Network of Queues with Inert Customers and Signals

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ABSTRACT

We introduce the concept of inert customers in a generalized queue with signals. A signal interacts with the customers when it arrives in a non empty queue. Then, it disappears instantaneously. Ordinary customers wait for service and eventually receive service if they do not interact with a signal beforehand. An inert customer does not receive service. It stays in the queue until an interaction with a signal makes it disappear. As it is present in the queue, it has an effect on the service capacity. We consider multiple classes of ordinary customers and a symmetric queueing discipline for ordinary and inert customers. In this paper we consider two types of interaction between signals and inert customers: the deletion of one inert customer and the deletion of a group of inert customers. Despite this deep modification of the model, the queue is still quasi-reversible. Therefore we prove that such a network of queues have a steady-state product form solution. We illustrate this approach with various examples consisting in PS and LIFO queues.

1. INTRODUCTION

Traditional queueing networks consist of queues and customers. Customers enter a queue. They wait for service and eventually receive it. At the end of their service they jump to another queue or leave the network. When the modeling assumptions are sufficiently simple (Poisson arrivals of new customers from the outside, exponential service, routing between queues based on a distribution of probability), we are able to find the steady-state distribution and this distribution has a product form. Such a result has a lot of applications for performance evaluation.

Generalized networks of queues (or G-networks) are based on a new paradigm: they contain queues, customers and signals [13]. Signals interact with one or several queues or with one or several customers. They disappear immediately after they try to interact. The interaction may fail but the vanishing of the signal always occurs. Despite this deep modification of the stochastic processes, the G-networks studied so far have also a product form solution. Many results have been published for various interactions between signals and queues or customers: negative customers which delete a usual customer [13], triggers which move a customer from one queue to another one [14], batch deletion [15], catastrophes which flush all the customers out of the queue [9] or resets which make the queue size jumps to an arbitrary state with a distribution of probability [18]. Signal may also change the class of a customer [4] or change its phase when the service distribution is PH [5]. It must be pointed out that the flow equations for these networks are not as simple as the flow equation for Jackson network. They are nor linear neither contracting. Thus the existence of a solution to the flow equation may be a problem. New techniques have been developed to prove the existence of a solution to the flow equation [19] and new algorithms have been developed [7, 11].

In this paper, we slightly change the paradigm again, adding a new actor. We consider a new type of customers which do not interact with the server even if they are queued. They are called inert customers. They are present in the queue and they follow the service discipline even if they do not receive service: in some sense, they do not use the service received. Thus, a part of the service capacity is wasted. For instance, in a PS queue, the total service capacity varies with the number of inert customers as the fraction of service capacity that they receive is wasted. An inert customer has a positive sojourn time (this is different from a signal which always disappears immediately) but it does not have, strictly speaking a service time.

However, an inert customer interacts with the signals entering the queue. In this paper we consider two types of signals: negative customers as defined by Gelenbe in [13] and catastrophes or iterated deletions [9, 10]. Such a signal can cancel a group of consecutive inert customers. Thus in a LIFO queue, if the first customer is an inert one, the queue is blocked until the arrival of a negative signal which deletes the inert customer or the arrival of the new positive customer which enters the queue in the first position.

Positive customers can becomes signals or inert customers at the completion of their service in a queue. Thus, we obtain new types of interactions between queues in such a model and we extend the range of applications of G-network of queues. We consider multiple classes of usual customers and a symmetric scheduling discipline as defined by Kelly [23]. As usual with multiclass queues with negative customers or signals [5, 8], the interaction between signals and customers in the queue must mimic the service discipline.

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The rest of the paper is as follows. In section 2, we introduce a short introduction to the extension of quasi-reversibility which can be used to study queues with signals. G-networks have also motivated new fundamental researches in queuing theory to explain why they have product form. Basically, two major theory have emerged since the seminal paper by Gelenbe [13] on negative customers: the extension of quasi-reversibility developed by Chao and his co-authors [3] and the Reversed Compound Agent Theorem (RCAT in the following) and its extensions proved by Harrison and his colleagues [22, 21, 2]. RCAT allows to give a more detailed description of the interactions between automata as it is based on a stochastic process algebra (a subset of Performance Evaluation Process Algebra, i.e. PEPA). However as we only deal with queues, we still use Chao's quasi-reversibility theory, which needs a more compact description of the model. In section 3, we introduce the model of inert customers and we prove in section 4 the product form solution for the steady-state distribution of networks of symmetric queues with inert customers and signals under some technical constraints. Section 5 is devoted to three examples to illustrate the new models which can be addressed by the interaction between queues, signals, inert customers and usual customers.

2. QUASI-REVERSIBLE QUEUES WITH SIGNALS

We first introduce the definition of quasi-reversibility of Chao, Miyazawa and Pinedo given in [3]. Then, we present the model of networks of quasi-reversible queues with instantaneous movement. This presentation follows the introduction on quasi reversibility already published in [5].

2.1 Definition of quasi-reversibility

In [3], Chapter 3, Chao, Miyazawa and Pinedo have given a new definition of quasi-reversibility. They first study a queue without signal, then they consider a more general definition to deal with signals and triggers. A signal may cause instantaneous movement in the queue: a signal arrives and changes the state of the queue, it can also simultaneously trigger a departure with some probability. We consider here this second definition to include simultaneous events in the network.

Consider a queue where the queue-content evolves as a continuous time Markov chain on state space $\mathcal{S}$. For a pair of states $(\vec{x}, \vec{y})$, we decompose the transition rate function $q(\vec{x}, \vec{y})$ of queue into three types with their own rates: $q_u^A(\vec{x}, \vec{y}), u \in T$; $q_u^D(\vec{x}, \vec{y}), u \in T$; $q^I(\vec{x}, \vec{y})$, where T is the set of the classes of arrivals and departures, which is supposed to be countable. "A", "D" and "I" stand for "arrival", "departure" and "internal". The transition rate of the queue is the sum of the rates:

$$q(\vec{x}, \vec{y}) = \sum_{u \in T} q_u^A(\vec{x}, \vec{y}) + \sum_{u \in T} q_u^D(\vec{x}, \vec{y}) + q^I(\vec{x}, \vec{y}), \quad \vec{x}, \vec{y} \in \mathcal{S}.$$

The transition rate functions q_u^A, q_u^D and q^I generate the point processes corresponding to class u arrivals, class u departures and the internal transitions, respectively.

