On the convergence of the best-response algorithm in routing games

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ABSTRACT

We investigate the convergence of sequential best-response dynamics in a routing game over parallel links. Each player controls a nonnegligible portion of the total traffic, and seeks to split its flow over the links of the network so as to minimize its own cost. We prove that best-response operators are lipschitz continuous, which implies that a sufficient condition for the convergence of the best-response dynamics is that the joint spectral radius of Jacobian matrices of best-response operators be strictly less than unity. We establish the specific structure of these Jacobian matrices for our game, and show that this condition is met in two cases: (a) two-player game for an arbitrary number of links and for a wide class of cost functions; and (b) for arbitrary numbers of players and links in the case of linear latency functions. For latency functions satisfying reasonable convexity assumptions, we conjecture that the proposed sufficient condition is met for arbitrary numbers of players and links.

Keywords

Best response dynamics, routing games, Nash equilibrium, joint spectral radius

1. INTRODUCTION

Game theory has emerged as a fundamental tool for the design and analysis of decentralized resource allocation mechanisms in networks. It has found applications in as diverse areas as load-balancing in server farms [9], power control and spectrum allocation in wireless networks [8], or congestion control in the Internet [15].

In recent years, substantial research effort has been devoted to the study of non-cooperative routing games in which each origin/destination flow is controlled by an autonomous agent that decides how its own traffic is routed through the network (cf. [18] and reference therein). Apart from the gain in scalability with respect to a centralized routing, there are wide-ranging advantages to such a decentralized routing scheme, including ease of deployment and robusteness to failures and environmental disturbances. However, several questions arise when seeking to design and implement such a non-cooperative routing scheme.

One of the most studied one pertains to the inefficiency of non-cooperative routing mechanisms. Indeed, in general, the Nash equilibrium resulting from the interactions of many self-interested agents does not correspond to an optimal routing solution. Numerous works have therefore focused on obtaining performance guarantees for non-cooperative routing schemes [20, 2]. This is usually done by evaluating the Price of Anarchy, a standard measure of the inefficiency of decentralized algorithms introduced by Koutsoupias and Papadimitriou [14]. A small value of the Price of Anarchy indicates that, in the worst case, the gap between a Nash Equilibrium and the optimal solution is not significant, and thus that good performances can be achieved even without a centralized control.

In this work, we address a different question: do uncoordinated routing agents converge to a Nash equilibrium? Thus, rather than the quality of the resulting routing strategy, we are concerned with the convergence of autonomous routing agents to a Nash equilibrium under some "natural" dynamics. More precisely, we address this question assuming the well-known (myopic) best-response dynamics. Best-response dynamics play a central role in game theory [5]. For instance, the Nash equilibrium concept is implicitly based on the assumption that players follow best-response dynamics until they reach a state from which no player can improve his utility. In a game, the best-response of player is defined as its optimal strategy conditioned on the strategies of the other players. It is, as the name suggests, the best response that the player can give for a given strategy of the others. Best-response dynamics then consists of players taking turns in some order to adapt their strategy based on the most recent known strategy of the others (without considering the effect on future play in the game). In this paper, we will restrict ourselves to the sequential (or round robin) best-response dynamics, where players play in a cyclic manner according to a pre-defined order.

The focus of this paper is the convergence of sequential best-response dynamics in a network of parallel links, shared by a finite number of selfish users. Each user controls a non-negligible portion of the total traffic, and seeks to split his flow over the links of the network so as to minimize his own

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cost. This model was introduced in the seminal paper of Orda *et al.* [19], where it shown that there exists a unique Nash equilibrium under reasonable convexity assumptions on the edge latency functions. The users may have different traffic demands. When all users control the same amount of traffic, the convergence to the Nash equilibrium follows from the fact that the symmetric game is a potential game, that is, the Nash equilibrium corresponds to the minimum of a convex optimization problem [10]. For the asymmetric game, convergence results are available only in some special cases. In [19], the convergence to the unique Nash equilibrium of the two-player routing game was proved when there are only two parallel links. As pointed out by the authors, the convergence proof is not readily extendible to more general cases. Altman *et al.* also study the two-link case [1]. Assuming linear latency functions for the links, they prove the convergence of the sequential best-response dynamics for any number of players. More recently, Mertzios has proven that, for the large class of edge latency functions introduced in [19], the two-player routing game converges to the unique Nash equilibrium in a logarithmic number of steps [17]. His proof of convergence relies on a potential-based argument. Namely, he shows that the amount of flow that is reallocated in the network at each step is strictly decreasing. Unfortunately, this argument does not seem to readily extend to more than two players. We also refer to [13, 12] for convergence results on related, but different, problems.

Contributions: We propose a different approach to study the convergence of best-response dynamics. The key idea to prove the convergence is to study the Jacobian matrices of best-response functions, and to analyze how long products of such matrices grow as a function of the number of best-response updates. One of the most prominent quantities characterizing the growth rate of matrix products is the so-called joint or generalized spectral radius. We show that the best-response function is Lipschitz, and establish the specific structure of their Jacobian matrices for our game. Then, a sufficient condition for the convergence of the best-response dynamics is that their joint spectral radius be strictly less than unity. We thus obtain a purely structural sufficient condition that allows to reduce the analysis of the convergence of the sequential best-response dynamics to the analysis of the joint spectral radius of certain matrices. This condition is used to prove the convergence of the two-player game for an arbitrary number of links. We also prove the convergence to the Nash equilibrium for arbitrary numbers of players and links in the case of linear latency functions. Furthermore, although we were not able to prove it, we conjecture that the proposed sufficient condition is valid for any numbers of players and links.

