (1-\epsilon_i)s_i^*, .., \epsilon_i s_N^* + (1-\epsilon_i)s_N^*), p)$$
$$< \bar{U}_i(s_i^*, (\epsilon_i s_1^* + (1-\epsilon_i)s_1^*, .., \epsilon_i s_i + (1-\epsilon_i)s_i^*, .., \epsilon_i s_N^* + (1-\epsilon_i)s_N^*), p).$$

By exploring the linearity of $\bar{U}_i$, we get

$$\epsilon_i \bar{U}_i(s_i, (s_1^*, .., s_i, .., s_N^*), p) + (1-\epsilon_i)\bar{U}_i(s_i, \mathbf{s}^*, p)$$
$$< \epsilon_i \bar{U}_i(s_i^*, (s_1^*, .., s_i, .., s_N^*), p) + (1-\epsilon_i)\bar{U}_i(s_i^*, \mathbf{s}^*, p).$$

Since we have $\epsilon_i > 0$ and we suppose $\bar{U}_i(s_i^*, \mathbf{s}^*, p) = \bar{U}_i(s_i, \mathbf{s}^*, p)$, the above inequality yields:

$$\bar{U}_i(s_i, (s_i, \mathbf{s}^*_{-i}), p) < \bar{U}_i(s_i^*, (s_i, \mathbf{s}^*_{-i}), p),$$

and so condition (8).
Now we prove that the proposition 1 implies the definition 3. We have for all i and for all $\mathbf{s} \neq \mathbf{s}^*$

$$\bar{U}_i(s_i, \mathbf{s}^*, p) \leq \bar{U}_i(s_i^*, \mathbf{s}^*, p). \quad (18)$$

If this inequality is strict for all i, then condition (6) holds for $\epsilon_i = 0$, and thus for sufficiently small ϵ_i. If there exists i such that the comparison in (7) is an equality, then we obtain $\bar{U}_i(s_i, (s_i, \mathbf{s}^*_{-i}), p) < \bar{U}_i(s^*_i, (s_i, \mathbf{s}^*_{-i}), p)$ (condition (8)). We multiply both sides by ϵ_i, and by observing that $\bar{U}_i(s_i, \mathbf{s}^*, p) = \bar{U}_i(s^*_i, \mathbf{s}^*, p)$, we add $(1 - \epsilon_i)\bar{U}_i(s_i, \mathbf{s}^*, p)$ to the left side and $(1 - \epsilon_i)\bar{U}_i(s^*_i, \mathbf{s}^*, p)$ to the right side, we get condition (6).

B. PROOF OF PROPOSITION 2

Let us first show that the definition 4 implies the proposition 2. Since condition (9) holds for any sufficiently small ϵ, we can take $\epsilon \to 0$, we get $\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p)$, and so condition (10). Now we establish the condition (11) in the proposition 2. By exploring the linearity of $\bar{U}_i$, the condition (9) can be rewritten

$$\begin{aligned} &\epsilon \sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}, p) + (1 - \epsilon) \sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) \\ &< \epsilon \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}, p) + (1 - \epsilon) \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p). \end{aligned} \tag{19}$$

Since $\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) = \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p)$ and $\epsilon > 0$, we obtain $\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}, p) < \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}, p)$. Hence, the second part of the proposition 2 is established.

Now let us prove that the proposition 2 implies the definition 4. We have for all $\mathbf{s} \neq \mathbf{s}^*$,

$$\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) \leq \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p).$$

If this inequality is strict, then condition (9) holds for $\epsilon = 0$, and thus for sufficiently small ϵ. If $\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p) = \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p)$, then $\sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}, p) < \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}, p)$. By multiplying both sides of this inequality by ϵ, and adding $(1 - \epsilon) \sum_{i\in\Gamma} \bar{U}_i(s_i, \mathbf{s}^*, p)$ to the left side, and $(1 - \epsilon) \sum_{i\in\Gamma} \bar{U}_i(s^*_i, \mathbf{s}^*, p)$ to the right side, we obtain the condition (9).