Suppose that q admits a stationary distribution π. Furthermore, assume that when a class u arrives and changes the state of the queue from $\vec{x}$ to $\vec{y}$, it instantaneously triggers a class v departure with probability $f_{u,v}(\vec{x}, \vec{y})$, where:

$$\sum_v f_{u,v}(\vec{x}, \vec{y}) \leq 1, \ u \in T, \ \vec{x}, \vec{y} \in \mathcal{S}.$$

With probability $1 - \sum_v f_{u,v}(\vec{x}, \vec{y})$ the class u arrival does not trigger any departure. Note that this departure provokes an arrival of a customer or a signal in another queue. Thus we have a synchronised transition between two (or more) queues. The function $f_{u,v}(\vec{x}, \vec{y})$ is the *triggering probability.* When $\sum_v f_{u,v} \equiv 0$ the effect of a signal on a queue is only local. There is no synchronised movements between queues due to signal.

Chao, Miyazawa and Pinedo have defined the quasi-reversibility of queues with instantaneous movement as follows.

DEFINITION 1. *If there exist two sets of non-negative numbers $\{\alpha_u, u \in T\}$ and $\{\beta_u, u \in T\}$ such that: for all $\vec{x} \in \mathcal{S}$, $u \in T$,*

$$\sum_{\vec{y} \in \mathcal{S}} q_u^A(\vec{x}, \vec{y}) = \alpha_u, \quad (1)$$

$$\sum_{\vec{y} \in \mathcal{S}} \pi(\vec{y}) \Big[q_u^D(\vec{y}, \vec{x}) + \sum_{v \in T} q_v^A(\vec{y}, \vec{x}) f_{v,u}(\vec{y}, \vec{x}) \Big] = \beta_u \pi(\vec{x}), \quad (2)$$

then the queue with signal is said to be quasi-reversible with respect to $\{q_u^A, f_{u,v}, \ u, v \in T\}$, $\{q_u^D, u \in T\}$ and $\{q^I\}$. *The non-negative numbers α_u and β_u are called the arrival rate and departure rate of class u customers.*

They proved that this definition of queue without instantaneous movements is equivalent to the previous definition of quasi-reversibility given by Kelly in [23]. This implies that the arrival processes and the departure (triggered and non-triggered) of class u customers are Poisson.

2.2 Network of quasi-reversible queues with signals and instantaneous movement

Consider the network of N queues. Each queue is a quasi-reversible queue with signals as described above. The set of arrival and departure classes is T (we may have a set T_i for each queue i, however, we can take $T = \cup_i T_i$). Let $\vec{x}_i$ be the state of queue i with state space $\mathcal{S}_i$. The Poisson source has index 0 and for the sake of simplification, we assume that the source has only one state denoted as 0. We use the same notation for the source and the queue. For each queue, we need to specify the arrival effects, the departure transition rate, the internal transition rate and the triggering probability. For queue i, we introduce the function p_{iu}^A, q_{iu}^D, q_i^I and $f_{iu,v}$ on the state space $\mathcal{S}_i$:

- $p_{iu}^A(\vec{x}_i, \vec{y}_i)$ = the probability that a class u arrival at queue i changes the state from $\vec{x}_i$ to $\vec{y}_i$, where it is assumed that $\sum_{\vec{y}_i \in \mathcal{S}_i} p_{iu}^A(\vec{x}_i, \vec{y}_i) = 1$, $\vec{x}_i \in \mathcal{S}_i$;
- $q_{iu}^D(\vec{x}_i, \vec{y}_i)$ = the rate at which class u departures change the state of queue i from $\vec{x}_i$ to $\vec{y}_i$;
- $q_i^I(\vec{x}_i, \vec{y}_i)$ = the rate at which internal transitions change the state of queue i from $\vec{x}_i$ to $\vec{y}_i$;
- $f_{iu,v}(\vec{x}_i, \vec{y}_i)$ = the triggering probability that when a class u arrivals at queue i and the state change from $\vec{x}_i$ to $\vec{y}_i$, it simultaneously induces a class v departure, where $\sum_{v \in T} f_{iu,v}(\vec{x}_i, \vec{y}_i) \leq 1$, $i \leq N$, $u \in T$, $\vec{x}_i, \vec{y}_i \in \mathcal{S}_i$.

For source 0, we set $p^A_{0u}(0,0) = 1$, $q^D_{0u}(0,0) = \beta_{0u}$, $q^I_0(0,0) = 0$ and $f_{0u,v} \equiv 0$. Here, β_{0u} is the arrival rate to the network from the outside (the source).

In the previous section, a queue is defined by three rates q^A_u, q^D_u and q^I. If a queue of the network is initially defined by q^A_u, q^D_u and q^I, then the arrival effect function may be defined as:

$$p^A_u(\vec{x},\vec{y}) = \frac{q^A_u(\vec{x},\vec{y})}{\sum_z q^A_u(\vec{x},\vec{z})} \ ,$$

and q^D_u and q^I are the departure and internal transition functions.

The dynamics of the network are described as follows. Customers of class u arrive to the network from the outside (the source) according to a Poisson process with rate β_{0u}, and are routed to queue i as a class v arrival with probability $r_{0u,iv}$. A class u departures from queue i, either trigger or non-trigger, enters queues j as a class v arrival with probability $r_{iu,jv}$. It is assumed that:

$$\sum_{j=0}^{N} \sum_{v} r_{iu,jv} = 1, \ i = 0, 1, \ldots N, \ u \in T.$$

Furthermore, whenever there is a class u arrival at queue i, either from the outside or from other queues, it causes the state of the queue change from $\vec{x}_i$ to $\vec{y}_i$ with probability $p^A_{iu}(\vec{x}_i,\vec{y}_i)$, it also triggers a class u departure with probability $f_{iu,v}(\vec{x}_i,\vec{y}_i)$, and it triggers no departure from queue i with probability $1 - \sum_{v\in T} f_{iu,v}(\vec{x}_i,\vec{y}_i)$, $i = 0, 1, \ldots, N$.

The transition rate function of the network is denoted by $q(\vec{x},\vec{y})$, $\vec{x},\vec{y} \in \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_N$ (note that we accept the case where $q(\vec{x},\vec{x}) \neq 0$).

We now study the stationary distribution of the network. Assuming quasi-reversibility for each queue in isolation, which will be defined below, we state that the stationary distribution of the network process has product form.

Consider for each queue i the following auxiliary process:

$$q_i^{(\vec{\alpha}_i)}(\vec{x}_i,\vec{y}_i) = \sum_{u\in T} \Big(\alpha_{iu} p^A_{iu}(\vec{x}_i,\vec{y}_i) + q^D_{iu}(\vec{x}_i,\vec{y}_i)\Big) + q^I_i(\vec{x}_i,\vec{y}_i),$$

where $(\vec{\alpha}_i) = (\alpha_{iu}, u \in T)$ are considered as dummy parameters and their values are determined by the traffic equations.

Suppose that $q_i^{(\vec{\alpha}_i)}$ has a stationary distribution $\pi_i^{(\vec{\alpha}_i)}$. Note that this is always true for the source 0, as for all $\vec{\alpha}_0$, $\pi_0^{(\vec{\alpha}_0)(0)} = 1$. We now require that $q_i^{(\vec{\alpha}_i)}$ be quasi-reversible.