The paper is organized as follows. In Section 2, we describe the non-cooperative routing game under investigation and introduce best-response dynamics as well as some notations. In Section 3, we outline the non-linear spectral radius approach to convergence, and present several properties of the best-response function, and compute the structure of its Jacobian matrix. In Section 4, we state our main result, and prove the convergence of the best-response function for the two-player game with general cost functions and of the K-player game with linear cost functions.

Certain proofs have been omitted due to constraints on the page length. Please refer to [7] for proofs not included in this manuscript.

2. PROBLEM STATEMENT

2.1 Notations

In the following, $\mathbb{R}_+$ denotes the set of non-negative real numbers. Recall that the 1-norm of a vector $\mathbf{x} \in \mathbb{R}^S$ is $\|\mathbf{x}\|_1 = \sum_{i=1}^S |x_i|$. For $\mathbf{x} \in \mathcal{X}$, $\mathcal{B}_o(\mathbf{x}, r)$ will denote the open ball of radius r centered at point $\mathbf{x}$, i.e., $\mathcal{B}_o(\mathbf{x}, r) = \{\mathbf{z} \in \mathcal{X} : \|\mathbf{x} - \mathbf{z}\|_1 < r\}$. Let $\mathbf{1}$ denote the column vector $(1, 1, \ldots, 1)^T$.

We let I and 0 denote the identity and the zero matrices, respectively (their sizes will be clear from the context). A matrix A is positive, and we write $A \geq 0$, if and only if $a_{i,j} \geq 0$, $\forall i, j$, and that it is negative if $-A$ is positive. We recall that the 1-norm of a matrix A is $\|A\|_1 = \max_j \sum_i |a_{ij}|$. denote by $\sigma(A)$ the spectrum of the matrix A, i.e., $\sigma(A) = \{\lambda \in \mathbb{R} : \exists \mathbf{x} \neq 0, A\mathbf{x} = \lambda \mathbf{x}\}$, by $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ its spectral radius, and we recall that $\rho(A) \leq \|A\|_1$. If $A_1, \ldots, A_n$ is a collection of matrices, we denote by $\prod_{i=1}^n A_i$ the product $A_n A_{n-1} \ldots A_1$.

For any function f that is differentiable at point $\mathbf{x}$, we denote by $\mathrm{D}f(\mathbf{x})$ its Jacobian matrix at $\mathbf{x}$.

2.2 Non-cooperative routing game

We investigate a non-cooperative routing game with K routing agents and S links in which each routing agent can control how its own traffic is routed over the parallel links. This routing game is depicted on Figure 1.

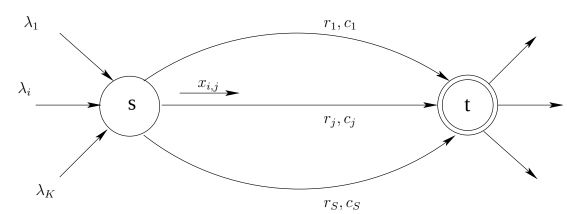

Figure 1: Traffic classes route their packets over parallel links.

Denote by $\mathcal{S} = \{1, \ldots, S\}$ the set of links. Link $j \in \mathcal{S}$ has capacity r_j and a holding cost c_j per unit time is incurred for each packet sent on this link. We let $\pi_j = c_j / r_j$ denote the cost per unit capacity for link j.

We let $\mathcal{C} = \{1, \ldots, K\}$ be the set of routing agent and λ_i be the traffic intensity of routing agent i. We shall also refer to routing agent i as traffic class i, or user i. Each class can control how its own traffic is split over the parallel links and seeks to minimize its own cost. Let $\mathbf{x}_i = (x_{i,j})_{j \in \mathcal{S}}$ denote the routing strategy of class i, with $x_{i,j}$ being the amount of traffic it sends over link j. We let $\mathcal{X}_i$ denote the set of routing strategies for class i, i.e., the set of vectors $\mathbf{x}_i \in \mathbb{R}^S$ such that $0 \leq x_{i,j} < r_j$ for all $j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} x_{i,j} = \lambda_i$.

A strategy profile is a choice of a routing strategy for each user such that the stability condition $\sum_{i \in \mathcal{C}} x_{i,j} < r_j$ is satisfied for all links $j \in \mathcal{S}$. It is thus a vector $\mathbf{x} = (\mathbf{x}_i)_{i \in \mathcal{C}}$ belonging to the product strategy space $\bigotimes_{i \in \mathcal{C}} \mathcal{X}_i$ such that $\sum_{i \in \mathcal{C}} x_{i,j} < r_j$, for all $j \in \mathcal{S}$. It will be assumed throughout the paper that $\sum_{i \in \mathcal{C}} \lambda_i < \sum_{j \in \mathcal{S}} r_j$, so that $\mathcal{X} \neq \emptyset$.

The optimization problem solved by class i, which depends on the routing decisions of the other classes, can be formulated as follows:

$$\text{minimize } T_i(\mathbf{x}, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{S}} \pi_j \, x_{i,j} \, \phi(\rho_j) \qquad \text{(BR-}i\text{)}$$

subject to

$$\mathbf{x} \in \mathcal{X}_i, \qquad (1)$$

$$y_j = x_{i,j} + \sum_{k \neq i} x_{k,j}, \quad \forall j \in \mathcal{S}, \qquad (2)$$

$$\rho_j = y_j / r_j, \quad \forall j \in \mathcal{S}, \qquad (3)$$

$$\rho_j < 1, \quad \forall j \in \mathcal{S}, \qquad (4)$$

In the above formulation, y_j represents the total traffic offered to link j, ρ_j is the utilization rate of this link, and ϕ is the cost associated to the link when there is a traffic of y_j flowing through it. In transportation or communication networks, ϕ models the delay on the road or the link. The total cost incurred by user i is then the sum of the cost of individual links weighted by the amount of traffic the user sends on each of the links. Thus, given the strategies of the others, user i seeks to minimize its total cost subject to flow conservation and stability constraints.