We always have:

$$\sum_{\vec{y}_i \in \mathcal{S}_i} \alpha_{iu} p^A_{iu}(\vec{x}_i,\vec{y}_i) = \alpha_{iu}, \qquad i = 1, \ldots N, \ u \in T.$$

Hence, the quasi-reversibility of $q_i^{(\vec{\alpha}_i)}$ for $i = 1, \ldots, N$ is equivalent to the existence of a set of non-negative numbers $\beta_{iu}, u \in T$ such that:

$$\sum_{\vec{y}_i} \pi_i^{(\vec{\alpha}_i)}(\vec{y}_i) \Big[q^D_{iu}(\vec{y}_i,\vec{x}_i) + \sum_{v\in T} \alpha_{iv} p^A_{iv}(\vec{y}_i,\vec{x}_i) f_{iv,u}(\vec{y}_i,\vec{x}_i) \Big] = \beta_{iu} \pi_i^{(\vec{\alpha}_i)}(\vec{x}_i), \quad (3)$$

for all $\vec{x}_i \in \mathcal{S}_i$, $i = 1, \ldots, N$ and $u \in T$.

Queue i in isolation is said to be quasi-reversible with $\vec{\alpha}_i$ if (3) is satisfied.

α_{iu} and β_{iu} are the arrival and the departure rates of class u customers at queue i. Then, we have the following traffic equations:

$$\alpha_{iu} = \sum_{j=0}^{N} \sum_{v} \beta_{jv} r_{jv,iu} \ , \qquad i = 0, 1, \ldots, N. \qquad (4)$$

We need the following condition to ensure that the network process is regular:

$$\sum_{i=1}^{N} \sum_{x_i \in \mathcal{S}_i} \pi_i^{(\vec{\alpha}_i)} \sum_{\vec{y}_i \in \mathcal{S}_i} q_i^{(\vec{\alpha}_i)}(\vec{x}_i,\vec{y}_i) < \infty.$$

Finally the following theorem (see [3] for a proof) established that the network has a product form stationary distribution:

THEOREM 1. *If each queue i with signals, $i = 1, \ldots, N$, is quasi-reversible with $\vec{\alpha}_i$ that is the solution to the traffic equations (4), then the queueing network with signal has the product form stationary distribution*

$$\pi(\vec{x}) = \prod_{i=1}^{N} \pi_i^{(\vec{\alpha}_i)}(\vec{x}_i),$$

where $\pi_i^{(\vec{\alpha}_i)}$ is the stationary distribution of $q_i^{(\vec{\alpha}_i)}$, $i = 1, \ldots, N$.

3. INERT CUSTOMERS AND THEIR INTERACTION WITH SIGNALS

In this section, we present the model of a multiclass queue with inert customers. The discipline is symmetric as described by Kelly in [23] but it is extended to take into account signals and inert customers.

- The set of classes for usual customers is given by $\mathcal{C}$. Customers of type c arrive according to a Poisson process of rate λ_c and ask for an exponential service of rate μ_c. Let us denote by λ the sum $\sum_{c\in\mathcal{C}} \lambda_c$.
- In the queue, there are also inert customers. Their service rate is equal to 0 but they follow the same queueing discipline as the usual customers.

  Thus they stay in the queue until they interact with a signal. The set of inert customers is given by $\mathcal{C}_0$ and we note $\lambda_0 = \sum_{c\in\mathcal{C}_0} \lambda_c$.
- The arrival rate of signals is λ^-.

As usual with symmetric disciplines, the state of the queue is modeled by a vector $\vec{x} = (x(1), \cdots, x(n))$, where n is the length of the queue and $x(l)$ is the class of customer in position l. All the customers, usual or inert, are explicitly represented.

We restrict ourselves to service disciplines which are symmetric with a service effort equal to 1, as defined by Kelly in [23]. This limitation on the service effort precludes to represent Infinite Server queue which are not consistent with a closed form solution when they contain negative customers or signals. Remember that FIFO is not a symmetric discipline. Let us more precisely describe the symmetric discipline for customers and signals. If there are n customers in the queue, then:

- A proportion $\gamma(l,n)$ of the total service effort is directed to customer in position l ($1 \leq l \leq n$). When its service is completed, customers in positions $l+1, l+2, \cdots, n$ move to positions $l, l+1, \cdots, n-1$, respectively.

- Upon arrival, a customer moves to position l ($1 \leq l \leq n+1$) with probability $\gamma(l, n+1)$. Customers previously in positions $l, l+1, \cdots, n$ move to positions $l+1, l+2, \cdots, n+1$. respectively.

- Upon arrival, a signal will chose a customer in position l as a "target" with probability $\gamma(l, n)$. The signal may succeed or fail to delete the targeted customer depending of its class. The target customer is cancelled with probability $q_{x(l)}$.

Denote by $\vec{x} \ominus (x(l), l)$ the state obtained by a departure of customer in position l and denote by $\vec{x} \oplus (c, l)$ the state obtained by an arrival of a customer of class c into position l.

The function γ will be called the proportional function. It determines the service discipline. The function γ verifies

$$\sum_{l \leq n} \gamma(l, n) = 1.$$

Note that the inert customers in the queue only disappear when they interact with signals. Thus, we need to receive signals in a queue if the arrival of inert customers is positive. This is a necessary condition for the existence of a stationary regime, but this is not sufficient.

DEFINITION 2. *A queue with inert customers is well-formed if $\lambda^- > 0$ when $\lambda_0 > 0$.*

In the following of this paper, we assume that all the queues are well formed. Then, we have the following theorem

THEOREM 2. *Consider a queue with inert customers as above. If*

$$\rho_c = \frac{\lambda_c}{\mu_c + \lambda^- q_c} < 1, \text{ for } c \in \mathcal{C} \tag{5}$$

$$\rho_c = \frac{\lambda_c}{\lambda^- q_c} < 1, \text{ for } c \in \mathcal{C}_0, \tag{6}$$

then the queue is stable and the steady state is given by

$$\pi(x(1), \cdots, x(n)) = C \prod_{l=1}^{n} \rho_{x(l)}. \tag{7}$$

The queue is quasi-reversible in sense defined by Chao et al in [3] with respect to departure of class c usual customers and cancellation of class c' inert customers ($c \in \mathcal{C}$ and $c' \in \mathcal{C}_0$), with rates respectively equal to: $\mu_c \rho_c$ and $\lambda^- q_{c'} \rho_{c'}$.

PROOF. We need to verify the balance equations:

$$\sum_{\vec{y}} \pi(\vec{x}) Q(\vec{x}, \vec{y}) = \sum_{\vec{y}} \pi(\vec{y}) Q(\vec{y}, \vec{x}) \tag{8}$$

The left-hand side is given by:

$$L = \pi(\vec{x}) \Big(\lambda + \lambda_0 + \mathbf{1}_{n>0} \sum_{l=1}^{n} \gamma(l, n) \{ \mathbf{1}_{x(l) \in \mathcal{C}} \mu_{x(l)} + \lambda^- q_{x(l)} \} \Big),$$

while the right-hand side is:

$$\begin{aligned} R &= \mathbf{1}_{n>0} \textstyle\sum_{l=1}^{n} \pi(\vec{x} \ominus (x(l), l)) \lambda_{x(l)} \gamma(l, n) \\ &+ \pi(\vec{x} \oplus (c, l)) \gamma(l, n+1) \{ \mathbf{1}_{c \in \mathcal{C}} \mu_c + \lambda^- q_c \}. \end{aligned}$$

We substitute the form of π and using the condition on the proportional function γ, one has that

$$R = \pi(\vec{x}) \Big\{ \mathbf{1}_{n>0} \sum_{l=1}^{n} \frac{\lambda_{x(l)} \gamma(l, n)}{\rho_{x(l)}} + \sum_{c \in \mathcal{C}} \rho_c (\mathbf{1}_{c \in \mathcal{C}} \mu_c + \lambda^- q_c) \Big\}.$$

The solution for ρ_c (i.e. Equations 5 and 6) implies that $R = L$. The balance equations are satisfied.

The departure rate for positive class c-customers ($c \in \mathcal{C}, \mu_c > 0$) is given by:

$$\frac{\sum_l \pi(\vec{x} \oplus (c, l)) \mu_c \gamma(l, n+1))}{\pi(\vec{x})} = \rho_c \mu_c$$

and the rate of inert class c-customers ($c \in \mathcal{C}_0, \mu_c = 0$) which are cancelled by signal is given by:

$$\frac{\sum_l \pi(\vec{x} \oplus (c, l)) \lambda^- \gamma(l, n+1)) q_c}{\pi(\vec{x})} = \rho_c \lambda^- q_c.$$

Hence, the queue is quasi-reversible. This completes the proof. □

This is not really a surprise as we have already proved in [5] that the queue is quasi-reversible when the service rates are positive (i.e. when we do not have inert customers in the queue). However, adding inert customers in the model allows to represent phenomena which were not previously taken into account.

For instance, in a LIFO queue, if an inert customer is at the head of the queue, the service capacity of the queue is completely lost until a new customer or a signal arrives. This allows to model a failure of the server (see Section 5 for more details).

For a Processor Sharing queue, the service capacity varies with the fraction of usual customers in the queue. Indeed, the fraction given to the inert customers is wasted. As the number of inert customers is governed by other queues in the network, we have a model of a control of the bandwidth of an abstracted networking element.

We can consider any other symmetric discipline with the limitation that function γ verifies $\sum_{l \leq n} \gamma(l, n) = 1$.

4. ANALYTICAL SOLUTION FOR STEADY-STATE

We now consider a network of N multi-class queues with inert customers. We slightly change the notation, adding a queue index. At queue i, the arrival rate for class c-customers is λ_c^i, while the rate for signal is $\lambda^{i,-}$. The service rate is now μ_c^i. The queueing discipline is symmetric for all queues. But it is not necessarily the same for all queues. For LIFO queues, it is possible to make the signal loops, leading to the deletion of a group of customers. Thus we distinguish between general symmetric discipline and LIFO queues with iterated deletion (also called catastrophes in this case).

The state of the network are given by $\vec{x} = (\vec{x}^1, \cdots, \vec{x}^N)$, where $\vec{x}^i$ is the state of queue i.

Upon arrival to queue i of size n, a signal will chose a customer in position l as a target with probability $\gamma^i(l, n)$. The target will be cancel with probability $q^i_{\vec{x}^i(l)}$, i.e. depending upon it's class.

After finishing service at queue i, a customer of class c ($c \in \mathcal{C}$) jumps to queue j as a customer of class k with probability $p^{i,j}_{c,k}$ ($k \in \mathcal{C} \cup \mathcal{C}_0$, therefore the customer may be either an usual one or an inert one), as a signal with probability $p^{i,j}_{c,-}$, or can leave the network with probability d^i_c. We have the following condition for the routing of usual customers:

$$\sum_{j,k} p^{i,j}_{c,k} + \sum_j p^{i,j}_{c,-} + d^i_c = 1$$

.

An inert customer c in queue i ($c \in \mathcal{C}_0, \mu^i_c = 0$), after a successful interaction with a signal, can move to queue queue j as a customer of class k (either inert or usual depending on k) with probability $p^{i,j}_{c,k}$, as a signal with probability $p^{i,j}_{c,-}$, or can leave the network with probability d^i_c. We have three conditions on the routing matrices:

- Normalisation of the routing probabilities for inert customers:
$$\sum_{j,k} p^{i,j}_{c,k} + \sum_j p^{i,j}_{c,-} + d^i_c = 1$$
.
- Self Loops with signals : if queue i is not LIFO then $p^{i,i}_{c,-} = 0$ for $c \in \mathcal{C}_0$. Intuitively the signal is not allowed to loop in a queue if the scheduling discipline is not LIFO. This technical constraint (which can be removed) allows to give a simple description for LIFO queues which cannot be easily generalized for other scheduling discipline.
- Self Loops with inert customers : $p^{i,i}_{c,c} = 0$ for $c \in \mathcal{C}_0$. Intuitively, an inert customer is not allowed to reenter the same queue. This is a technical constraint to avoid dummy transitions.

We now define some relations between queues based on these routing matrices.

DEFINITION 3. *We build three relations among the queues in the network based on the routing of usual customers, signals and inert customers. In the following i, j and k are queues indices.*

- *$i\mathcal{P}_c j$ hold if there is a positive probability for an usual customer to move from queue i to queue j following a sequence of transitions as an usual customer. The customer at queue j has class c. We write $i\mathcal{P}j$ if there exists a customer class c such that $i\mathcal{P}_c j$.*
- *$i\mathcal{N}j$ holds if and only if there exists a queue k and a class c of usual customer such that $i\mathcal{P}_c k$ and $p^{k,j}_{c,-} > 0$.*
- *$i\mathcal{I}j$ holds if and only if there exists a queue k and two classes c, c' of customer, $c \in \mathcal{C}$ and $c' \in \mathcal{C}_0$, such that $i\mathcal{P}_c k$ and $p^{k,j}_{c,c'} > 0$.*

As usual index 0 represents the outside of the network.

Using these relations we can define the topology of the networks we will consider in this paper. Note that more complex topologies can be considered as well: for instance, one can build mixed topology which will be open for customers and signals and closed for inert customers. But such a generalization, even it is straightforward cannot be investigated here because it requires a substantial number of definitions which cannot be introduced here for the sake of conciseness.