Assumption 1. *We shall make the following assumptions on the cost function ϕ:*

(A_1) $\phi : [0, 1) \to [0, \infty)$,

(A_2) $\lim_{\rho \to 1^-} \phi(\rho) = +\infty$,

(A_3) *continuous, strictly increasing, convex function, and is twice continuously differentiable.*

Remark 1. *At first glance, it appears that the assumptions are not loose enough to include polynomial cost functions, which are widely used in transportation networks. However, it will be shown in Appendix A that any function satisfying*

(B_1) $\phi : [0, \infty) \to [0, \infty)$,

(B_2) $\lim_{\rho \to \infty} \phi(\rho) = +\infty$, *and*

(B_3) (A_3),

has an equivalent function which satisfies assumptions (A_1)–(A_3). Two functions are said to be equivalent if the solution of (BR-i) *with one function is also the solution of* (BR-i) *with the other. Thus, results obtained for functions satisfying (A_1)–(A_3) will be applicable to functions that satisfy (B_1)–(B_3).*

We note that Problem (BR-i) is well-defined for all points $\mathbf{x} \in \mathcal{X}$ since $\sum_{k \neq i} x_{k,j} < r_j$ for all links j.

2.3 Nash equilibrium

A Nash equilibrium of the routing game is a strategy profile from which no class finds it beneficial to deviate unilaterally. Hence, $\mathbf{x}^* \in \mathcal{X}$ is a Nash Equilibrium Point (NEP) if $\mathbf{x}_i^*$ is an optimal solution of problem (BR-i) for all classes $i \in \mathcal{C}$, that is, if

$$\mathbf{x}_i^* = \arg\min_{\mathbf{z} \in \mathcal{X}_i} T_i(\mathbf{z}, \mathbf{x}_{-i}^*), \quad \forall i \in \mathcal{C},$$

where $\mathbf{x}_{-i}^*$ is the vector of strategies of all players other than player i at the NEP.

It follows from our assumptions on the function ϕ, that the link cost functions are a special case of type-B functions, as defined in reference [19]. As proved in Theorem 2.1 of this reference, this implies the existence of a unique NEP for our routing game. In the following, we shall denote by $\mathbf{x}^*$ this Nash equilibrium point.

2.4 Best response dynamics

The best-response of player is defined as its optimal strategy conditioned on the strategies of the other players. It is, as the name suggests, the best response that the player can give for a given strategy of the others. Let $x^{(u)} : \mathcal{X} \to \mathcal{X}$, defined as

$$x^{(u)}(\mathbf{x}) = \left(\arg\min_{\mathbf{z} \in \mathcal{X}_u} T_u(\mathbf{z}, \mathbf{x}_{-u}), \mathbf{x}_{-u} \right), \qquad (5)$$

be the best-response of user u to the strategy $\mathbf{x}_{-u}$ of the other players. From the definition of T_u, it can be shown that for each $\mathbf{x} \in \mathcal{X}$, there is a unique $x^{(u)}(\mathbf{x})$. Given a point $\mathbf{x} \in \mathcal{X}$, the strategy profile $x^{(u)}(\mathbf{x})$ describes the strategies of all the players after the best response of user u.

Best-response dynamics then consists of players taking turns in some order to adapt their strategy based on the most recent known strategy of the others (without considering the effect on future play in the game).

Define a *round* to be a sequence of best-responses in which each player plays exactly once. Once an order is fixed in the first round, it is assumed to be the same in each subsequent round. The order in which the players best-respond in the first-round can be arbitrary. Let us fix this order to be 1, 2, . . ., K.

Define $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ as

$$\hat{x}^{(1)}(\mathbf{x}) = x^{(K)} \circ x^{(K-1)} \circ \ldots \circ x^{(1)}(\mathbf{x}), \qquad (6)$$

be the point reached from $\mathbf{x}$ after one round of play. One can recursively define

$$\hat{x}^{(n)}(\mathbf{x}) = \hat{x}^{(1)} \circ \hat{x}^{(n-1)}(\mathbf{x}), \qquad (7)$$

which is the point reached after n rounds.

The best-response dynamics can then be defined as the sequence $\{\hat{x}^{(n)}(\mathbf{x}_0)\}_{n \geq 1}$ corresponding to the strategy of players after each round of best-response when $\mathbf{x}_0$ is the initial strategy. A NEP has the property that each player's strategy is a best-response to strategies of the other players. Therefore if $\mathbf{x}_0$ is a NEP then sequence will remain at $\mathbf{x}_0$.

The main question we seek to answer is: do the best-response dynamics for the routing game converge from any starting point? If they converge, then they converge to the Nash equilibrium point.

3. THE NON-LINEAR SPECTRAL RADIUS APPROACH

A usual method to prove the convergence of iterates of an operator $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ is to show that this operator is a contraction. For this, one needs to find a suitable norm, say $\|\cdot\|$, for which there exists a constant $c \in [0, 1)$ such that

$$\|\hat{x}^{(1)}(\mathbf{x}) - \hat{x}^{(1)}(\mathbf{y})\| \leq c\|\mathbf{x} - \mathbf{y}\|,$$

for every pair of points $\mathbf{x}$ and $\mathbf{y}$ in the set $\mathcal{X}$. The contraction condition says that the distance between iterates of the function starting from two different points decreases with each iteration. The constant c depends on the norm, and for a continuously differentiable operator, it can be computed as $\sup_{\mathbf{x}} \|D\hat{x}^{(1)}(\mathbf{x})\|$, which is the supremum of the Jacobian over all points in the domain of the operator. It is then sufficient to find a norm in which the above condition is satisfied.

For the best-response function, it turns out that it is non-trivial to find such a norm, independently of the starting point, in which the distance decreases with every iteration. Instead, as will be seen later it will be sufficient to find a norm in which the distance decreases asymptotically and not with every iteration. This weaker condition can be formalized using the notion of the *non-linear spectral radius* described below.

For a function $f : \mathcal{X} \to \mathcal{X}$, define the set

$$\mathcal{J}(f) = \{Df(x) : f \text{ is differentiable at } x\}.$$

which is the set of Jacobian matrices of the function f evaluated at all points at which f is differentiable.

DEFINITION 1. *The non-linear spectral radius of a function $f : \mathcal{X} \to \mathcal{X}$ is defined as [16]:*

$$\bar{\rho}(f) = \limsup_{n\to\infty} \sup_{A_i \in \mathcal{J}(f)} \left\| \prod_{i=1}^{n} A_i \right\|^{1/n}.$$

The non-linear spectral radius of f is related to the notion of *joint spectral radius* of a set $\mathcal{M}$ of matrices which is defined as:

$$\hat{\rho}(\mathcal{M}) = \limsup_{n\to\infty} \sup_{M_i \in \mathcal{M}} \left\| \prod_{i=1}^{n} M_i \right\|^{1/n}, \tag{8}$$

and is independent of the induced matrix norm. It measures the worst case growth rate of a sequence of linear transformations that are taken from the set $\mathcal{M}$. It can been seen that the non-linear spectral radius of f is in fact the joint spectral radius of the set of Jacobian matrices of f, $\mathcal{J}(f)$.

When there is only one matrix in $\mathcal{M}$, from Gelfand's formula it follows that the joint spectral radius is equal to the spectral radius of that matrix. For a set with several matrices, there is an equivalent result in terms of the *generalized spectral radius* of $\mathcal{M}$ which is defined as:

$$\rho(\mathcal{M}) = \limsup_{n\to\infty} \sup_{M_i \in \mathcal{M}} \rho\left(\prod_{i=1}^{n} M_i\right)^{\frac{1}{n}}, \tag{9}$$

where $\rho(A)$ is the spectral radius of the matrix A. If $\mathcal{M}$ is bounded then the generalized spectral radius and the joint spectral radius of $\mathcal{M}$ are equal [4].

Consider a linear dynamical system of the form

$$x_{n+1} = A_{i(n)} x_n,$$

where the matrices $A_i \in \mathcal{M}$ can be chosen differently in each step. Such a system is called a switched linear system. When all the matrices are the same, one can determine the stability of such a system by checking whether the spectral radius of this matrix is less than 1 or not. In case of switched linear systems, the same condition with the joint spectral radius in place of the spectral radius can be used to ascertain the stability of the system, see for example [21].

For non-linear operators, the following convergence criterion was stated in [16].

THEOREM 1 ([16] THEOREM 1). *If $f : \mathcal{X} \to \mathcal{X}$ is Lipschitz-continuous and has a non-linear spectral radius smaller than 1, then the iterates of f are globally asymptotically stable. Moreover, the rate of exponential decay, r, satisfies $0 < r \le -\log(\bar{\rho}(f))$.*

Thus, instead of requiring the best-response to be a contraction, one can show the convergence of the best-response dynamics by showing that:

1. $\hat{x}^{(1)}$ is Lipschitz-continuous; and
2. $\bar{\rho}(\hat{x}^{(1)}) < 1$.

In the rest of this section, first we shall show a few properties of the best-response function, and then compute the structure of its Jacobian matrices, before arriving at our main result.

3.1 Properties of the best-response function

The purpose of this section is to establish various properties of best-response function, mainly related to its continuity and differentiability. Let us define

$$\mathcal{S}_u(\mathbf{x}) = \{j \in \mathcal{S} : x_{u,j}^{(u)}(\mathbf{x}) > 0\} \tag{10}$$

as the set of links used by player u in its best-response to the strategies $\mathbf{x}_{-u}$ of other players. We have the following result.

THEOREM 2. *The best-response function $x^{(u)}$ of player u is Lipschitz-continuous on $\mathcal{X}$ with*

$$\|x^{(u)}(\mathbf{z}) - x^{(u)}(\mathbf{w})\|_1 < 2\,\|\mathbf{z} - \mathbf{w}\|_1, \quad \forall \mathbf{z}, \mathbf{w} \in \mathcal{X}. \tag{11}$$

COROLLARY 1. *Since the best-response over one round, $\hat{x}^{(1)}$, is a composition of best-responses of each of the players (cf. (5)), it then follows that $\hat{x}^{(1)}$ is Lipschitz continuous.*

REMARK 2. *The continuity of the best-response functions is a direct consequence of Berge's Theorem on the continuity of correspondances [3] (see also page 64 of [6]). However, Lipschitz continuity requires some more work than that.*

Once the Lipschitz continuity of $\hat{x}^{(1)}$ has been established, it remains to be shown that its non-linear spectral radius is smaller than 1. For this, we shall investigate the points at which the $\hat{x}^{(1)}$ is differentiable and compute the structure of its Jacobian.