DEFINITION 4. *A queueing network with signals and inert customers is open and well-formed if the following conditions hold:*

1. *for all queue i, $0\mathcal{P}i$ or $0\mathcal{I}i$,*
2. *for all queue i, $i\mathcal{P}0$ or there exists queue index j such that $i\mathcal{N}j$,*
3. *for all queue i, if there exists a queue index j such that $j\mathcal{I}i$, then there exists a queue index k such that $k\mathcal{N}i$ and $0\mathcal{P}k$.*

Intuitively, the first condition means that all the queues receive customers (inert or usual). Condition number 2 implies that customers leave the queues, either as a signal or as a customer. The last condition implies that if queue receives inert customers, it also receives signals to cancel them.

We further assume that if we allows the looping of signal in a LIFO queue (say i), we have the following property for the deletion probability: $q^i_c = 0$ for all classes c in $\mathcal{C}$ and $q^i_c = 1$ for all classes c in $\mathcal{C}_0$. In other words, the signal has no effect on usual customers and always delete an inert customer if it at the head of the LIFO queue. As a consequence, upon arrival to a LIFO queue, the signal may cancel all consecutive inert customers at the head of the buffer.

Applying the theorem of network of quasi-reversible queues, we have the following result which is proved by the quasi-reversibility property for the elementary queues previously established.

THEOREM 3. *Consider an open well-formed network of queues with inert customers and signals. Let us denote by Λ^i_c the global arrival rate of class c customers at queue i (if $c \in \mathcal{C}$ they are usual customers and if $c \in \mathcal{C}_0$ they are inert). Similarly, $\Lambda^{i,-}$ is the arrival rate of signals. Consider the following traffic equations:*

$$\Lambda^i_c = \lambda^i_c + \sum_j \sum_{k|\mu^j_k>0} p^{j,i}_{k,c}\mu^j_k\rho^j_k \quad (9)$$

$$+ \sum_j \sum_{k|\mu^j_k=0} p^{j,i}_{k,c}\Lambda^{j,-}q^j_k\rho^j_k, \quad (10)$$

$$\Lambda^{i,-} = \lambda^{i,-} + \sum_j \sum_{k|\mu^j_k>0} p^{j,i}_{k,-}\mu^j_k\rho^j_k \quad (11)$$

$$+ \sum_j \sum_{k|\mu^j_k=0} p^{j,i}_{k,-}\Lambda^{j,-}q^j_k\rho^j_k, \quad (12)$$

$$(13)$$

where for each class, ρ_c is determined by:

- *for class c usual customers ($c \in \mathcal{C}$),*
$$\rho^i_c = \frac{\Lambda^i_c}{\mu^i_c + \Lambda^{i,-}q^i_c}.$$
- *for inert customers of class c ($c \in \mathcal{C}_0$) and queue i which are not LIFO,*
$$\rho^i_c = \frac{\Lambda^i_c}{\Lambda^{i,-}q^i_c},$$
- *for LIFO queues and inert customers of class c, ρ^i_c is given by*
$$\Lambda^i_c = \Lambda^{i,-}\rho^i_c q^i_c \left(1 + \sum_{l=1}^{\infty} (\rho^i_c q^i_c p^{(i,i)}_-)^l\right).$$

If the traffic equations have a solution such that $\rho^i_c < 1$, then the network is stable and the steady state is given by:

$$\pi(\vec{x}) = \prod_{i=1}^{n} \pi(\vec{x}^i), \quad (14)$$

where

$$\pi(\vec{x}^i) = \pi(\vec{x}^i(1), \cdots, \vec{x}^i(n^i)) = C \prod_{l=1}^{n^i} \rho_{\vec{x}^i(l)}.$$

5. SOME EXAMPLES

We consider several examples to illustrate the effects of the interactions between inert customers and signals. We first present in two examples of networks with some queues which only contains inert customers. Such a queue will be denoted in this paper as a tank to put more emphasis on the fact that the server is not really acting on the customer. We consider various possible interactions between tanks and ordinary queues. As the inert customers in the tank do not receive service, they are only moved by signals. They can move between tanks or between a tank and a queue. For the sake of simplicity we only consider one class of usual customers in that model and one class of inert customers. However it is possible to study more complex models of tanks and queues with multiple classes of both types of customers and some scheduling discipline.

In the last part of this section, we consider queues with usual customers and inert customers. These examples are based on PS queues with signals acting as negative customers to delete inert customers and LIFO queues with catastrophes restricted to inert customers.

In all the figures in this section, the usual customers are depicted in white while the signals are represented as black boxes and the inert customers are grey boxes. The routing of usual customers are drawn as dotted lines while signals and inert customers routing are represented with black lines.

5.1 Network of queues and tanks with simple deletion

We introduce the concept of a tank to put more emphasis on the behavior of inert customers. A tank is queue which only contains inert customers. Therefore the power capacity of the server is always lost. The only arrivals to a tank consist in inert customers and signals. Inert customers arrive at the queue and stay into the queue without receiving any service. When a signal enters a tank, it interacts with the inert customers. Signals are emitted by queues when customers complete their services and leave the queue. For the sake of simplicity we assume without any loss of generality that inert customers and signals de not arrive from the outside of the network. Symmetrically, we assume that the ordinary queues only contains usual customers. We make the same assumptions about the nature of the processes as in the previous sections. We assume that $\mathcal{Q}$ is the set of queues and $\mathcal{T}$ the set of tanks. We add the following assumptions to the ones already made in section 2 and 3.

- We only consider one class of usual customers. Thus we do not have to consider the scheduling discipline in the queues and the service rate in a queue is the same for all the usual customers.
- At the completion of the service the customers may join another queue as a customer, or move to a tank either as a signal or as an inert customers.
- The routing probabilities are denoted as $p^{j,i}$ for a routing of a customer from i to j. We have

$$p^{j,i} = 0 \;\; if \; j \in \mathcal{T} \;\; or \;\; i \in \mathcal{T}$$

- Routing of a signal (denoted as $p^{j,i}_-$) only occurs from a queue to a tank. Therefore:

$$p^{j,i}_- = 0 \;\; if \; j \notin \mathcal{Q} \;\; and \;\; i \notin \mathcal{T}$$

- Routing of an inert customer (denoted as $p^{j,i}_s$) only occurs from a queue to a tank. Therefore:

$$p^{j,i}_s = 0 \;\; if \; j \notin \mathcal{Q} \;\; and \;\; i \notin \mathcal{T}$$

We have depicted in Fig. 1 a small network with three queues and two tanks. To put more emphasis on the distinction between tanks and queues, we do not represent the server for a tank.

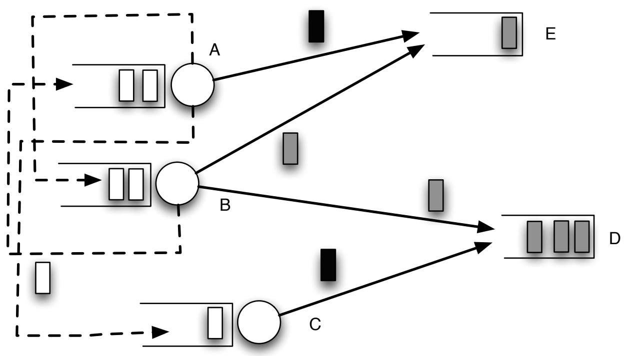

Figure 1: A small network with queues and tanks.

To illustrate the behavior of the network we write now the balance equation at steady-state.

$$\begin{aligned}
\pi(x)\left(\textstyle\sum_{i\in\mathcal{Q}} \lambda^i + (\mu_i + \lambda^{i,-})1_{x_i>0}\right) = \\
\textstyle\sum_{i\in\mathcal{Q}} \lambda^i \pi(x - e_i) 1_{x_i>0}) \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \lambda^{i,-} \pi(x + e_i) \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \mu^i \pi(x + e_i) d^i \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \sum_{j\in\mathcal{Q}} \mu^i \pi(x + e_i - e_j) p^{i,j} 1_{x_j>0}) \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \sum_{j\in\mathcal{T}} \mu^i \pi(x + e_i - e_j) p^{i,j}_- 1_{x_j>0}) \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \sum_{j\in\mathcal{T}} \mu^i \pi(x + e_i + e_j) p^{i,j}_s \quad + \\
\textstyle\sum_{i\in\mathcal{Q}} \sum_{j\in\mathcal{T}} \mu^i \pi(x + e_i) p^{i,j}_s 1_{x_j=0}
\end{aligned}$$

The left hand side represents the evolution of the queue. Indeed the evolution of the population in the tanks are always provoked by the evolution of the queue. As usual the right hand side is much more complex: the first three terms correspond to arrival of customer, arrival of a signal and completion of the service of a customer which leaves the network. The fourth term describes the transition of a customer between two queues at a completion of its service. The fifth term represents the completion of a customer in a queue which jumps at a tank as an inert customer. Finally, the sixth and the seventh terms model the completion of service of a customer in a queue which joins a tank as a signal to delete one inert customer and succeeds (term number 6) or fails (term number 7).

As the network with queues and tanks is clearly a restricted case of the model of queues with inert customers, we know that the steady-state distribution has a product form solution which is detailed in the following corollary.

COROLLARY 1. *Assume that the Markov chain of the network with queues and tanks described in this section is ergodic. Let ρ_i be the solution of the flow equation:*

$$\rho_i = \frac{\lambda_i + \sum_{j \in \mathcal{Q}} \mu_j \rho_j p^{j,i}}{\mu_i + \lambda^{i,-}} \quad if \quad i \in \mathcal{Q},$$

and

$$\rho_i = \frac{\sum_{j \in \mathcal{Q}} \mu_j \rho_j p_s^{j,i}}{\sum_{j \in \mathcal{Q}} \mu_j \rho_j p_i^{j,i}} \quad if \quad i \in \mathcal{T}.$$

If for all i in $\mathcal{Q}$ and $\mathcal{T}$, the inequality $\rho_i < 1$ hold, then the network has a product form steady-state distribution:

$$\pi(x) = \prod_{i \in \mathcal{Q}} (1-\rho_i)\rho_i^{x_i} \prod_{j \in \mathcal{T}} (1-\rho_j)\rho_j^{x_j}.$$

PROOF. The multiplicative form of the solution comes from the theorem previously stated. We just have to compute the normalization constant. Such a result comes from the fact that the solution is multiplicative, that all the states are reachable and that the stationary solution for an isolated queue has a geometric distribution. □

Due to the assumptions about the creation and routing of inert customers and signals, we can prove important results for the solution of the flow equations.

PROPERTY 1. *The solutions of the flow equations exist.*

PROOF. We decompose the problem into two parts. First, the solution exists for the index $i \in \mathcal{Q}$, as the flow equation in that case is contracting (as the flow equation of a Jackson network). The existence of the solution for the flow in the tanks follows immediately as they are obtained directly from the flows in the queues. Indeed, note that the definition of ρ_i for $i \in \mathcal{T}$ only requires the values of ρ_i for $i \in \mathcal{Q}$. □

PROPERTY 2. *The computation of the ρ_i follows directly. In a first step we compute ρ_i for $i \in \mathcal{Q}$ using a fixed point iteration as for a Jackson network. Then we compute directly the values of ρ_i for $i \in \mathcal{T}$.*

Finally, due to closed form solution, one can obtain many technical results for the tanks.

COROLLARY 2. *As the solution for the population of inert customers in a tank at steady state is a geometric with rate ρ_i, all the results proved for M/M/1 queue can be generalized to queues with tanks. The only difference remains in the computation of the values or the rate ρ_i.*

5.2 Network of queues and tanks with iterated deletions

We now assume that the tanks are paired during the interaction with signals. To illustrate the idea, let us consider the small network described in Fig. 2. In this figure, A and B are ordinary queues while C and D are tanks. The interaction between signals and tanks is an iterated deletion of inert customers which occurs as follows:

1. First, we assume that the signal arrives to tank C. The pair C and D is ordered.

2. At its arrival in C, the signal deletes an inert customer in C if there is any and jumps to tank D as a signal. If tank C is empty at its arrival, the signals disappears.

3. The signal arriving in tank D proceeds in almost the same manner. If tank D is empty at its arrival, the signals disappears. Otherwise, the signal deletes an inert customer in D and jumps to tank C as a signal.

4. The iterated deletion of inert customers in tank C and D continue until the smallest queue becomes empty.

5. All these destructions are immediate.

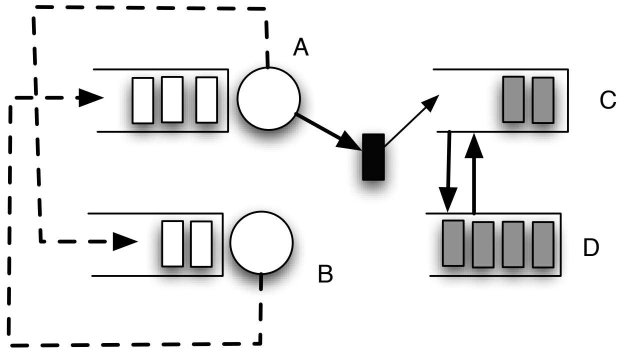

Figure 2: A small network with two paired tanks. The arrows between tank C and D are used to represent the interaction with the signal.

In Fig. 2 we have represented 2 inert customers in tank C and four inert customers in tank D while a signal arrives to tank C. The effect of this arrival is depicted in Fig. 3 where we have depicted the population in the network immediately after the interaction between the signal and the paired tanks.