We note that, according to Rademacher's theorem [11], a consequence of Theorem 2 is that the best-response function $x^{(u)}$ is Fréchet-differentiable almost everywhere in $\mathcal{X}$; that is, the points in $\mathcal{X}$ at which $x^{(u)}$ is not differentiable form a set of Lebesgue measure zero. To compute the points at which the derivative is defined, we shall need the following definitions:

- Let

$$g_{i,j}(\mathbf{x}) = \frac{\partial T_i}{\partial x_{i,j}}(\mathbf{x}) = \pi_j \left(\phi(\frac{y_j}{r_j}) + \frac{x_{i,j}}{r_j}\phi'(\frac{y_j}{r_j})\right), \tag{12}$$

  where $y_j = \sum_k x_{k,j}$, be the marginal cost of player i on link j under strategy profile $\mathbf{x}$.

  We say that link j is *marginally used* by user u at point $\mathbf{x}$ whenever the flow of user u on that link is 0 although the marginal cost of that player on that link is minimum, that is

$$x_{u,j} = 0 \text{ and } g_{u,j}(\mathbf{x}) = \min_{k\in\mathcal{S}} g_{u,k}(\mathbf{x}). \quad (13)$$

- we say that the set $\mathcal{S}_u(\mathbf{x})$ is *locally stable* at point $\mathbf{x}$ if it does not change for an infinitesimal variation on the strategies of the other players, that is

$$\exists\epsilon > 0, \forall \mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon), \mathcal{S}_u(\mathbf{x}) = \mathcal{S}_u(\mathbf{z}). \quad (14)$$

From our assumptions on the function ϕ, the continuity of the best-response functions imply that of the marginal costs $g_{i,j}$ defined in (12) under the best-response dynamics. In the following, we say that no link is marginally used by user u in its best-response at point $\mathbf{x}$ if there is no link that is marginally used by user u at point $x^{(u)}(\mathbf{x})$. The two notions introduced above are related through the following result.

LEMMA 1. *if there is no link that is marginally used by player u in its best-response at point $\mathbf{x}$, then the set of links $\mathcal{S}_u(\mathbf{x})$ is locally stable at point $\mathbf{x}$.*

Our first result regarding the differentiability of best-response functions is the following.

PROPOSITION 1. *The best-response function $x^{(u)}$ is differentiable at every point $\mathbf{x} \in \mathcal{X}$ where the set $\mathcal{S}_u(\mathbf{x})$ is locally stable.*

In the next section, we show that the Jacobian matrices have a very specific form at point at which the best-response function is locally stable, and hence differentiable.

3.2 Structure of the Jacobian matrices

The Jacobian matrix of $\hat{x}^{(1)}$ is the product of Jacobian matrices of best-responses of individual players. So, we shall start by computing the Jacobian of the best-response functions of individual players.

Consider a player u and a point $\mathbf{x} \in \mathcal{X}$ at which $x^{(u)}$ is differentiable. The Jacobian matrix of this function is then the block matrix

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \end{pmatrix},$$

where the (i,j)-block $\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_j}(\mathbf{x})$ measures the sensitivity of the strategy of player i obtained after the best response of player u with respect to a change in the strategy of player j.

The best-response of a player u is sensitive only to the strategies of the other players $v \neq u$, and these sensitivities are reflected by the block matrices $\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_v}$ which appear in the uth row of the Jacobian matrix. Recalling that

$$\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = \left(\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,j}}(\mathbf{x})\right)_{i\in\mathcal{S}, j\in\mathcal{S}}, \quad (15)$$

we shall distinguish between links $i \notin \mathcal{S}_u(\mathbf{x})$ and links $i' \in \mathcal{S}_u(\mathbf{x})$. We assume in the following that the set $\mathcal{S}_u(\mathbf{x})$ is locally stable (cf. Section 3.1), and thus that it does not change for an infinitesimal variation on the strategy $\mathbf{x}_v$ of player $v \in \mathcal{C}$.

LEMMA 2. *For all links $i \notin \mathcal{S}_u(\mathbf{x})$,*

$$\frac{\partial x_{u,i}^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = 0, \quad \forall v \in \mathcal{C}, \quad (16)$$

For links $i \in \mathcal{S}_u(\mathbf{x})$, we have:

LEMMA 3. *There exist a vector $\boldsymbol{\theta} \in \mathbb{R}_+^S$ and a vector $\boldsymbol{\gamma} \in \mathbb{R}_+^S$ satisfying $\gamma_i = 0$ for all $i \notin \mathcal{S}_u(\mathbf{x})$ and $\sum_{i\in\mathcal{S}} \gamma_i = 1$ such that*

$$\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} = \begin{cases} \theta_i(\gamma_i - 1) & \text{if } k = i, \\ \theta_k \gamma_i & \text{otherwise,} \end{cases} \quad (17)$$

for all players $v \neq u$ and all links $i \in \mathcal{S}_u(\mathbf{x})$ and $k \in \mathcal{S}$.

PROOF. See Appendix B. □

REMARK 3. *The vectors $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ depend upon the strategy profile $\mathbf{x}$ and upon the player u that updates its strategy. We have not made this dependence explicit in order to simplify the notation.*

Further, the vector $\boldsymbol{\theta}$ has the following important property which will be helpful in establishing the desired inequality on the non-linear spectral radius of $\hat{x}^{(1)}$.

LEMMA 4. *There exists a constant $q < 1$ such that*

$$\frac{1}{2} \leq \theta_k \leq q, \quad \forall k \in \mathcal{S}, \ \forall \mathbf{x} \in \mathcal{X}, \ \forall u \in \mathcal{C}. \quad (18)$$

The structure of the Jacobian matrices of the best-response functions is summarized in the following result.