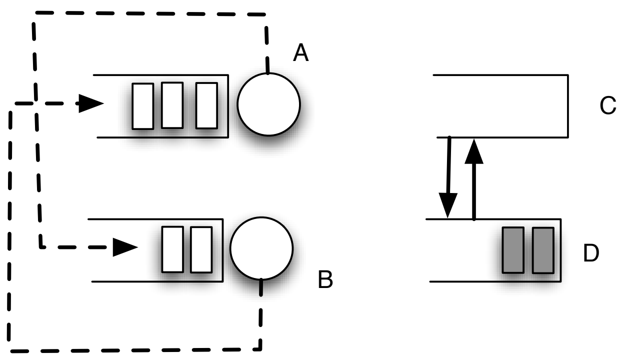

Figure 3: Same network after the interaction between the signal and the customers in the paired tanks.

It must be clear that the pair of tanks is ordered. If we consider the same state and a signal entering in tank D which is paired with C in the order D followed by C, the resulting state will be one inert customer in D and 0 in C. Clearly, the transition is not the same.

All the assumptions about the routing and the rates are the same as in the previous example. The networks of queues and paired tanks has a product form steady state distribution (when the associated Markov chain is ergodic) as it is an example of network with signals and inert customers. However in that case the flow equationsand the definition of the rates are more complex. We now detailed how they can be stated. We begin with the global balance equation.

$$\begin{aligned}\pi(x)\left(\sum_{i\in\mathcal{Q}}\lambda^i+(\mu_i+\lambda^{i,-})1_{x_i>0}\right) = & \\ \sum_{i\in\mathcal{Q}}\lambda^i\pi(x-e_i)1_{x_i>0}) & + \\ \sum_{i\in\mathcal{Q}}\lambda^{i,-}\pi(x+e_i) & + \\ \sum_{i\in\mathcal{Q}}\mu^i\pi(x+e_i)d^i & + \\ \sum_{i\in\mathcal{Q}}\sum_{j\in\mathcal{Q}}\mu^i\pi(x+e_i-e_j)p^{i,j}1_{x_j>0}) & + \\ \sum_{i\in\mathcal{Q}}\sum_{j\in\mathcal{T}}\mu^i\pi(x+e_i-e_j)p_-^{i,j}1_{x_j>0}) & + \\ \sum_{i\in\mathcal{Q}}\sum_{(j,k)\in\mathcal{T}\times\mathcal{T}}\mu^i p_s^{i,j}1_{x_j=0}\gamma(x) & + \\ \sum_{i\in\mathcal{Q}}\sum_{(j,k)\in\mathcal{T}\times\mathcal{T}}\mu^i p_s^{i,j}1_{x_k=0}\delta(x), &\end{aligned}$$

where $\gamma(x) = \sum_{m=0}^{\infty}\pi(x+e_i+me_j+me_k)$ and $\delta(x) = \sum_{m=0}^{\infty}\pi(x+e_i+e_j+me_j+me_k)$. The only differences with the balance equation for the previous case is the sixth and seventh terms. Here (j,k) represents an ordered pair of tanks. It means that the signal enters tank j at the first step. We have put more emphasis on this order as we explicitly mention index j in routing probability $p_s^{i,j}$.

To simplify the description of the steady state distribution, we consider the very simple network depicted in Fig. 2. We assume that the only source of signals is queue A. Queues A and B receives customer from the outside. They exchange customers after their service and send inert customers to tanks. Eventually a customer is changed into a signal after leaving queue A and it jumps to the pair C and D beginning its interaction with C.

Corollary 3. *We assume that the Markov chain associated with the network is ergodic. If there exist a solution to the flow equation*

$$\begin{aligned}\rho^A &= \frac{\lambda^A+\mu^B\rho^B p^{B,A}}{\mu^A}\\ \rho^B &= \frac{\lambda^B+\mu^A\rho^A p^{A,B}}{\mu^B}\\ \rho^C &= \frac{\mu^A\rho^A p^{A,C}+\mu^B\rho^B p^{B,C}}{\frac{\mu^A\rho^A p_-^{A,C}}{1-\rho^C\rho^D}}\\ \rho^D &= \frac{\mu^A\rho^A p^{A,D}+\mu^B\rho^B p^{B,D}}{\frac{\mu^A\rho^A p_-^{A,C}\rho^C}{1-\rho^C\rho^D}}\end{aligned}$$

such that $\rho_i < 1$ for all i, then the steady-state has a product form distribution:

$$\pi(x) = \prod_{i\in\{A,B,C,D\}}(1-\rho_i)\rho_i^{x_i}.$$

Property 3. *The computation of the ρ_i is numerically difficult as the system is unstable. One must use the algorithm introduced in [11] to numerically find the solution if it exists.*

5.3 Network of PS queues

For the sake of readability, we now assume that all the queues have the same queueing discipline. Remember that this is not mandatory for the steady-state solution as the quasi-reversibility is established for arbitrary symmetric queues with signals and inert customers. We consider a network of PS queues. All the queues receive both usual and inert customers. For a PS queue, we consider a simpler and more usual state representation: for queues i, it is sufficient to represent the number of class c customer. Let $y_c(i)$ be the number of customers of class c (either usual or inert) in queue i.

Remember that every customer in the queue receives with the PS scheduling discipline an equal share of $1/n$ if there are n customers in the queue. But the service given to the inert customers is wasted as they do not use it. More formally the total service rate is

$$\mu_i(y(i)) = \frac{\sum_{c\in\mathcal{C}} y_c(i)\mu_c^i}{\sum_{c\in\mathcal{C}} y_c(i) + \sum_{c\in\mathcal{C}_0} y_c(i)}$$

We assume that the signal will have no effect on the usual customer but it deletes an inert customer with probability 1 if the inert customer is selected.

Thus we can model a system where the service capacity may vary after receiving some information for an other part of the network. For instance consider the following network depicted in Fig. 4. Queues A is the queue whose service capacity is controlled by queue B and C. They all receive usual customers from the outside. Queue B and C exchange customers to represent that they build agreement on the controller action. Queue B sends eventually an inert customer to queue A while queue C sends signal to queue B to delete the inert customers. Note however that the deletion success is proportional to the ratio of inert customers in queue A. If there is no inert customer in the queue, the signal has

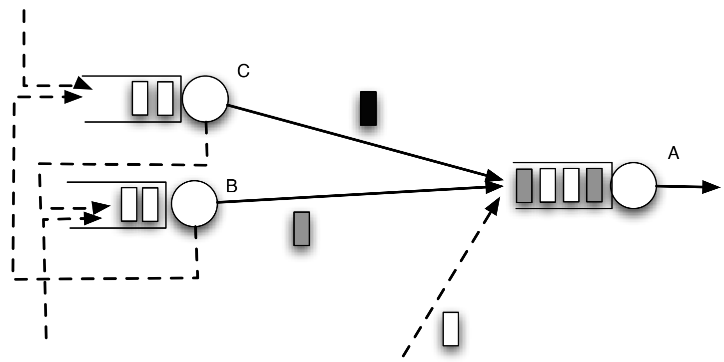

Figure 4: A simple network with a control of service capacity based on the presence of inert customers.

no effect, while its probability of success is more important when the population of inert customers in large. We have depicted in Fig. 5 the evolution of the processing power of queue A for an example of a sample path, assuming that we have one class of usual customer (white box) and one class of inert customer (grey box) and that $\mu_c^i = 1$. Remember that the signals are depicted as black boxes. The population in queue A and the arrivals and departure are also reported. The effect of some events are shown.