THEOREM 3. *The Jacobian matrix of the best response function $x^{(u)}$ of player $u \in \mathcal{C}$ has the following form*

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} I & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ M_u(\mathbf{x}) & \dots & 0 & \dots & M_u(\mathbf{x}) \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & I \end{pmatrix},$$

and $M_u(\mathbf{x}) = \Psi(\Gamma B - I)\Theta$, where

- *B is the $S \times S$ matrix with 1 in every entry, i.e., $B = \mathbf{1}^T\mathbf{1}$,*
- *$\Gamma = diag(\boldsymbol{\gamma})$ and $\Theta = diag(\boldsymbol{\theta})$, the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\theta}$ being those defined in Lemma 3,*
- *Ψ a positive diagonal matrix such that $\Psi_{i,i} = 1$ if $i \in \mathcal{S}_u(\mathbf{x})$, and $\Psi_{i,i} = 0$ otherwise.*

PROOF. The proof is broken down in three steps. Firstly, the uth row follows directly from Lemma 3. Secondly, the strategies of all players except player u do not change following the best response of player i. Therefore, for all $i \neq u$ and all $v \in \mathcal{C}$, we have

$$\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = \begin{cases} I & \text{if } v = i, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

This explains the appearance of the identity matrix on the diagonal and the 0 matrix in other columns of each row except the row corresponding to the player doing the best-response (that is, row u).

Finally, since the best response of player u at point $\mathbf{x}$ is insensitive to her strategy at that point and depends only on the strategies of the other player, we can conclude that, for all $u \in \mathcal{C}$,

$$\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x}) = 0. \tag{20}$$

This explains why the diagonal block in the uth row is 0. □

COROLLARY 2. *The Jacobian matrix of $\hat{x}^{(1)}$ has the form*

$$\mathrm{D}\hat{x}^{(1)}(\mathbf{x}) = \prod_{u=K}^{1} \mathrm{D}x^{(u)}(\mathbf{x}),$$

where the index u goes down from K to 1.

4. CONVERGENCE OF BEST-RESPONSE DYNAMICS

In this section, we shall first formulate a conjecture on the non-linear spectral radius of $\hat{x}^{(1)}$ on which the main result of this paper hinges. Then, this conjecture will be shown to be true for two particular cases : *(a)* two-player routing games; *(b)* K player routing games with linear link cost function, ϕ.

CONJECTURE 1. *For a fixed K and S, let $\hat{\mathcal{J}}$ be the set of matrices of the form given in Corollary 2. Then, the joint spectral radius of $\hat{\mathcal{J}}$ is strictly less than* 1.

On the extensive numerical experiments that we conducted, the above conjecture was indeed true.

The main result of this paper is then:

THEOREM 4. *If Conjecture 1 is true, then the best-response dynamics* (7) *for the routing game* (BR-i) *converges to the unique Nash equilibrium of the game.*

While we were unable to prove the conjecture, and hence the convergence of best-response dynamics, in its generality, we can show its validity for two non-trivial cases – the two player game, and the K player game with linear link cost function, which we show below.

4.1 Two-player routing game

First, we shall prove a general result related to the Joint spectral radius of a certain class of matrices. The claimed result on the convergence of the best-response for the two-player game will then follow directly from that result.

Let $\mathcal{D}^+$ be the set of positive diagonal matrices, and $\mathcal{G}$ be the set of diagonal matrices $\Gamma \in \mathcal{D}^+$ whose diagonal entries satisfy in addition

$$\sum_{i=1}^{S} \gamma_i = 1. \tag{21}$$

For any natural number $k \geq 0$, the above two types of diagonal matrices are used to define the set $\mathcal{M}$ of $S \times S$ matrices as follows. $\mathcal{M}$ is the set of matrices M that can be written as $M = (\Gamma B - I)\,\Theta$ for some matrices $\Gamma \in \mathcal{G}$ and $\Theta \in \mathcal{D}^+$. We also define $\mathcal{M}^{(k)}$ for $k \geq 0$ as the set of matrices that can be written as the product of k matrices belonging to $\mathcal{M}$, where by convention $\mathcal{M}^{(0)}$ contains only the identity matrix.

For $q \in (0, 1)$, we say that a matrix M is in the set $\mathcal{M}_q$ if $M = (\Gamma B - I)\,\Theta \in \mathcal{M}$ and in addition $\|\Theta\|_1 \leq q$. We similarly define $\mathcal{M}_q^{(k)}$ as the set of matrices that can be written as the product of k matrices belonging to $\mathcal{M}_q$. We note that the set $\mathcal{M}_q$ is obviously bounded.

According to Theorem 3 and Lemma 4, the Jacobian matrices of the best-response functions of players 1 and 2 have the following simple form:

$$\mathrm{D}x^{(1)}(\mathbf{x}) = \begin{pmatrix} 0 & \Psi_1 M_1 \\ 0 & I \end{pmatrix}, \quad \text{and} \quad \mathrm{D}x^{(2)}(\mathbf{x}) = \begin{pmatrix} I & 0 \\ \Psi_2 M_2 & 0 \end{pmatrix}, \tag{22}$$

where $M_1, M_2 \in \mathcal{M}_q$ for some $q < 1$ and where Ψ_1, Ψ_2 are diagonal matrices with 0-1 entries on the diagonal. Using Corollary 2, the Jacobian of the best-response function over one round has the form

$$\mathrm{D}\hat{\mathbf{x}}^{(1)} = \begin{pmatrix} 0 & \Psi_1 M_1 \\ 0 & \Psi_2 M_2 M_1 \end{pmatrix},$$

where $M_1, M_2 \in \mathcal{M}_q$. It then follows that the structure of the product of n Jacobian matrices has the following form.

LEMMA 5. *If $J_1, J_2, \ldots, J_n \in \mathcal{J}$, then*

$$\prod_{i=1}^{n} J_i = \begin{pmatrix} 0 & \Psi_1 X_1^{(2n-1)} \\ 0 & \Psi_2 X_2^{(2n)} \end{pmatrix}, \tag{23}$$

where Ψ_1, Ψ_2 are positive diagonal matrices with 0-1 entries on the diagonal, $X_1^{(2n-1)} \in \mathcal{M}_q^{(2n-1)}$, and $X_2^{(2n)} \in \mathcal{M}_q^{(2n)}$.

Lemma 5 shows that the behaviour of a large product of Jacobian matrices is governed by the asymptotic behaviour of the matrices $X_1^{(n)}, X_2^{(n)}$. These matrices are themselves the product of matrices that belong to $\mathcal{M}_q$. This suggests to first characterize the asymptotic growth rate of products of matrices in $\mathcal{M}_q$. Our key result regarding this characteriztion is stated in theorem 5.

THEOREM 5. *For any $k \geq 1$ and any matrix*

$$M = \prod_{i=1}^{k} (\Gamma^{(i)} B - I)\,\Theta^{(i)}$$

in $\mathcal{M}^{(k)}$, it holds that

$$\rho(M) \leq \prod_{i=1}^{k} \theta_{max}^{i}, \tag{24}$$

where $\theta_{max}^{i} = \max_{1 \leq j \leq S} \theta_j^{(i)}$ for all $i = 1, \ldots, k$.

The above theorem holds for any product of matrices belonging to $\mathcal{M}$. If we now restrict our attention to matrices belonging to $\mathcal{M}_q$, we obtain the following immediate corollary.

COROLLARY 3. *For any product $M_n M_{n-1} \ldots M_1$ of matrices belonging to $\mathcal{M}_q$, we have $\rho(M_n M_{n-1} \ldots M_1) \leq q^n$, implying that $\rho(\mathcal{M}_q) \leq q$.*

We are now in position to prove that sequential best-response dynamics converges to the unique Nash equilibrium $\mathbf{x}^*$.

THEOREM 6. *For the two player routing game over parallel links, the sequential best-response dynamics converges to the unique Nash equilibrium for any initial point* $\mathbf{x}_0 \in \mathcal{X}$.

4.2 K player games with linear link cost functions

Consider $\phi(x) = x$, a delay function which is often used in congestion games to model delays in road networks. From (43), it follows that $\theta_k = 1/2$. Thus, the matrix M_u in Theorem 3 is of the form $\frac{1}{2}(\Gamma B - I)$ for some $\Gamma \in \mathcal{G}$.

THEOREM 7. *For the* K *player routing game over parallel links and linear delay function, the sequential best-response dynamics converges to the unique Nash equilibrium for any initial point* $\mathbf{x}_0 \in \mathcal{X}$.

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APPENDIX

A. EQUIVALENCE OF FUNCTIONS

Let ψ be a function satisiying assumptions (B_1)–(B_3). If the link cost function is ψ, then the player i will solve the following optimization problem:

$$\text{minimize } T_i(\mathbf{x}, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{S}} c_j \, x_{i,j} \, \psi(y_j) \qquad \text{(BRpsi-}i\text{)}$$

subject to

$$\sum_{j \in \mathcal{S}} x_{i,j} = \lambda_i, \qquad (25)$$

$$y_j = x_{i,j} + \sum_{k \neq i} x_{k,j}, \quad \forall j \in \mathcal{S}, \qquad (26)$$

$$x_{i,j} \geq 0, \quad \forall j \in \mathcal{S}, \qquad (27)$$

Note that there is no capacity associated with a link, and there is no constraint of the type $y_j < r_j$.

Let $r > \bar{\lambda}$ be a constant. Define the function ϕ as

$$\phi(x) = \begin{cases} \psi(rx) & \text{if } x \leq \rho; \\ \psi(rx) + \frac{1}{1-x} + \frac{1-x}{(1-\rho)^2} - \frac{2}{1-\rho} & \text{if } \rho < x < 1, \end{cases} \qquad (28)$$

where $\rho = \frac{\bar{\lambda}}{r}$. It can be verified that the ϕ satisaties assumptions (A_1)–(A_3).

We now show that the best-response of a player i when it solves (BRpsi-i) is the solution of (BR-i) with ϕ as in (28), and $r_j = r, \forall j \in \mathcal{S}$. If $\mathbf{z} \in \mathcal{X}_i$ is a solution of (BR-i), then according to the KKT conditions:

$$\frac{c_j}{r}\left[\phi\left(\frac{y_j}{r}\right) + z_j\phi'\left(\frac{y_j}{r}\right)\frac{1}{r}\right] = \mu_i \quad \text{if } z_j > 0,$$
$$\frac{c_j}{r}\phi\left(\frac{y_j}{r}\right) > \mu_i \quad \text{if } z_j = 0.$$

Using the relation (28), the above conditions can be replaced by:

$$c_j\left[\psi(y_j) + z_j\psi'(y_j)\right] = r\mu_i \quad \text{if } z_j > 0, \tag{29}$$
$$c_j\psi(y_j) > r\mu_i \quad \text{if } z_j = 0. \tag{30}$$

where we have used the fact that, $\forall j \in \mathcal{S}$, $y_j/r \leq \bar{\lambda}/r = \rho$.

It can be verified that (29)–(30) are the KKT conditions for (BRpsi-i). Hence, both (BR-i) and (BRpsi-i) have the same best response dynamics.

B. PROOF OF LEMMA 3

PROOF. The proof is based on two observations: *(i)* at a best-reponse strategy, the change in marginal cost of player u due to a change in the strategy of player v is the same in every link that is used at the best-response strategy; and *(ii)* the total flow is conserved for player u irrespective of the change in the strategy of player 1.