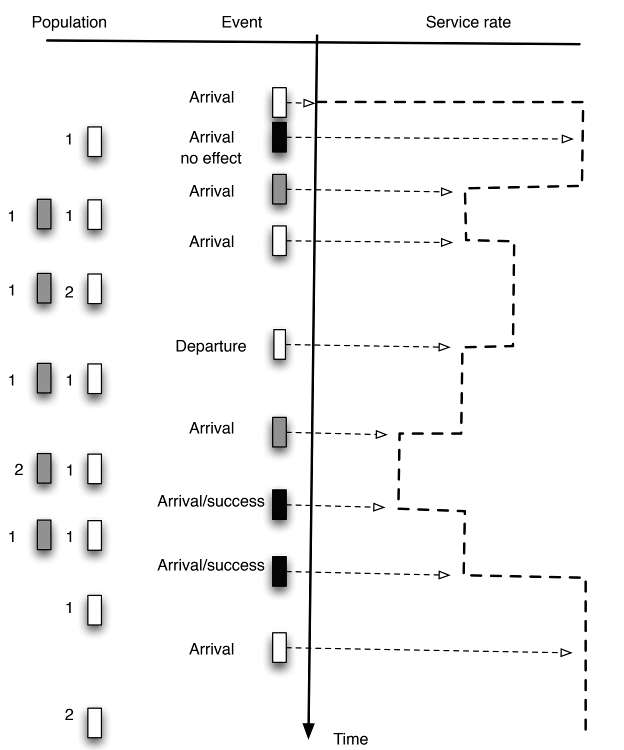

Figure 5: Service rate and population on a simple path.

5.4 Network of LIFO queues with iterated deletion of inert customers

We assume that all the queues are LIFO. The arrivals and the service takes place at the head of the queue. We have to distinguish between the queues where the signal propagates with a loop and the queues where such a phenomenon does not occur. More precisely, in the first case, we assume the following $p_-^{i,i} = 1$ (i.e. the signal loops on the queue almost surely), $q^{i,c} = 0$ if $c \in \mathcal{C}$ (i.e. when the signal tries to delete an usual customer, it always fails) and $q^{i,c} = 1$ if $c \in \mathcal{C}_0$ (i.e. when the signal tries to delete an inert customer, it always succeeds). When $p_-^{i,i} = 0$, we do not make any assumptions about $q^{i,c}$. Finally we de not consider models where the routing probability is not 0 or 1. Thus, a signal entering a LIFO queue where the signal can loop (with the former assumptions about the probabilities) will cancel all consecutive inert customers at the head of the buffer until it fails because the queue is now empty, or the customer at the head is an usual one. This behaviour is illustrated in Fig. 6 where we have depicted the effect of two events on two possible states on a queue. These event are first the arrival of a signal and then the end of service of the customer at the head of the queue. We consider a queue where the signal loops. In the left part of the figure, the initial state is a queue with 2 inert customers at the head and two usual customers at the back end. The signal will delete both inert customers because it loops and delete an inert customer at each iteration until it fails because the customer at the head is an usual one. In the right part of Fig. 6, the initial state is a queue with the same number of customers but no in the same order. As the customer at the head is an usual one, the signals fails and it does not have any effect on the queue.

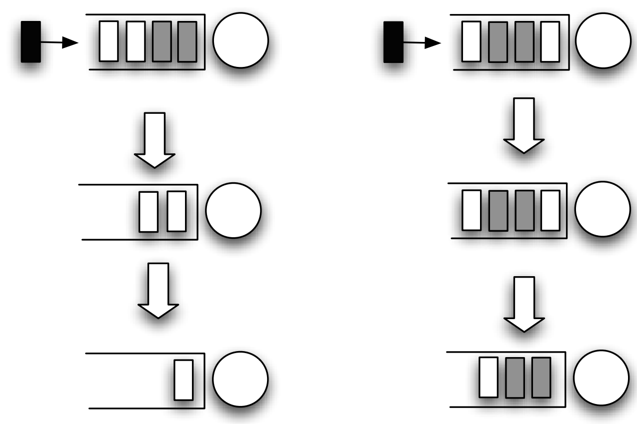

Figure 6: The effect of a signal followed by a service on a LIFO queue with usual and inert customers.

Applying the previous results, under the stability condition $\rho_c < 1$, the steady state is given by

$$\pi(x(1), \cdots, x(n)) = C \prod_{l=1}^{n} \rho_{x(l)},$$

where, for class c usual (or positive) customers ($c \in \mathcal{C}$), ρ_c is given by

$$\rho_c = \frac{\lambda_c}{\mu_c + \lambda^- q_c},$$

and for class c inert ($c \in \mathcal{C}_0$), ρ_c is equal to $\lambda_c/(\lambda^- q_c)$ if the queue does not allow the signal to loop, and is determined by

$$\lambda_c = \frac{\rho_c \lambda^s}{1 - \rho_c}$$

if the signal loops. Note that for $c \in \mathcal{C}_0$, there always exist a solution $\rho_c < 1$ to the equation determined ρ_c (which depends on the parameters: $\lambda_c, \lambda^-, q_c$).

6. CONCLUDING REMARKS

We can generalize our model with several classes of signals with distinct routing matrices and class-dependent probability of success during the interaction with inert and usual customers. Thus, we can represent models with routing based on the type of signals, like in Kelly networks.

Generalized networks have received a lot of attention in the literature in the last 20 years. Many applications have been presented for performance evaluation [12] but also for the modeling of neural networks [17]. Stochastic neural networks have been used for the Cognitive Packet Networks architecture [20] and also to optimize audio and video networks subjective quality of service [25] as there exists a very efficient learning algorithm [16]. They have been successful as a modeling and optimization tool in several domains: texture generation [1] and gene regulatory network [24], see [6] for a recent list of papers and applications. All these applications are mainly based on the iteration between simple signals (in general negative customers) and ordinary customers. We hope that these extensions of the theory allows to increase the applications of this modeling paradigm.

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