Recall that

$$g_{u,i}(x^{(u)}(\mathbf{x})) := \frac{\partial T_u}{\partial x_{u,i}}(x^{(u)}(\mathbf{x})).$$

is the marginal cost of player u at link i under strategy profile $x^{(u)}(\mathbf{x})$, i.e., after the best-response of player u.

For $i \in \mathcal{S}_u(\mathbf{x})$, from the KKT conditions, the best-response strategy of player u, $\mathbf{x}_u^{(u)}$, is such that the marginal cost is the same in all the links that receive a positive traffic at this strategy. That is,

$$g_{u,i}(x^{(u)}(\mathbf{x})) = \mu(\mathbf{x}_{-u}) \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{31}$$

where the constant μ depends upon the strategies of the players but not on the index of the link. The local stability of $\mathcal{S}_u(\mathbf{x})$ implies that the set of links used by user u does not change for an infinitesimal variation on the strategies of the other players. This leads to our first observation which is that the change in the marginal cost of player u at its best-response strategy due to the change in the strategy of player $v \neq u$ at link k is the same at all links that receive a positive flow of player u. Thus,

$$\frac{\partial g_{u,i}}{\partial x_{v,k}}(x^{(u)}(\mathbf{x})) = \mu_2, \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{32}$$

where μ_2 depends upon the strategies of the players. We have not made this dependence explicit in order to simplify the notation.

For a function of the form $h(f(x), x)$, its derivative with respect to x is given by

$$\frac{dh(f(x),x)}{dx} = \frac{dh(f,x)}{df}\frac{df}{dx} + \frac{dh(f,x)}{dx},$$

where in the first term on the RHS, h is treated to as a function of f only, whereas in the second term it is treated as a function of x only.

Since $x_{u,i}^{(u)}$ is a function of $x_{v,k}$, we can use the above formula to rewrite (32) as

$$\frac{dg_{u,i}}{dx_{u,i}}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} + \frac{dg_{u,i}}{dx_{v,k}} = \mu_2, \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{33}$$

where the partial derivates are replaced by full derivates in order to indicate that the function is differentiated in one variable while treating the other as constant.

The particular form of the cost function given in problem ((BR-i)) permits a simplification of the LHS of the above equation by noting that the marginal cost in a link depends only on the traffic that is routed to that link. Thus,

$$\frac{dg_{u,i}}{dx_{u,i}}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} + \delta_k(i)\frac{dg_{u,i}}{dx_{v,k}} = \mu_2, \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{34}$$

where $\delta_k(i)$ is unity if $i = k$, and is zero otherwise.

The value of μ_2 can be computed using the second observation that the total flow of player u is conserved irrespective of the strategy of player v. That is,

$$\sum_{i \in \mathcal{S}_u(\mathbf{x})}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} = 0 \tag{35}$$

We thus obtain

$$\begin{aligned}\mu_2 &= \left(\sum_{l \in \mathcal{S}_u(\mathbf{x})}\delta_k(l)\frac{dg_{u,l}}{dx_{v,k}}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)\left(\sum_{l \in \mathcal{S}_u(\mathbf{x})}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)^{-1} \\ &= \left(\frac{dg_{u,k}}{dx_{v,k}}\left(\frac{dg_{u,k}}{dx_{u,k}}\right)^{-1}\right)\left(\sum_{l \in \mathcal{S}_u(\mathbf{x})}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)^{-1},\end{aligned} \tag{36}$$

and

$$\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} = \theta_k\left(\gamma_i - \delta_k(i)\right), \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{37}$$

where

$$\theta_k = \frac{dg_{u,k}}{dx_{v,k}}\left(\frac{dg_{u,k}}{dx_{u,k}}\right)^{-1}, \tag{38}$$

and

$$\gamma_i = \left(\sum_{l \in \mathcal{S}_u(\mathbf{x})}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)^{-1}\left(\frac{dg_{u,i}}{dx_{u,i}}\right)^{-1}. \tag{39}$$

We will now show that $0 < \theta_k < 1$ and $0 < \gamma_i < 1$. We have

$$g_{u,k} = \pi_k\left(\phi(\rho_k) + \frac{x_{u,k}}{r_k}\phi'(\rho_k)\right). \tag{40}$$

Thus, since ϕ is an increasing and convex function,

$$\frac{dg_{u,k}}{dx_{v,k}} = \frac{\pi_k}{r_k}\left(\phi'(\rho_k) + \frac{x_{u,k}}{r_k}\phi''(\rho_k)\right) > 0, \tag{41}$$

independently of the player $v \neq u$, and

$$\frac{dg_{u,k}}{dx_{u,k}} = \frac{\pi_k}{r_k}\left(2\phi'(\rho_k) + \frac{x_{u,k}}{r_k}\phi''(\rho_k)\right) > 0. \tag{42}$$

Thus, from (38), $\theta_k > 0$ and

$$\theta_k = \frac{\phi'(\rho_k) + \frac{x_{u,k}}{r_k}\phi''(\rho_k)}{2\phi'(\rho_k) + \frac{x_{u,k}}{r_k}\phi''(\rho_k)} < 1. \tag{43}$$

We thus obtain that θ_k is independant of v and that $0 < \theta_k < 1$. Similarly, we note that γ_i is positive and smaller than unity due to the fact that $\frac{dg_{0,l}}{dx_{0,k}^{(1)}}$ is postive for all l. To conclude the proof, we note that $\sum_{i \in \mathcal{S}_u(\mathbf{x})} \gamma_i = 1$ from the definition of the vector $\boldsymbol{\gamma}$ in (39). Thus, letting $\gamma_i = 0$ for $i \notin \mathcal{S}_u(\mathbf{x})$, we obtain $\sum_{i \in \mathcal{S}} \gamma_i = 1$. □